NOTES ON GROTHENDIECK RING OF VARIETIES

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ABSTRACT. In this notes, we mainly concerned about $K_0(\mathrm{Var}_k)$. We will introduce the *Grothendieck spectrum* of varieties in [19, 20], and also the main results of [19]. And we will introduce some results on $K_0(\mathrm{Var}_k)$, and its relation with birational geometry.

CONTENTS

1. INTRODUCTION

Let Var_k denote the category of varieties over a field k. By a variety X over k, we mean a reduced, separated and finite type scheme over k. We define the Grothendieck ring of Var_k as follows.

Definition 1.1. $K_0(\mathrm{Var}_k) := \mathbb{Z}[\mathrm{Var}_k]/R$. $\mathbb{Z}[\mathrm{Var}_k]$ is the free abelian group generated by isomorphism classes of varieties over k. And R is the subgroup of $\mathbb{Z}[\mathrm{Var}_k]$ generated by $[X] - [Y] - [X \setminus Y]$, where Y is a closed subvariety of X. And the multiplication of $K_0(\mathrm{Var}_k)$ is defined to be $[X] \cdot [Y] := [(X \times_k Y)_{\mathrm{red}}]$. We denote the class of $\mathbb{A}^1_k$ in $K_0(\mathrm{Var}_k)$ as $\mathbb{L}$, called *Lefschetz motive.* $\mathcal{M}_k := K_0[\mathrm{Var}_k][\mathbb{L}^{-1}]$.

We interested in this ring because we can define *motivic measure* and *motivic zeta function*, and it has many applications in birational geometry, for more details of its application one can see [4, 7, 9, 11, 12].

Definition 1.2. A *motivic measure* valued in ring A is a ring homomorphism

$$\mu : K_0(\mathrm{Var}_k) \to A.$$

So, μ satisfies

$$\mu([X]) = \mu([X \setminus Y]) + \mu([Y])$$
$$\mu([X \times Y]) = \mu([X]) \cdot \mu([Y]).$$

Example 1.3 ($\mathbb{Z}-$valued motivic measure)**.** For any variety X over k, let

$$\mu([X]) = \sum_i (-1)^i \dim_{\mathbb{Q}_l} \mathrm{H}^i_c(X \times_k k_s, \mathbb{Q}_l),$$

where $(l, \mathrm{char}(k)) = 1$.

So, form the theory of étale cohomology, if $k = \mathbb{C}$, then $\mu([X]) = \chi_{\mathrm{top}}(X)$.

Example 1.4. Let $k = \mathbb{F}_q$, where $q = p^k$. Let

$$N_r(X) = |X(\mathbb{F}_{q^r})|.$$

N gives a $\mathbb{Z}-$valued measure on $K_0(\mathrm{Var}_{\mathbb{F}_q})$. This relates to 1.3 by étale cohomology theory.

Example 1.5. Let $k = \mathbb{C}$. By the theory of Deligne's mixed Hodge theory, the Hodge-Deligne polynomial gives a measure, i.e.

$$\begin{aligned} Hdg : K_0(\mathrm{Var}_k) &\to \mathbb{Z}[u,v] \\ [X] &\mapsto \sum_{p,q\geq 0, 0\leq k\leq 2\dim X} (-1)^k h^{p,q}(\mathrm{H}^k(X,\mathbb{Q}))u^p v^q, \end{aligned}$$

where $h^{p,q}$ is the Hodge number. Recall that a *weight n pure Hodge structure* is a $\mathbb{Q}-$vector space V together with a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \oplus_{p+q=n} V^{p,q},$$

and $\overline{V^{q,p}} = V^{p,q}$. A *mixed Hodge structure* is a finite dimensional $\mathbb{Q}-$vector space V together with weight filtration and Hodge filtration, i.e.

$$\begin{aligned} 0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W^{2k} = V. \\ V \otimes \mathbb{C} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m = 0. \end{aligned}$$

F^* induces a filtration on $\mathrm{Gr}_l^W = W_l/W_{l-1}$, i.e. $(F^p \cap W_l \otimes \mathbb{C} + W_{l-1}\otimes\mathbb{C})/W_{l-1}\otimes\mathbb{C}$. And we required that there is a weight l pure Hodge structure on $F^p(\mathrm{Gr}_l^W)$. Deligne proved for any $X \in \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}})$, there is a mixed Hodge structure for $\mathrm{H}^k(X,\mathbb{Q})$, i.e. there exists $(\mathrm{H}^*(X,\mathbb{Q}), W_*, F^*)$ is a mixed Hodge structure. The Hodge number $h^{p,q} := \dim_{\mathbb{C}} \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^k(X,\mathbb{Q})$. So, if x is a smooth projective variety over $\mathbb{C}$, we take the trivial weight filtration $W_l = 0, l \neq 2k$, $W_{2k} = \mathrm{H}^{2k}(X,\mathbb{Q})$. By Hodge decomposition $\mathrm{H}^k(X,\mathbb{C}) = \oplus_{p+q=k} \mathrm{H}^p(X,\Omega^q)$, take $F^l = \oplus_{p+q=n,p\geq l}(\mathrm{H}^p(X,\Omega^q))$. Therefore $h^{p,q} = \dim_{\mathbb{C}} \mathrm{H}^p(X,\Omega^q)$. (This is the Hodge number in the classical case.)

Kontsevich use this measure to prove a result in birational geometry which was motivated by the topological mirror symmetry test of string theory.

Proposition 1.6. [2, Corollary 6.29] *Let X_1, X_2 be two birational equivalent Calabi-Yau varieties(we only require canonical divisor trivial), then X_1 and X_2 have same Hodge numbers.*

Definition 1.7. We say an element $\tau \in K_0(\mathrm{Var}_k)$ is dimension d if there exists an expression

$$\tau = \sum a_i[X_i]$$

with $\dim X_i \leq d, \forall i$, and no expression with all $\dim \leq d-1$.

The dimension function gives a filtration on $\mathcal{M}_k$, i.e.

$$\cdots \supseteq F^{-1}\mathcal{M}_k \supseteq F^0\mathcal{M}_k \supseteq \cdots$$

$F^i\mathcal{M}_k$ is the subgroup generated by $[X]\cdot\mathbb{L}^{-k}$ with $\dim[V] - k \leq i$, i.e. the subgroup with all elements $\dim \leq i$. Then he defined $\hat{\mathcal{M}}_k$ to be the completion of this filtration, i.e. $\lim_i \mathcal{M}_k/F^i\mathcal{M}_k$.

Sketch of proof of Proposition 1.6. First, we need to extend Hodge measure to

$$\mathcal{M}_k \to \mathbb{Z}[u,v,(uv)^{-1}].$$

Then we can extend Hodge measure to

$$E : \hat{\mathcal{M}}_k \to \mathbb{Z}[u,v,(uv)^{-1}, \frac{1}{uv-1}, \frac{1}{(uv)^2-1}, \ldots].$$

For any $X \in \mathrm{Obj}\,\mathrm{Var}_{\mathbb{C}}$ we can define the arcs space $J_\infty(X)$. Let $f : X \to Y$ be a proper birational morphism, with discrepancy $D = K_Y - f^*K_X = \sum_{i=1}^r a_i D_i$. The motivic integral of pair (Y, D) is

$$\int_{J_\infty(Y)} F_D \mathbb{L}^n d\mu = \sum_{J \subseteq \{1,\ldots,r\}} [D_J^\circ] \cdot (\prod_{j \in J} \frac{\mathbb{L} - 1}{(\mathbb{L})^{a_j+1} - 1}). \tag{1.0.1}$$

Let $f : X_1 \dashrightarrow X_2$. By Hironaka's theorem we have two projective birational morphisms $f_1 : Y \to X_1, f_2 : Y \to X_2$, such that $f = f_2 \circ f_1^{-1}$. Let $D_i = K_Y - f_1^* K_{X_i}, i = 1, 2$. Since the canonical divisors are trivial, thus $D_1 \sim D_2$.

