Algebraic K-theory and K-theory of assembler

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K_0

- The story begin with the study of finite locally free sheaves on schemes, for simplicity we may consider the affine case.
- Let R be a commutative ring, $P(R)$ be the set of all finitely generated projective R-modules.
- Now, consider $P(R)/\sim$ be the set of isomorphism classes of $P(R)$. $(P(R)/\sim, \bigoplus)$ is a commutative monoid.

Definition

The 0th K theory of R, $K_0(R) := (P(R)/\sim)^{\mathrm{gp}}$.

Remark

The finiteness condition is essential, if we consider all projective module, we will get 0 by Eilenberg Swindle argument. If we consider $M(R)$, all finitely generated $R-$module, we will get G-theory. In generally, $G_0(R)$ is different from $K_0(R)$.

K_1 and K_2

Let $\mathrm{GL}(R) = \bigcup_{n \geq 1} \mathrm{GL}_n(R)$, $E(R) \subseteq \mathrm{GL}(R)$ be the subgroup generated by elementary matrices.

Definition

$K_1(R) := \mathrm{GL}(R)^{\mathrm{ab}} \simeq \mathrm{GL}(R)/E(R)$.

Definition

Let $\mathrm{St}(R)$ be the Steinberg group of R, roughly
$\mathrm{St}(R) =< x_{i,j}(r) >_{i>0,j>0,i\neq j,r\in R} / \sim$.
There is a natural surjection $\phi : \mathrm{St}(R) \to E(R)$. Now, $K_2(R) := \ker(\phi)$.

Is there $K_3, K_4, \ldots$

Higher algebraic K-theory

- Roughly speaking, higher algebraic K-theory is also the output of group completion, but the input equipped with higher structure in the sense of HTT.
- By abuse of notation, we denote $P(R)$ to be the category of finitely generated projective R-modules.

Definition

A monoidal category is a category $\mathcal{C}$ with a functor $\bigoplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $0 \in \mathrm{Obj}\,\mathcal{C}$. And natural isomorphisms

$$\alpha_{X,Y,Z} : (X \oplus Y) \oplus Z \cong X \oplus (Y \oplus Z),$$

$$\lambda_X : 0 \oplus X \cong X,$$

$$\rho_X : X \oplus 0 \cong X.$$

And some compatible conditions that you could imagine should be satisfied.

Higher algebraic K-theory

Definition

A symmetric monoidal category is a monoidal category $\mathcal{C}$, together with natural isomorphism $\gamma_{X,Y} : X \oplus Y \cong Y \oplus X$. And some compatible conditions that you could imagine should be satisfied.

It is clear that $P(R)$ is a symmetric monoidal category. using $P(R)$ we can actually get an E_∞-monoid, a higher version of commutative monoid.

Definition (Segal)

Γ is the category whose objects are pointed finite sets, whose morphism $S \to T$ is a map $\theta : S \to \mathcal{P}(T)$ such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint for $\alpha \neq \beta$. The composition of $\theta : S \to \mathcal{P}(T)$ and $\varphi : T \to \mathcal{P}(U)$ is $\psi : S \to \mathcal{P}(U)$ where $\psi(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \varphi(\beta)$. We denote $\mathbf{n}$ to be the set $\{1, 2, \ldots, n\}_+$.

Higher algebraic K-theory

Definition

An E_∞-monoid is a contravariant functor $\mathcal{C}$ form Γ to category of spaces such that:

- $\mathcal{C}(\emptyset_+)$ is contractable.
- for each n, the morphism $p_n : \mathcal{C}(\mathbf{n}) \to \underbrace{\mathcal{C}(\mathbf{1}) \times \cdots \times \mathcal{C}(\mathbf{1})}_{n}$ induced by $i_k : \mathbf{1} \to \{k\} \subseteq \mathbf{n}$ is homotopy equivalence.

Definition

A grouplike E_∞-monoid is an E_∞-monoid such that $\pi_0(\mathcal{C}(\mathbf{1}))$ is a group.

Theorem (Segal)

The category of grouplike E_∞-monoid is equivalence to the category of connective spectra.

