NOTES ON COHOMOLOGY OF $\mathbf{SL}_n(\mathbb{Z})$

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In this short notes, we study the restriction map $\mathrm{res} : \mathrm{H}^*(\mathrm{SL}_n(\mathbb{Z}), M) \to \mathrm{H}^*(H, M)$, where H is a finite index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ and M is an $\mathrm{SL}_n(\mathbb{Z})$ module.

To answer the question, we need find all finite index subgroups of $\mathrm{SL}_n(\mathbb{Z})$. This problem was solved by Mennicke and Bass–Lazard–Serre [2], more specifically, they proved following theorem.

Theorem 0.1. [2, Theorem 14.1] *The kernel of the reduction map $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/l)$ is denoted by $\Gamma_n(l)$. A subgroup H of $SL_n(\mathbb{Z})$ is called congruence subgroup if there exsits $l \in \mathbb{Z}\backslash\{0\}$ such that $\Gamma_n(l) \subseteq H$. For $n \geq 3$, all finite index subgroups of $SL_n(\mathbb{Z})$ are congruence subgroups.*

For the case $n = 2$ Theorem 0.1 fails.

Let $\Gamma = [\mathrm{SL}_2(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z})] \subseteq \mathrm{SL}_2(\mathbb{Z})$. Then by [7, Chapter I 4.3, Proposition 18] Γ is a free group, and its rank is 2 [4, page 248, example 3]. We have computed $H_*(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ in homework 6, and we have free resolution for $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$ [4, page 18, example 4.3].

$$\mathrm{H}_*(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \mathbb{Z}, & * = 0, \\ \mathbb{Z}/12, & * > 0, \text{odd}, \\ 0, & * > 0, \text{even}. \end{cases} \quad \mathrm{H}_*(\Gamma) = \begin{cases} \mathbb{Z}, & * = 0, \\ \mathbb{Z}^{\oplus 2}, & * = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Since $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$, for a trivial $\mathrm{SL}_2(\mathbb{Z})$-module M, using the resolution free resolution for $\mathbb{Z}$ over $\mathbb{Z}[\Gamma]$ and Mayer-Vietoris sequence we get follows.

$$\mathrm{H}^*(\Gamma) = \begin{cases} M, & * = 0, \\ M \oplus M, & * = 1, \\ 0, & \text{otherwise}. \end{cases}$$

$$\cdots \longrightarrow \mathrm{H}^{n-1}(\mathbb{Z}/2, M) \longrightarrow \mathrm{H}^*(\mathrm{SL}_2, M(\mathbb{Z})) \longrightarrow \mathrm{H}^n(\mathbb{Z}/4, M) \oplus \mathrm{H}^n(\mathbb{Z}/6, M) \longrightarrow \mathrm{H}^n(\mathbb{Z}/2, M) \longrightarrow \cdots$$

Thus, by the formula of cohomology of finite cyclic groups we have follows.

$$0 \longrightarrow M \longrightarrow M \oplus M \longrightarrow M \longrightarrow \mathrm{H}^1(\mathrm{SL}_2(\mathbb{Z}), M) \longrightarrow M_4 \oplus M_6 \longrightarrow M_2 \longrightarrow \cdots$$

Take M is torsion-free.

$$\mathrm{H}^*(\mathrm{SL}_2(\mathbb{Z}), M) = \begin{cases} \mathbb{Z}, & * = 0, \\ 0, & \text{odd}, \\ M/4 \oplus M/3, & \text{otherwise}. \end{cases}$$

Thus, for a finite order element $\alpha \in \mathrm{H}^*(\mathrm{SL}_2(\mathbb{Z}), M), * > 0$, we have $\mathrm{res}(\alpha) = 0$, where $\mathrm{res} : \mathrm{H}^*(\mathrm{SL}_2(\mathbb{Z}), M) \to \mathrm{H}^*(\Gamma, M)$.