By standard methods in birational geometry, we actually can show $D_1 = D_2$.

By *transformation rule* (analogue of change variables), we have

$$\int_{J_\infty(X_1)} F_0 \mathbb{L}^n d\mu = \int_{J_\infty(Y)} F_{D_1} \mathbb{L}^n d\mu = \int_{J_\infty(X_2)} F_0 \mathbb{L}^n d\mu.$$

Now, by the formula in 1.0.1, the left hand side is $[X_1]$ and the right hand side is $[X_2]$. So $[X_1] = [X_2] \in \hat{\mathcal{M}}_k$.

And we have the following commutative diagram

$$\begin{tikzcd}
K_0(\mathrm{Var}_k) \arrow[r] \arrow[d, "Hdg"] & \mathcal{M}_k \arrow[r] \arrow[d] & \hat{\mathcal{M}}_k \arrow[d, "E"] \\
\mathbb{Z}[u,v] \arrow[r, hook] & \mathbb{Z}[u,v,(uv)^{-1}] \arrow[r, hook] & \mathbb{Z}[u,v,(uv)^{-1}, \frac{1}{uv-1}, \frac{1}{(uv)^2-1}, \ldots]
\end{tikzcd}$$

Therefore $E[X_1] = E[X_2]$. So, if $Hdg[X_1] \neq Hdg[X_2]$, by the commutativity of the above diagram, we can have $E[X_1] \neq E[X_2]$, but that gives a contradiction. So X_1 and X_2 have same Hodge polynomial, thus they have same Hodge numbers. $\square$

We thus have some questions relate to the above constructions.

Question 1.8. Is the localization map $K_0(\mathrm{Var}_k) \to \mathcal{M}_k$ injective? Or equivalently, is $\mathbb{L}$ a zero divisor?

Question 1.9. More generally, is $K_0(\mathrm{Var}_k)$ a domain?

We can define the "filtration" for $K_0(\mathrm{Var}_k)$.

$$F_n := \mathbb{Z}[X \,|\, \dim X \leq n] / < [X] - [X \setminus Y] - [Y] >,$$

where the generators are isomorphism classes of varieties with dimension $\leq n$, Y is a closed subvariety of X. There is a map, namely

$$\begin{aligned} \psi_n : F_n &\to K_0(\mathrm{Var}_k) \\ [X] &\to [X]. \end{aligned}$$

Question 1.10. Is ψ_n injective?

In [13], Poonen proved that $K_0(\mathrm{Var}_k)$ is not a domain. We will explain that in section 2. It turns out Question 1.8 and Question 1.10 are related. In [19], Inna proved that if $\mathbb{L}$ is a zero divisor then ϕ_n is not injective for some n. And in [3], Borisov proved that $\mathbb{L}$ is a zero divisor, So ϕ_n is not injective for some n. We will explain Question 1.10 in section 3. There is also a question relates to Question 1.8 and Question 1.10, but we still need some basic facts to illustrate it.

There are some basic facts.

Proposition 1.11.

(1) $[\varnothing] = 0 \in K_0(\mathrm{Var}_k)$.
(2) $[\mathrm{Spec}\, k] = 1 \in K_0(\mathrm{Var}_k)$.

(3) $[\mathbb{P}^n_k] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n \in K_0(\mathrm{Var}_k)$.

Proof.

(1) $[X] = [X \setminus X] + [X]$, so $[\varnothing] = 0$.

(2) $[X] = [X \times \operatorname{Spec} k] = [X] \cdot [\operatorname{Spec} k] = [\operatorname{Spec} k] \cdot [X]$.

(3) By induction on n, we have $\mathbb{P}^n = \mathbb{A}^n_k \bigsqcup \mathbb{P}^{n-1}_k$, for $n = 1$ by (2) we have $[\mathbb{P}^1_k] = 1 + \mathbb{L}$, so $[\mathbb{P}^n] = 1 + \cdots + \mathbb{L}^{n-1} + \mathbb{L}^n$. □

Definition 1.12. Let $X \in \operatorname{Obj} \mathrm{Var}_k$, Y is called locally closed subvariety of X if Y is a intersection of open subset and closed subset.

Proposition 1.13. *If $X \in \operatorname{Obj} \mathrm{Var}_k$ and $X = X_1 \bigsqcup \ldots \bigsqcup X_r$, where X_i is locally closed subvariety of X. Then*

$$[X] = \sum_{i=1}^{r} [X_i] \in K_0(\mathrm{Var}_k).$$

Proof. By induction on $\dim X$ and then by induction on the number of irreducible components of X of maximal dimension. When $\dim X = 0$, this is trivial. Let $X = \bigcup_{i=0}^{k} Z_i$ be the irreducible decomposition.(Note X is finite type over k.) Assume $\dim Z_{i_0} = \dim X, 0 \leq i_0 \leq k$ Take the generic point η of Z_{i_0}, suppose $\eta \in X_j, 1 \leq j \leq r$. Then $Z_{i_0} = \overline{\eta} \subseteq \overline{X_j}$. Write $X_j = V \cap \overline{X_j}$, V is open in X. Then $X \setminus \bigcup_{i \neq i_0} Z_i = \bigcap_{i \neq i_0} (X \setminus Z_i) \subseteq X$ is open. And $X \setminus \bigcup_{i \neq i_0} Z_i \subseteq Z_{i_0}$, so $V \cap X \setminus \bigcup_{i \neq i_0} Z_i \subseteq Z_{i_0} \cap X_j$. Let $U = V \cap X \setminus \bigcup_{i \neq i_0} Z_i$, then U is open in X. So we have $[X] = [U] + [X \setminus U], [X_j] = [U] + [X_j \setminus U]$.

(1) If $\dim X \setminus U < \dim X$, and we have $X \setminus U = X_j \setminus U \bigsqcup_{i \neq j} X_j$. Then by induction $[X \setminus U] = [X_j \setminus U] + \sum_{i \neq j} [X_j]$, therefore $[X] = \sum_i [X_i]$.

(2) If $\dim X \setminus U = \dim X$, then we claim $|\{Z'_j | \dim Z'_j = \dim X \setminus U, Z'_j\}| < |\{Z_j | \dim Z_j = \dim X\}|$. Then by induction we also have $[X \setminus U] = [X_j \setminus U] + \sum_{i \neq j} [X_j]$. Then $[X] = \sum_i [X_i]$. Suppose the claim is false, assume

$$m = |\{Z'_j | \dim Z'_j = \dim X \setminus U, Z'_j \text{ is irreducible component}\}|,$$

$$n = |\{Z_j | \dim Z_j = \dim X\}|.$$

Then $m \geq n$. Any Z'_j is also a irreducible component in X, because $\dim Z_{i_0} = \dim X$, so $n = m + 1 \leq m$, a contradiction. □

Definition 1.14. Let X, Y be two varieties over k, X, Y are piecewise isomorphic if there are decomposition $X = \bigsqcup_{i=1}^n X_i, Y = \bigsqcup_{i=1}^n Y_i$ such that $X_i \simeq Y_i$.

By Proposition 1.13, if two varieties X, Y are piecewise isomorphic, then $[X] = [Y] \in K_0(\mathrm{Var}_k)$. The natural question comes.

Question 1.15. If $X, Y \in \operatorname{Obj} \mathrm{Var}_k$, and $[X] = [Y] \in K_0(\mathrm{Var}_k)$, does X and Y are piecewise isomorphic? This also called cut-and-paste conjecture. See [9, Question 1.2].

Remark 1.16. It is clear that if X and Y are piecewise isomorphic then X and Y are bijective as sets, but it is not clear that they are isomorphic as topological spaces. Moreover, it is incorrect that they are **isomorphic as schemes** in general. Basically because for locally closed morphism $i : X \to Y$ we only have sujective morphism

$$i^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

And actually there is a categorical way to understand piecewise isomorphism. In [3] Borisov gives a counterexample of Question 1.15, and in [19] Inna also proved cut-and-paste conjecture fails using 1.10.

Let Schemes be the category of scheme and $\mathrm{Schemes}_0$ be the full subcategory of Schemes whose local rings are fields. There is a functor cons : Schemes $\to$ $\mathrm{Schemes}_0$ is the right adjoint with $i : \mathrm{Schemes}_0 \to \mathrm{Schemes}$.

Proposition 1.17. [10, Proposition 2] *Let $X, Y \in \operatorname{Obj}\operatorname{Var}_k$, then X, Y are piecewise isomorphic if and only if $X^{\mathrm{cons}} \simeq Y^{\mathrm{cons}}$.*

Proposition 1.18. *Let $f : X \to Y$ be a proper morphism of smooth varieties over k, which is a blow up with a smooth center $Z \subseteq Y$ of co-dimension d. Then $[f^{-1}(Z)] = [Z][\mathbb{P}_k^{d-1}]$.*

2. $K_0(\mathrm{Var}_k)$ is not a domain

In this section, we will prove following theorem.

Theorem 2.1. [13, Theorem 1] *Suppose k is characteristic 0, then $K_0(\mathrm{Var}_k)$ is not a domain.*

We need some lemmas.

Lemma 2.2. *Let k be a field with $\mathrm{char}(k) = 0$. There exists an abelian variety A defined over k such that $\operatorname{End}_k(A) = \operatorname{End}_{\bar{k}}(A) \simeq \mathcal{O}$, where $\mathcal{O}$ is ring of algebraic integers with class number 2.*

Proof. By checking the database [18], we know that there is $f \in S_2(\Gamma_0(590))^{\mathrm{new}} \subseteq S_2(\Gamma_1(590))^{\mathrm{new}}$ such that the Fourier expansion of f is

$$f = q + (-1)q^2 + \sqrt{10}q^3 + \sum_{n=4}^{\infty} a_n q^n, q = e^{2\pi i z}.$$

And the field $K = \mathbb{Q}(a_1, a_2, \ldots) = \mathbb{Q}(\sqrt{10})$. By [16, Theorem 1] and [5, Theorem 6.6.6, Definition 6.6.3], there is an abelian variety A_f defined over $\mathbb{Q}$ which is a quotient of $\operatorname{Jac}(X_1(590))$ And $\operatorname{End}(A) \otimes \mathbb{Q} \simeq K$. And we know $\operatorname{End}(A)$ is an order of $\operatorname{End}(A) \otimes \mathbb{Q}$, and $\operatorname{End}(A)$ containing $a_3 = \sqrt{10}$.(See [5, Definition 6.6.3].) Therefore $\operatorname{End}(A)$ containing $\mathbb{Z}[\sqrt{10}]$, and $\operatorname{End}(A)$ is rank 2 free abelian group, so $\operatorname{End}(A) \simeq \mathbb{Z}[\sqrt{10}]$. And A_f is semistable because 590 is square free, so by [14, Corollary 1.4], $\operatorname{End}_{\bar{k}}(A) = \operatorname{End}(A) \simeq \mathbb{Z}[\sqrt{10}]$. The class number of $\mathbb{Z}[\sqrt{10}]$ is 2 (By standard method in number theory, for instance Minkowski bound. Or checking the database [18].) ☐

Let A be an abelian variety, denote $\mathcal{O} = \operatorname{End}(A)$. There is a fully faithful functor:

$$T : \operatorname{fgproj} \mathcal{O}\mathrm{mod} \to \mathrm{Abvar}_k.$$

$\operatorname{fgproj} \mathcal{O}\mathrm{mod}$ is the category of finitely generated projective $\mathcal{O}-$module. Abvar_k is the category of abelian varieties over k.

Roughly speaking, for a projective $\mathcal{O}-$module M, we have a presentation

$$\mathcal{O}^m \xrightarrow{\psi} \mathcal{O}^n \longrightarrow M \longrightarrow 0.$$

Then ψ gives a map from A^m to A^n, now $T(M) := \operatorname{coker}(\psi_*)$. One can prove that $T(M)$ is well defined.

Let X, Y be two smooth projective geometrically integral varieties. X is stably birational to Y if $X \times \mathbb{P}^m$ is birational to $Y \times \mathbb{P}^n$ for some $m, n \in \mathbb{N}$. The set SB_k be the equivalent class of this relation. Let $\mathbb{Z}[SB_k]$ be the free abelian group generated by SB_k. And let $\mathbb{Z}[\mathrm{Abvar}_k]$ be the free group generated by isomorphism classes of abelian varieties. By [9, Theorem 2.3], there is an unique morphism $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[SB_{\mathbb{C}}]$ and by taking the Albanese functor we can have a morphism $\mathbb{Z}[SB_{\mathbb{C}}] \to \mathbb{Z}[\mathrm{Abvar}_{\mathbb{C}}]$. And actually these map can also defined for $k \models \mathrm{ACF}_0$, basically because the method only require resolution of singularities and weak factorization of birational maps.

Proof of Theorem 2.1. By Lemma 2.2, we can take a fractional ideal $I \subseteq \mathbb{Q}(\sqrt{10})$, such that $I \neq 0 \in Cl(\mathbb{Z}[\sqrt{10}])$. Because $\mathbb{Z}[\sqrt{10}]$ is Dedekind domain, by the structure theory of modules over Dedekind domain, we know $I \oplus I \simeq \mathbb{Z}[\sqrt{10}] \oplus \mathbb{Z}[\sqrt{10}]$. Because $[I] \cdot [I] = 0 \in Cl(\mathbb{Z}[\sqrt{10}])$, and ranks are same. Let $B = T(I)$, then $A, B \in \operatorname{Obj} \mathrm{Abvar}_k$. And $B \times_k B \simeq T(I \oplus I) \simeq T(\mathbb{Z}[\sqrt{10}] \oplus \mathbb{Z}[\sqrt{10}]) \simeq A \times_k A$. But $B_{\bar{k}} \simeq T(I) \not\simeq T(\mathbb{Z}[\sqrt{10}]) \simeq A_{\bar{k}}$. Therefore $[A \times_k A] = [B \times_k B] \in K_0(\mathrm{Var}_k)$, i.e. $([A] + [B])([A] - [B]) = 0 \in K_0(\mathrm{Var}_k)$. We claim that $[A] + [B] \neq 0, [A] - [B] \neq 0$. Consider the map

$$K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{Var}_{\bar{k}}) \longrightarrow \mathbb{Z}[SB_{\bar{k}}] \longrightarrow \mathbb{Z}[\mathrm{Abvar}_{\bar{k}}].$$

$[A] - [B] \neq 0$ because $[A] \neq [B] \in \mathbb{Z}[\mathrm{Abvar}_{\bar{k}}]$, thus $[A] - [B] \neq 0 \in K_0(\mathrm{Var}_k)$, and $[A] + [B] \neq 0$ because A is not trivial abelian variety. Therefore $K_0(\mathrm{Var}_k)$ is not a domain. ☐

3. Cut-and-paste conjecture fails

In this section, we give an answer for Question 1.15 and Question 1.10. More precisely, we will prove following theorem.