For $n \geq 3$, since every finite index subgroup $H \subseteq \mathrm{SL}_n(\mathbb{Z})$ is congruence subgroup, i.e., there exsits $\Gamma_n(l) \subseteq H \subseteq \mathrm{SL}_n(\mathbb{Z})$. By the functoriality of the restriction map, the original question is equivalent to the following question.

Question 0.2. *For a finite order element $\alpha \in H^*(SL_n(\mathbb{Z}), M)$, can one find some $l \in \mathbb{Z} \setminus \{0\}$ such that $res : H^*(SL_n(\mathbb{Z}), M) \to H^*(\Gamma_n(l), M)$ sends α to 0.*

Assume t is the order of α, $t \in \mathbb{Z}_{>1}$, then $t | \#\mathrm{SL}_n(\mathbb{Z}/l)$. Since $\mathrm{cores} \circ \mathrm{res} \alpha = \#SL_n(\mathbb{Z}/l)\alpha = 0$. And also t can not invertible in M, otherwise t is invertible in $\mathrm{H}^*(\mathrm{SL}_n(\mathbb{Z}\ M)$, then $\alpha = 0$, contradiction.

By Chinese remainder theorem we have

$$\#\mathrm{SL}_n(\mathbb{Z}/l) = \prod_{p|l} \#\mathrm{SL}_n(\mathbb{Z}/p^{k_p}).$$

By induction, one gets

$$\#\mathrm{SL}_n(\mathbb{Z}/p^{k_p}) = p^{(n^2-1)k_p} \prod_{i=2}^{n} (1 - \frac{1}{p^i}).$$

Next, we list some results to partially answer Question 0.2.

Recall that $\mathrm{SL}_n(\mathbb{Z})$ acts on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$. X is a contractible manifold of dimension $n(n+1)/2-1$. The $\mathbb{Q}$ -rank of $\mathrm{SL}_n(\mathbb{Q})/\mathrm{Radical} = \mathrm{SL}_n(\mathbb{Q})$ is $n-1$ given by diagonal matrices. For $l > 2, \Gamma_n(l) \subseteq \mathrm{SL}_n(\mathbb{Z})$ is torsion-free [4, page 40, exercises 3]. Passing to the cohomology of the compactification $\tilde{X}$ of X we have the duality theorem.

Theorem 0.3. [3, Theorem 14.1, 14.2] *Let N=n(n+1)/2-1-(n-1)=n(n-1)/2.*

$$H^i(\Gamma_n(l), \mathbb{Z}[\Gamma_n(l)]) = \begin{cases} 0 & i \neq N, \\ D, & i = N. \end{cases}$$

D is a free abelian group of infinite rank.

For a $\Gamma_n(l)$-module A, cap product induces isomorphism.

$$H^q(\Gamma_n(l), A) \simeq H_{N-q}(\Gamma_n(l), D \otimes A).$$

Recall the Steinberg module for a division ring Λ. Denote F the quotient filed of Λ. The Tits building $\mathscr{T}(n, \Lambda)$ is a simplicial complex. A p complex is a chain of proper subspace of $V = F^n$, $0 \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_q \subseteq V$.

Theorem 0.4. [6, Theorem 2] *$\mathscr{T}(n, \Lambda)$ has homotopy type of bouquet of $(n-2)$-spheres. Then $St(n)$ is defined to be $H_{n-2}(\mathscr{T}(n, \mathbb{Z}))$, it is a free abelian group and has a natrual $GL_n(\mathbb{Q})$ action. $St(n,p) = H_{n-2}(\mathscr{T}(n, \mathbb{F}_p))$, it is a free abelian group and has a natrual $GL_n(\mathbb{F}_p)$ action.*

Theorem 0.5. [5, Theorem 1.2 and corollary]

$$H^N(\Gamma_n(3), \mathbb{Z}) \simeq St(n, 3).$$

Thus by Theorem 0.3, 0.4 and 0.5 we have following theorem.