Theorem 3.1. *Let k be a field with $\mathrm{char}(k) = 0$, there exists $X, Y \in \operatorname{Obj} \mathrm{Var}_k$ such that $[X] = [Y]$ but X and Y are **not** piecewise isomorphic.*

And actually we will prove a more general result.

Theorem 3.2. *Let k be a field with $\mathrm{char}(k) = 0$, there is $n \in \mathbb{N}$ such that ψ_n in 1.10 is not injective.*

In order to understand the structure of $K_0(\mathrm{Var}_k)$, in [19] Inna defined a special Grothendieck site to study $K_0(\mathrm{Var}_k)$, namely *assembler*, and for any assembler we can define a spectrum for the assembler, and the ith homotopy group of the spectrum is called ith K-theory for the assembler.

Definition 3.3. Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a sieve in $\mathcal{C}$ if for any morphism $A \to B \in \mathrm{Mor}(\mathcal{C})$ with $B \in \mathcal{D}$ then $A \in \mathcal{D}$.

Remark 3.4. The above definition is slightly different with [17, Tag 00YX], but when we consider the sieve in $\mathcal{C}/U$, the above definition will coincide with [17, Tag 00YX]. However, the above definition allow us to talk about the localization sequence, see Theorem 3.20.

Definition 3.5. A Grothendieck topology on a category is a collection $J(C)$ of sieves in $\mathcal{C}/C$ for all $C \in \operatorname{Obj} \mathcal{C}$, such that the following conditions hold:

- If $S \in J(C)$ and $f : B \to C \in \mathrm{Mor}(\mathcal{C})$ then $f^*S \in J(B)$.
- Let $S \in J(C)$ and T be any sieve in $\mathcal{C}/C$. $f^*T \in J(B)$ for any $f : B \to C \in \operatorname{Obj} S$ then $T \in J(C)$.
- $\mathcal{C}/C \in J(C)$.

Remark 3.6. For any $C \in \mathcal{C}$ given $\{f_i : C_i \to C\}$ we have a sieve S containing f_i see [17, Tag 00YC]. And there is a topology on $\mathcal{C}$ containing S, we called this topology is generated by the coverage $\{f_i : C_i \to C\}$. A category with a Grothendieck topology is called a Grothendieck site.

Given a family of morphisms in $\mathcal{C}$ $\{f_i : A_i \to A\}_{i \in I}$ we say it is a covering family if the full subcategory containing

$$\{g : X \to A | \exists i \in I, h : x \to A, h \circ f_i = g\}$$

is in $J(A)$.

Remark 3.7. One could use covering family to define Grothendieck pretopology, and for a pretopology there is a Grothendieck topology associated to that pretopology. The category of sheaves over the associated topology is equivalent to the category of sheaves on that pretopology, that is why we usually define "Grothendieck topology" to be the pretopology. See [17, Tag 00ZC]. But in our case, we need the real Grothendieck topology.

Definition 3.8. An assembler $\mathcal{C}$ is a small Grothendieck site satisfying following conditions:

- All the morphsims in $\mathcal{C}$ are monomorphism.
- For any $A \in \operatorname{Obj} \mathcal{C}$ any two finite disjoint covering families have a common refinement which itself a finite disjoint family.
- $\mathcal{C}$ has initial object $\varnothing$, the empty covering family is a covering family for $\varnothing$.

We denote $\mathcal{D}^\circ$ to be the full subcategory of noninitial objects in $\mathcal{D}$.

Example 3.9. For our case, we denote $\mathcal{V}_k$ to be the assembler whose objects are varieties over k and morphisms are locally closed embedding. And the topology of $\mathcal{V}_k$ is generated by the coverage $\{Y \hookrightarrow X, X \setminus Y \hookrightarrow X\}$, where Y is a closed subvariety of X.

Definition 3.10. Let $\mathcal{C}, \mathcal{D}$ be two assemblers, $F : \mathcal{C} \to \mathcal{D}$ is a morphism of assemblers if F is a morphism of sites and F preserve initial object and disjointness.

One can prove the category of assemblers have products and coproducts. Actually, if $\{\mathcal{C}_x\}_{x \in X}$ is $X-$tuples of assemblers. Then the class of objects of $\bigvee_{x \in X} \mathcal{C}_x$ is $\{\varnothing\} \cup \coprod_{x \in X} \operatorname{Obj} \mathcal{C}_x^\circ$, morphisms are clear. The class of objects of $\prod_{x \in X} \mathcal{C}_x$ is $\prod_{x \in X} \operatorname{Obj} \mathcal{C}_x$, morphisms are clear.

Definition 3.11. Let $\mathcal{C}$ be an assembler. We define $W(\mathcal{C})$ to have objects $\{A_i\}_{i \in I}$, where I is a finite set and all $A_i \in \operatorname{Obj} \mathcal{C}^\circ$. A morphism $f : \{A_i\}_{i \in I} \to \{B_i\}_{j \in J}$ is a map $f : I \to J$ and morphisms $f_i : A_i \to B_{f(i)}$. Such that $\{f_i : A_i \to B_j\}_{i \in f^{-1}(j)}$ is a finite disjoint covering family in $\mathcal{C}$.

For a pointed set X, $X \wedge \mathcal{C}$ is denoted to be the assembler $\bigvee_{x \in X \setminus \{*\}} \mathcal{C}_x$.

We define the K-theory spectrum of an assembler to be a spectrum associated to a $\Gamma-$space.

Definition 3.12. [15, Definition 1.1] Γ is the category whose objects are finite sets, whose morphism $S \to T$ is a map $\theta : S \to \mathcal{P}(T)$ such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint for $\alpha \neq \beta$. The composition of $\theta : S \to \mathcal{P}(T)$ and $\varphi : T \to \mathcal{P}(U)$ is $\psi : S \to \mathcal{P}(U)$ where $\psi(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \varphi(\beta)$. We denote $\mathbf{n}$ to be the set $\{1, 2, \ldots, n\}_+$.

Definition 3.13. A Γ-category is a contravariant functor $\mathcal{C}$ form Γ to categories such that:

- $\mathcal{C}(\emptyset_+)$ is equivalent to the category with one object and one morphism.
- for each n, the functor $p_n : \mathcal{C}(\mathbf{n}) \to \underbrace{\mathcal{C}(\mathbf{1}) \times \cdots \times \mathcal{C}(\mathbf{1})}_{n}$ induced by $i_k : \mathbf{1} \to \{k\} \subseteq \mathbf{n}$ is an equivalence of categories.

A Γ-space is a functor from Γ to the category of simplicial sets, satisfies some conditions that you could imagine. Actually one could extend Γ-space as a bisimplicial set. Namely we have a functor $\Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$, [15, Corollary 2.2] says if $\mathcal{C}$ is a $\Gamma-$category then $S \to |N(\mathcal{C}(S))|$ is a Γ-space. And for any Γ-space A there is an Ω-spectrum $\mathbf{B}A$, and actually $(\mathbf{B}A)_n \simeq \Omega(\mathbf{B}A)_{n+1}, n \geq 1$, and $(\mathbf{B}A)_0 \simeq \Omega A(S^1)$, $A(S^0) \to \Omega A(S^1)$ is a group completion on π_0. See [15, Section 4].

Definition 3.14. It turns out that $X : S \to W(S \wedge \mathcal{C})$ is a Γ- category for any assembler $\mathcal{C}$, see [20, Proposition 2.11 (3)]. So $X : S \to |N(W(S \wedge \mathcal{C}))|$ is a Γ-space, so We define the K-theory spectrum $K(\mathcal{C})$ to be the spectrum $\mathbf{B}X$.

Remark 3.15. The above definition is slightly different form [20, Definition 2.12], but they are equivalent, see [20, Theorem 2.13]. And the author think Definition 3.14 keeps the original idea of defining $K(\mathcal{C})$, roughly speaking, one need to find a spectrum for $\mathcal{V}_k$ whose 0-th homotopy group is $K_0(\mathrm{Var}_k)$. But in general, π_0 may not have group structure, from the theory of $\Gamma-$space we know there is a method making π_0 into a group(see [15, Section 4]), that is exactly how Definition 3.14 work. So, the main point is that $W(\mathcal{C})$ contains the whole scissors congruence information in $\mathcal{C}$.

There are some important results on $\pi_*(K(\mathcal{C}))$.

Theorem 3.16. *(1) The group $\pi_0(K(\mathcal{C}))$ is the free abelian group on objects of $\mathcal{C}$, under the relations that for any finite disjoint covering family $\{f_i : A_i \to A\}$ in $\mathcal{C}$,*

$$[A] = \sum_i [A_i].$$

(2) Every element in $\pi_1(K(\mathcal{C}))$ can be represented by following data:
- *a pair of finite tuples $\{A_i\}_{i\in I}, \{B_j\}_{j\in J}$ of objects in $\mathcal{C}$*
- *for $\epsilon = \pm 1$, a map of finite set $f_\epsilon : I \to J$ and for all $i \in I$, morphism $f_{\epsilon,i} : A_i \to B_{f_\epsilon(i)}$ such that for $\{f_{\epsilon,i} : A_i \to B_j\}_{i\in f_\epsilon^{-1}(j)}$ is a covering family.*

Sketch of proof. We only sketch the proof of (1) to get readers some feeling about this direction. We keep the notations used in Definition 3.14. Actually $\pi_0(\mathbf{B}X)$ is the group completion of $\pi_0(X(\mathbf{1}))$. $X(\mathbf{1}) = |N(W(\mathcal{C}))|$. And $\pi_0(X(\mathbf{1}))$ is a monoid, the operation is induced by

$$\pi_0(X(\mathbf{1})) \times \pi_0(X(\mathbf{1})) \simeq \pi_0(X(\mathbf{2})) \to \pi_0(X(\mathbf{1})).$$

The maps above are induced by following diagram :

$$W(\mathcal{C}) \times W(\mathcal{C}) \xrightarrow{P_2^{-1}} W(\mathcal{C} \vee \mathcal{C}) \xrightarrow{\mu} W(\mathcal{C}). \tag{3.0.1}$$

By the theory of simplicial homotopy theory, for instance see [6, Section 2], [8]. $\pi_0(X(\mathbf{1})) \simeq \pi_0(N(W(\mathcal{C}))) \simeq \operatorname{Obj} W(\mathcal{C})/\sim$. Therefore let $\{A_i\}_{i\in I}, \{B_j\}_{j\in J} \in \operatorname{Obj} W(\mathcal{C})$, the operation of $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ under 3.0.1 is $\{C_k\}_{k\in I\sqcup J}$ where $C_k = A_k$ if $k \in I$, $C_k = B_k$ if $k \in J$. The relations over $\pi_0(W(\mathcal{C}))$ is induced by 1-simplices of $N(W(\mathcal{C}))$, i.e. morphisms in $W(\mathcal{C})$. A morphism $f : \{A_i\}_{i\in I} \to \{B_j\}_{j\in J}$ with components morphisms $\{f_i : A_i \to B_j\}_{j\in f^{-1}(i)}$. $(A_i, B_j \in \mathcal{C}^\circ)$. The components are finite disjoint covering family, therefore f can be written as $\bigsqcup_{j\in J}\{A_i \to B_j\}_{i\in f^{-1}(j)}$. Therefore

$$[B_j] = \sum_{i\in f^{-1}(j)} [A_i] \in \pi_0(NW(\mathcal{C})).$$

Notice that $\{A_i\}_{i\in I} = \bigsqcup_{i\in I}\{A_i\}$, i.e.

$$[\{A_i\}_{i\in I}] = \sum_{i\in I} [A_i] \in \pi_0(NW(\mathcal{C})).$$

Therefore $\pi_0(X(\mathbf{1}))$ is the abelian group generated by **noninitial** objects in $\mathcal{C}$ modulo the relation stated in the theorem. By the relation stated in the theorem, $[\varnothing] = 0 \in \pi_0(K(\mathcal{C}))$. ($\mathrm{Hom}_{\mathcal{C}}(C, \varnothing) = \emptyset$.) That means $\pi_0(K(\mathcal{C}))$ is isomorphic to the group in the theorem. Done. ☐

Now let us focus on the assembler $\mathcal{V}_k$ in the Example 3.9. We denote $K_i(\mathcal{C})$ to be $\pi_i(K(\mathcal{C}))$. For $\mathcal{C} = \mathcal{V}_k$, the spectrum $K(\mathcal{V}_k)$ is called *Grothendieck spectrum* of varieties in [19].

Theorem 3.17. $K_0(\mathcal{V}_k) \simeq K_0(\mathrm{Var}_k)$.

Proof. The generators are same for those two groups, by Proposition 1.13 and Theorem 3.16 the relations are same. ☐

In the following, we fix a field, for instance $k = \mathbb{C}$. And we write $\mathcal{V}_k$ as $\mathcal{V}$.

Definition 3.18. Let $\mathcal{V}^n$ to be the full subcategory of varieties of dimension at most n, so actually $\mathcal{V}^n$ is a sub assembler of $\mathcal{V}$. Let $\mathcal{V}^{n,n-1}$ be the assembler whose underlying category is the full subcategory of $\mathcal{V}$ consisting of varieties of dimension exactly n and the empty variety. Let B_n be the set of the birational isomorphism classes of irreducible varieties over k of dimension n. For any $\alpha \in B_n$, define

$$\mathrm{Aut}(\alpha) = \mathrm{Aut}_k(k(X)),$$

where X is a representative of α.

Theorem 3.19. *There is a spectral sequence for $\pi_* K(\mathcal{V}) = K_*(\mathcal{V})$, the first page of the spectral sequence is*

$$E^1_{p,q} = \pi_p(K(\mathcal{V}^{q,q-1})) \simeq \bigoplus_{\alpha \in B_q} \pi_p(\Sigma^\infty_+ B \operatorname{Aut} \alpha) \Longrightarrow K_p(\mathcal{V}).$$

The qth graded piece of $\pi_p(\mathcal{V})$ is

$$E^\infty_{p,q} = \operatorname{im}(\pi_p(\mathcal{V}^q) \to \pi_p(\mathcal{V})) / \operatorname{im}(\pi_p(\mathcal{V}^{q-1}) \to \pi_p(\mathcal{V})).$$

Theorem 3.20 (Localization)**.** *Let $\mathcal{D}$ be a sub assembler of $\mathcal{C}$ that $\mathcal{D}$ is a sieve in $\mathcal{C}$ and $\mathcal{C}$ has complements for all objects of $\mathcal{D}$. Then*

$$K(\mathcal{D}) \to K(\mathcal{C}) \to K(\mathcal{C} \setminus \mathcal{D})$$

is a cofiber sequence. $\mathcal{C} \setminus D$ is the full subcategory of $\mathcal{C}$ containing all objects not in $\mathcal{D}^\circ$. a family $\{f_i : A_i \to A\}_{i \in I}$ in $C \setminus \mathcal{D}$ is defined to be a covering family if there exists a family of morphisms $\{f_j : A_j \to A\}_{j \in J}$ such that each $A_j \in \operatorname{Obj} D$ for $j \in J$ and such that $\{f_i : A_i \to A\}_{i \in I \sqcup J}$is a covering family in C.