Theorem 0.6. *If there is finite order element $\alpha \in H^i(SL_n(\mathbb{Z}), i \geq N, i = 1$. Take subgroup $\Gamma_n(3)$. Then $res(\alpha) = 0$ where $res : H^i(SL_n(\mathbb{Z})) \to H^i(\Gamma_n(3), \mathbb{Z})$. Or there is not finite order element in $H^i(SL_n(\mathbb{Z}), i \geq N, i = 1$.*

Actually the coeffiencts module M can taken to be a flat $\Gamma_n(3)$-module, under this hypothesis the Theorem 0.6 still holds. Since we have $H_{N-q}(\Gamma_n(3), D \otimes M) = H_{N-q}(P_* \otimes D \otimes M) = H_{N-q}(P_* \otimes D) \otimes M \simeq St(n,3) \otimes M$, where P_* is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[\Gamma_n(3)]$.

Remark 0.7. For $n = 3$, $N = 3$, $\mathrm{H}^i(\mathrm{SL}_3(\mathbb{Z}))$ is computed in [8, Section 2.2]. And it indeed has torsion, namely we have

$$\mathrm{H}^i(\mathrm{SL}_3(\mathbb{Z})) = \begin{cases} 6m\mathbb{Z}/2, & i = 12m+1 \\ 6m\mathbb{Z}/2, & i = 12m+2 \\ (6m+2)\mathbb{Z}/2, & i = 12m+3 \\ 2\mathbb{Z}/3 \oplus 2\mathbb{Z}/4 \oplus 6m\mathbb{Z}/2, & i = 12m+4 \\ (6m+1)\mathbb{Z}/2, & i = 12m+5 \\ (6m+4)\mathbb{Z}/2, & i = 12m+6 \\ (6m+3)\mathbb{Z}/2, & i = 12m+7 \\ \cdots . \end{cases}$$

For lower dimension cohomology, we list some resluts on stable cohomology of $\mathrm{SL}_n(\mathbb{Z})$ to answer Question 0.2.

$\mathrm{SL}_n(\mathbb{Z})$ are homology stable with $\mathbb{Z}$-coefficients[1]. $H_i(\mathrm{SL}_n(\mathbb{Z}),\mathbb{Z}) \cong H_i(\mathrm{SL}_{n+1}(\mathbb{Z}),\mathbb{Z})$ for $n \geq 2i+1$. If $M=\mathbb{Q},\mathbb{Z}[1/p]$ or $\mathbb{Z}/m$ for all integers m with $(m,p)=1$, then $H_i(\Gamma_n(p);M) \cong H_i(\Gamma_{n+1}(p);M)$ for $n \geq 2i+5$.

For n large, $H^1(SL_n(\mathbb{Z})) = \mathrm{H}^2(\mathrm{SL}_n(\mathbb{Z})) = 0$, $H^3(\mathrm{SL}(\mathbb{Z})) \simeq \mathbb{Z}/2$, see[Lemma 2.1] [1].

For n large, $\text{ res }: H^3(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \to H^3(\Gamma_n(p);\mathbb{Z})$ is injective [1, Section 3].

Theorem 0.8. [1, Corollary 3.3] $c_2(\mathrm{SL}_n(\mathbb{Z}))$ *generates* $H^4(\mathrm{SL}_n(\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}/24$. *For all odd primes p the restriction map*

$$\textit{res: } H^4(\mathrm{SL}_n(\mathbb{Z}),\mathbb{Z}) \to H^4(\Gamma_n(p),\mathbb{Z})$$

is zero for $n \geq 9$.

Theorem 0.9. [1, Lemma 2.2] *Using universal coefficents theorem, we have* $H^5(SL_n(\mathbb{Z}))$ *contains only 2- and 3-torsion.*

The above discussion answer the Question 0.2 for $n \geq 9$.

Remark 0.10. For other cases, the results in [1] may be improved. The methods in [5] rely on the interpretation of low-dimensional homology.

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