Theorem 3.21 (Devissage)**.** *Let $\mathcal{C}$ be an assembler and D a full subassembler. If $\forall A \in \mathcal{C}$ there exists a finite disjoint covering family $\{D_i \to A\}_{i \in I}$ such that $\forall i \in I, D_i \in \operatorname{Obj} D$, then the induced map*

$$K(\mathcal{D}) \to K(\mathcal{C})$$

is an equivalence of spectra.

Sketch proof of Theorem 3.19. By Theorem 3.20, the cofiber of the map $K(\mathcal{V}^{q-1}) \to K(\mathcal{V}^q)$ is $K(\mathcal{V}^q \setminus \mathcal{V}^{q-1}) = K(\mathcal{V}^{q,q-1})$. It is clear that we have a spectral sequence convergent to $K_*(\mathcal{V})$ by the standard method, namely consider following diagram :

$$\begin{tikzcd}
\cdots \arrow[r] & K_p(\mathcal{V}^{q-1,q-2}) \arrow[r, "\partial"] & K_{p-1}(\mathcal{V}^{q-2}) \arrow[r, "p"] \arrow[d] & K_{p-1}(\mathcal{V}^{q-2,q-3}) \arrow[r] & \cdots \\
\cdots \arrow[r] & K_p(\mathcal{V}^{q,q-1}) \arrow[r, "\partial"] & K_{p-1}(\mathcal{V}^{q-1}) \arrow[r, "p"] \arrow[d] & K_{p-1}(\mathcal{V}^{q-1,q-2}) \arrow[r] & \cdots \\
\cdots \arrow[r] & K_p(\mathcal{V}^{q+1,q}) \arrow[r, "\partial"] & K_{p-1}(\mathcal{V}^{q}) \arrow[r, "p"] & K_{p-1}(\mathcal{V}^{q,q-1}) \arrow[r] & \cdots
\end{tikzcd} \tag{3.0.2}$$

The above diagram gives a exact couple, this exact couple induce a spectral sequence with first page

$$E^1_{p,q} = \pi_p(K(\mathcal{V}^{q,q-1})) \Longrightarrow K_p(\mathcal{V}).$$

We only need to compute $\pi_p(K(\mathcal{V}^{q,q-1}))$.

Let $\mathcal{V}^{q,\hat{q}-1}$ be the full subassembler of $\mathcal{V}^{q,q-1}$ of all irreducible subvarieties. By Theorem 3.21, $K((\mathcal{V}^{q,\hat{q}-1}))$ is equivalence to $K(\mathcal{V}^{q,q-1})$. For any $\alpha \in B_q$ pick X_α represents α. Let $\mathcal{C}$ be the full subassembler of $\mathcal{V}^{q,\hat{q}-1}$ consisting of subvarieties of X_α for any α. Therefore for any irreducible variety X, suppose $X \sim X_\alpha$, then there are isomorphism $U \to U_\alpha$, where $U \subseteq X, U_\alpha \subseteq X_\alpha$. X is irreducible so $\dim(X \setminus U) < \dim X = q$, when $q = 0$, a $0-$dimensional variety is just finite set, so birational equivalence is isomorphic. By induction on the dimensional we know $U \to U_\alpha$ is a covering family in $(\mathcal{V}^{q,\hat{q}-1})$, so by Theorem 3.21, $K(\mathcal{V}^{\hat{q},q-1}) \simeq K(\mathcal{C})$. And if some Z had morphisms to X_α and X_β, because the morphsims in $\mathcal{V}^{q,q-1}$ are inclusion of dense open subsets, then $X_\alpha \simeq X_\beta$. So

$$\mathcal{C} \simeq \bigvee_{\alpha \in B_q} \mathcal{C}_{X_\alpha}.$$

And $K(\mathcal{C}_{X_\alpha}) \simeq K(\mathbb{S}_{\mathrm{Aut}(\alpha)}) \simeq \Sigma^\infty_+ B\,\mathrm{Aut}(\alpha)$,see [20, Theorem 4.8]. For a group G, $\mathbb{S}_G$ is the assembler with two objects $\varnothing, *$, a injective morhism $\varnothing \to *$, and $\mathrm{Aut}(*) = G$. By Therefore $E^1_{p,q} \simeq \bigoplus_{\alpha\in B_q} \pi_p(\Sigma^\infty_+ B\,\mathrm{Aut}(\alpha))$. ☐

So, we can compute $E^1_{0,*}, E^1_{1,*}$. We have $\Sigma^\infty_+ B\,\mathrm{Aut}(\alpha) \simeq \mathbb{S} \vee \Sigma^\infty B\,\mathrm{Aut}(\alpha)$, so

$$E^1_{0,q} \simeq \bigoplus_{\alpha\in B_q} \mathbb{Z},$$

and by Hurewicz isomorphism we have $E^1_{1,q} \simeq \bigoplus_{\alpha\in B_q} \mathbb{Z}/2 \oplus \mathrm{Aut}(\alpha)^{\mathrm{ab}}$. Now we need to compute the differentials in the spectral sequence. Note that $\partial(\mathbb{Z}/2) = 0$, where ∂ is same as in 3.0.2.

Theorem 3.22. *If there is a nonzero differential $d_r, r \geq 1$ between 1th column and 0th column, then there is $n \in \mathbb{N}$, ψ_n is not injective.*

Proof. We need a fact.

Fact. Let φ be a birational automorphism of irreducible variety X, suppose $\dim X = q$, and $[\varphi] \in E^1_{1,q} = K_1(\mathcal{V}^{q,q-1})$. And also suppose $\varphi : U \to V$ is isomorphism for $U, V \subseteq X$, then $\partial([\varphi]) = [X \setminus U] - [X \setminus V] \in K_0(\mathcal{V}^{q-1})$.

The proof of fact use Theorem 3.16 (2), and [20, Proposition 3.13]. With the fact, we can prove the theorem. Since $E^r_{1,*}$ is sub quotient of $E^1_{1,*}$, then there exists $\alpha \in B_*, \varphi \in \mathrm{Aut}(\alpha)$ such that $\partial(\varphi) \neq 0$, otherwise the differentials d_r between 1th column and 0th column are all zero. Suppose $\partial(\varphi) = [X \setminus U] - [X \setminus V] \neq 0 \in K_0(\mathcal{V}_q)$,(by the fact), but $[X] = [X \setminus U] + [U] \in K_0(\mathrm{Var}_k), [X] = [X \setminus V] + [V] \in K_0(\mathrm{Var}_k)$, and $U \simeq V$ so $[X \setminus V] - [X \setminus U] = 0 \in K_0(\mathrm{Var}_k)$. Therefore ψ_q is not injective. ☐

To analyses the differentials between 1th column and 0th column of the spectral sequence in Theorem 3.19, we need another spectral sequence.

Definition 3.23. For any assembler $\mathcal{C}$, denote $\nabla : \mathcal{C} \vee \mathcal{C} \to \mathcal{C}$ as the fold map which is given by identity on each component. A simplicial assembler is a simplicial object in the category of assemblers. For a simplicial assembler $\mathcal{C}_.$ we define

$$K(\mathcal{C}_.) = \mathrm{hocolim}_{[n]} K(\mathcal{C}_n).$$

Let $F : \mathcal{C}_. \to \mathcal{D}_.$ be a morphism of simplicial assemblers. The simplicial assembler $C_./F$ is defined by

$$(C_./F)_n = \mathcal{D}_n \vee \bigvee_{i=1}^{n} \mathcal{C}_n.$$

The face map d_i see [20, Definition 6.1].

We denote $L : \mathcal{V} \to \mathcal{V}$ to be the morphism of assemblers which send $X \in \mathrm{Obj}\,\mathrm{Var}_k$ to $X \times \mathbb{A}^1_k \in \mathrm{Var}_k$. We write

$$C = \mathrm{cofib}(K(\mathcal{V}) \xrightarrow{K(L)} K(\mathcal{V})).$$

It turns out, $C \simeq K(\mathcal{V}/L)$.([20, Theorem C].) We have the filtration:

$$\cdots \longrightarrow K(\mathcal{V}^n/L) \longrightarrow K(\mathcal{V}^{n+1}/L) \longrightarrow \cdots \longrightarrow K(\mathcal{V}/L).$$

The above filtration gives a spectral sequence.

Theorem 3.24. *We define $l : B_n \to B_{n+1}$, $l([X]) = [X \times \mathbb{A}^1_k]$. The spectral sequence derived above is*

$$\tilde{E}^1_{p,q} \simeq \bigoplus_{\beta \in B_q} \pi_p C_\beta \Longrightarrow K_p(\mathcal{V}/L),$$

where

$$C_\beta = \tilde{C}_\beta \vee \operatorname{cofib} \nabla_\beta,$$

$$\tilde{C} = \operatorname{cofib}(\bigvee_{\alpha \in l^{-1}\beta} \Sigma^\infty \operatorname{Aut}(\alpha) \to \Sigma^\infty \operatorname{Aut}(\beta)), \quad \nabla : \bigvee_{l^{-1}\beta} \mathbb{S} \to \mathbb{S}.$$

Sketch of Proof. We need to compute the cofiber of $K(\mathcal{V}^{n-1}/L) \to K(\mathcal{V}^n/L)$. Denote $\iota : \mathcal{V}^{n-1} \hookrightarrow \mathcal{V}^n, \tilde{\iota} : K(\mathcal{V}^{n-1}/L) \to K(\mathcal{V}^n/L)$. By definition we have $\mathcal{V}^n/L/\tilde{\iota} \simeq \mathcal{V}^n/\iota/\tilde{L}$. Therefore we need to compute the cofiber of :

$$K(\mathcal{V}^{n-1}/\iota) \xrightarrow{K(\tilde{L})} K(\mathcal{V}^n/\iota).$$

We have the commutative diagram:

$$\begin{tikzcd} K(\mathcal{V}^{n-1}/\iota) \arrow[r, "K(\tilde{L})"] \arrow[d, "\simeq"] & K(\mathcal{V}^n/\iota) \arrow[d, "\simeq"] \\ K(\mathcal{V}^{n-1,n-2}) \arrow[r, "K(L)"] & K(\mathcal{V}^{n,n-1}) \end{tikzcd}$$

As seen in the proof of Theorem 3.19, we have can decompose $\mathcal{V}^{n,n-1}$ by the birational isomorphism classes. Therefore the bottom row can be written as

$$\bigoplus_\beta (\oplus_{\alpha \in l^{-1}(\beta)} K(\mathcal{V}^{n-1.n-2}|\alpha) \to K(\mathcal{V}^{n,n-1}|\beta))).$$

So, $C_\beta = \operatorname{cofib} \bigoplus_{\alpha \in l^{-1}(\beta)} K(\mathcal{V}^{n-1.n-2}|\alpha) \to K(\mathcal{V}^{n,n-1}|\beta)$. And it actually isomorphic to

$$\operatorname{cofib}(\bigoplus_{\alpha \in l^{-1}(\beta)} \mathbb{S}_{\operatorname{Aut}(\alpha)} \to \mathbb{S}_{\operatorname{Aut}(\beta)}).$$

Consider the following commutative diagram:

$$\begin{tikzcd} \vee_{\alpha \in l^{-1}(\beta)} \mathbb{S} \arrow[r] \arrow[d, "\nabla"] & \vee_{\alpha \in l^{-1}(\beta)} \mathbb{S}_{\operatorname{Aut}(\alpha)} \arrow[r] \arrow[d, "L"] & \vee_{\alpha \in l^{-1}(\beta)} \mathbb{S} \arrow[d, "\nabla"] \\ \mathbb{S} \arrow[r] & \mathbb{S}_{\operatorname{Aut}(\beta)} \arrow[r] & \mathbb{S}. \end{tikzcd}$$

Therefore $C_\beta \simeq \tilde{C}_\beta \vee \operatorname{cofib} \nabla_\beta$. □

So

$$E^1_{1,q} = \bigoplus_{\beta \in B_q} \pi_1(\tilde{C}_\beta) \oplus \pi_1(\nabla_\beta).$$

It is clear if $l^{-1}(\beta) = \varnothing$, then $\pi_1(\nabla_\beta) \simeq \pi_1(\mathbb{S}) \simeq \mathbb{Z}/2\mathbb{Z}$. And $l^{-1}(\beta) \neq \varnothing$, $\pi_1(\nabla_\beta) = \tilde{\mathbb{Z}}^{\oplus l^{-1}(\beta)}$.

It is clear that the morphism $K(\mathcal{V}) \to K(\mathcal{V}/L)$ induce morphism of spectral sequences $E^1_{p,q} \to \tilde{E}^1_{p,q}$.

Definition 3.25. A birational morphism $\varphi : X \dashrightarrow Y$ between smooth projective of dimension n varieties over a field k is convenient if

$$[X \setminus U] - [Y \setminus V] \in \operatorname{im}(K_0(\mathcal{V}^{n-2}) \xrightarrow{L} K_0(\mathcal{V}^{n-1})),$$

where $\varphi : U \to V$ is a isomorphism. It can be proved that the above definition is independent of the choice of open subsets. And a field k is convenient if all birational morphisms are convenient.

Basically, the morphism between spectral sequences $E^1_{p,q} \to \tilde{E}^1_{p,q}$ is surjective on permanent cycles when work in convenient field.

Theorem 3.26. *Field with* $\operatorname{char}(k) = 0$ *is convenient.*

Sketch of proof. The proof use weak factorization theorem, weak factorization theorem says every birational isomorphism can be factored as blowups and blowdowns along smooth center. One can induction on the number of blowups and blowdowns. The simplest case is when $\varphi : X \dashrightarrow Y$ is blow up along the the smooth center $Z \subseteq Y$, so $\varphi : X \setminus f^{-1}(Z) \to Y \setminus Z$ is isomorphism. Suppose Z is codimension d, then by Proposition 1.18 we have,

$$[f^{-1}(Z)] - [Z] = [Z][\mathbb{P}^{d-1}] = [Z]([\mathbb{L}] + [\mathbb{L}]^2 + \cdots + [L]^{d-1}) \in \operatorname{im}(K_0(\mathcal{V}^{n-2}) \xrightarrow{L} K_0(\mathcal{V}^{n-1})).$$

Therefore φ is convenient. ☐

Theorem 3.27. *In a convenient field k, if $\mathbb{L}$ is zero divisor then there is a nonzero differential between* 1*th column and* 0*th column of the spectral sequence $E^1_{p,q}$ in Theorem 3.19.*

Proof. We need some facts.

Fact.

(1) $E^1_{p,q} \to \tilde{E}^1_{p,q}$ is surjective on the component $\pi_1(\tilde{C}_\beta)$.
(2) When $l^{-1}(\beta) \neq \varnothing$, the component $\pi_1(\nabla_\beta)$ is not permanent cycle.(This requires k is a convenient field).
(3) When $l^{-1}(\beta) = \varnothing$, $\pi_1(\nabla_\beta) = \mathbb{Z}/2$, this component are permanent cycle, and it is clear that $E^1_{p,q} \to \tilde{E}^1_{p,q}$ is surjective on this component.

Now suppose all the differentials of $E^1_{p,q}$ between 1th column and 0th column, then all the cycles in $E^1_{1,q}$ are permanent cycle. By the above facts, we have

$$\begin{tikzcd} & \bigcap_{r=1}^\infty Z^r_{1,q}/B^1_{1,q} \arrow[r, two heads] \arrow[d, two heads] & \bigcap_{r=1}^\infty \tilde{Z}^r_{1,q}/\tilde{B}^1_{1,q} \arrow[d, two heads] \\ E^\infty_{1,q} = & \bigcap_{r=1}^\infty Z^r_{1,q}/\bigcup_{r=1}^\infty B^1_{1,q} \arrow[r] & \tilde{E}^\infty_{1,q} = \bigcap_{r=1}^\infty \tilde{Z}^r_{1,q}/\bigcup_{r=1}^\infty \tilde{B}^1_{1,q} \end{tikzcd}$$

Notice by Theorem 3.16 (2), any element in $x \in K_1(\mathcal{V}/L)$ there is q such that $x \in \operatorname{im}(K_1(\mathcal{V}^q) \to K_1(\mathcal{V}/L))$. Consider the commutative diagram with row exact,

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{im}(K_1(\mathcal{V}^{n-1}) \to K_1(\mathcal{V})) \arrow[r] \arrow[d] & \operatorname{im}(K_1(\mathcal{V}^n) \to K_1(\mathcal{V})) \arrow[r] \arrow[d] & E^\infty_{1,n} \arrow[r] \arrow[d, two heads] & 0 \\ 0 \arrow[r] & \operatorname{im}(K_1(\mathcal{V}^{n-1}/L) \to K_1(\mathcal{V}/L)) \arrow[r] & \operatorname{im}(K_1(\mathcal{V}^n/L) \to K_1(\mathcal{V}/L)) \arrow[r] & \tilde{E}^\infty_{1,n} \arrow[r] & 0 \end{tikzcd}$$

By induction on n we can prove x has a preimage in $\operatorname{im}(K_1(\mathcal{V}^q) \to K_1(\mathcal{V}))$. Therefore $K_1(\mathcal{V}) \to K_1(\mathcal{V}/L)$ is surjective. we have the long exact sequence

$$\cdots \longrightarrow K_1(\mathcal{V}) \longrightarrow K_1(\mathcal{V}/L) \longrightarrow K_0(\mathcal{V}) \xrightarrow{\mathbb{L}} K_0(\mathcal{V}) \longrightarrow K_0(\mathcal{V}/L) \longrightarrow 0.$$

Thus $\mathbb{L}$ is not zero divisor. ☐

Now, we can prove the main results.

Proof of Theorem 3.2. By [3, Theorem 2.2] and remark on [3], $\mathbb{L}$ is a zero divisor in $K_0(\mathrm{Var}_k)$. By Theorem 3.27, there is a nonzero differential between 1th column and 0th column of the spectral sequence $E^1_{p,q}$ in Theorem 3.19. Now by Theorem 3.22, there is $n \in \mathbb{N}$, such that ψ_n is not injective. $\square$

Proof of Theorem 3.1. By Theorem 3.2, there is $n \in \mathbb{N}$ such that ψ_n is not injective, suppose $[X] = [Y] \in K_0(\mathrm{Var}_k)$ and $[X] \neq [Y] \in F_n = \mathbb{Z}[X| \dim X \leq n]/ < [X] - [X \setminus Y] - [Y] >$. Then X and Y is not piecewise isomorphic. Otherwise, there are decomposition $X = \bigsqcup_{i=1}^m X_i, Y = \bigsqcup_{i=1}^m Y_i$ such that $X_i \simeq Y_i$. $\forall 1 \leq i \leq n, \dim X_i = \dim Y_i \leq n$. Therefore we have $[X_i] = [Y_i] \in F_n$, and $[X] = \sum_{i=1}^m [X_i], [Y] = \sum_{i=1}^m [Y_i]$,(The proof of this is similar to Proposition 1.13.) but these give we $[X] = [Y] \in F_n$ which contradicts the hypothesis. Therefore X and Y are not piecewise isomorphic. $\square$

Appendix A.

In this appendix, we list some facts and definitions used in the main text.

Definition A.1. Consider the subgroups of $\mathrm{SL}_2(\mathbb{Z})$,

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \xrightarrow{\text{reduction}} \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})).$$

$$\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} |a \equiv d \equiv 1 \mathrm{mod} N, c \equiv 0 \mathrm{mod} N\}.$$

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} |c \equiv 0 \mathrm{mod} N\}.$$

A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called congruence subgroup if there is $N \in \mathbb{Z}, |N| > 1$ such that $\Gamma(N) \subseteq \Gamma$.

Definition A.2. Let $\mathbb{H}$ be the upper half complex plane, i.e. $\mathbb{H} = \{z \in \mathbb{C}| \operatorname{Im} z > 0.\}$ There is an $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathbb{H}$, namely $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), z \in \mathbb{H}$,

$$\gamma z = \frac{az+b}{cz+d} \in \mathbb{H}.$$

And for a meromorphic function $f : \mathbb{H} \to \mathbb{C}$, the weight $k \in \mathbb{Z}$ *slash operator* of γ is

$$f[\gamma]_k = (cz+d)^{-k} f(\gamma z).$$

Definition A.3. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, $k \in \mathbb{Z}$, a function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k with respect to Γ if following conditions hold.

(1) f is holomorphic.
(2) $\forall \gamma \in \Gamma$, $f[\gamma]_k = f$.
(3) $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
(4) The Fourier expansion of $f[\gamma]_k = \sum_{n=0}^{\infty} a_n q^n$ with $a_0 = 0$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Actually, a function f satisfies (1),(2),(3) is called modular form, if in addition f satisfies (4) we call f as cusp form. The space of modular forms is denoted by $\mathcal{M}_k(\Gamma)$, the space of cusp forms is denoted by $S_k(\Gamma)$.

Definition A.4. The quotient space $\Gamma \backslash \mathbb{H}$ is denoted by $Y(\Gamma)$. The compactification of $Y(\Gamma)$ is denoted by $X(\Gamma)$ which is a compact Riemann surface. We denote $X_1(N)$ to be $X(\Gamma_1(N))$, $X_0(N) = X(\Gamma_0(N))$.

Fact. $S_2(\Gamma) \simeq \Omega^1_{hol} X(\Gamma)$, $\mathrm{Jac}(X(\Gamma)) = \Omega^1_{hol} X(\Gamma)^\wedge / \mathrm{H}^1(X(\Gamma), \mathbb{Z})$.

Fact. The new forms $S_2(\Gamma_0(N))^{\text{new}}$ is a subspace of $S_2(\Gamma_0(N))$, and it is the orthogonal complement of old forms.

Theorem A.5. [1, Theorem 0.1.1] *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between complete nonsingular algebraic varieties X_1 and X_2 over an algebraically closed field K of characteristic zero, and let $U \subset X_1$ be an open set where ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowings up and blowings down with smooth irreducible centers disjoint from U, namely, there exists a sequence of birational maps between complete nonsingular algebraic varieties*

$$X_1 = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \dots \overset{\varphi_i}{\dashrightarrow} V_i \overset{\varphi_{i+1}}{\dashrightarrow} V_{i+1} \overset{\varphi_{i+2}}{\dashrightarrow} \dots \overset{\varphi_{l-1}}{\dashrightarrow} V_{l-1} \overset{\varphi_l}{\dashrightarrow} V_l = X_2$$

such that

(1) $\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \varphi_2 \circ \varphi_1$,
(2) φ_i *are isomorphisms on* U,
(3) either $\varphi_i : V_i \to V_{i+1}$ *or* $\varphi_i^{-1} : V_{i+1} \to V_i$ *is a morphism obtained by blowing up a smooth irreducible center disjoint from* U*.*

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