Let Mon_{E_∞} to be the category of E_∞-monoid, $\mathrm{Mon}_{E_\infty}^{\mathrm{gp}}$ be the category of grouplike E_∞-monoid.

Theorem

The inclusion $\mathrm{Mon}_{E_\infty}^{gp} \subset \mathrm{Mon}_{E_\infty}$ admits a left adjoint, called group completion.

Now, we can conclude as follows.

- First, we have a monoidal category $P(R)$.
- There is a way to get an E_∞-monoid from $P(R)$, namely consider the summing functor and compose the nerve functor.
- Then we use the group completion to get a grouplike E_∞-monoid.
- By the theorem of Segal, we actually get a connective spectrum, we denote this spectrum to be $K(R)$.

More concrete

Let us consider $S = P(R)^{\simeq}$ be the sub groupoid of $P(R)$. We know $BS = |NS|$ is a homotopy associative H space. There is a group completion $B(S^{-1}S)$ of BS, and $B(S^{-1}S)$ is the 0th space of $K(R)$ defined before.

Theorem

We have a homotopy equivalence $B(S^{-1}S) \simeq K_0(R) \times B(\mathrm{GL}(R))^+$.

Definition

The plus construction of pair $(B\,\mathrm{GL}(R)), E(R))$ is a space $B(\mathrm{GL}(R))^+$ and a map $f : B(\mathrm{GL}(R)) \to B(\mathrm{GL}(R))^+$ such that $\pi_1(B\,\mathrm{GL}(R)^+) \cong \mathrm{GL}(R)/E(R)$, and for any local system L on $B(\mathrm{GL}(R))^+$ $H_*(B(\mathrm{GL}(R)), f^*L) \cong H_*(B(\mathrm{GL}(R))^+, L)$. And $B(\mathrm{GL}(R))^+$ is initial object among these properties.

More concrete

- Let $\mathcal{M}$ be an exact category, i.e. a full sub category of an abelian category, with a collection of (admissible) short exact sequences satisfies some conditions.
- In our case $\mathcal{M}$ will be $P(R)$, and admissible exact sequences will be all exact sequences.
- Let Q be the Q-construction of $\mathcal{M}$, the objects of $\mathcal{M}$ can be represented by. And morphisms are defined in a more subtle way.
- Now, $\Omega|BQ\mathcal{M}|$ is homotopy equivalence to $K_0(R) \times B(\mathrm{GL}(R))^+$.
- And actually, algebraic K theory can be defined for a stable ∞-category using Waldhausen construction.
- So we can actually consider $K(\mathbb{S})$, by consider the ∞-category of (compact) $\mathbb{S}$-modules which is a stable ∞-category. And for commutative ring R, $\mathrm{H}R$ to be the Eilenberg-Maclane spectrum, we should consider the ∞-category of $\mathrm{H}R$-modules.

K-theory of assembler

Now let us consider the Grothendieck ring of varieties over k.

Definition

$K_0(\mathrm{Var}_k) := \mathbb{Z}[\mathrm{Var}_k]/R$. $\mathbb{Z}[\mathrm{Var}_k]$ is the free abelian group generated by isomorphism class of varieties over k. And R is the subgroup of $\mathbb{Z}[\mathrm{Var}_k]$ generated by $[X]-[Y]-[X\setminus Y]$, where Y is a closed subvariety of X. And the multiplication of $K_0(\mathrm{Var}_k)$ is defined to be $[X]\cdot[Y] := [(X\times_k Y)_{\mathrm{red}}]$. We denote the class of $\mathbb{A}^1_k$ in $K_0(\mathrm{Var}_k)$ as $\mathbb{L}$, called *Lefschetz motive*. $\mathcal{M}_k := K_0[\mathrm{Var}_k][\mathbb{L}^{-1}]$.

Definition

The Higher K-theory of varieties is a spectrum $K(\mathcal{V}_k)$, with $\pi_0(K(\mathcal{V}_k)) \cong K_0(\mathrm{Var}_k)$.

K-theory of assembler

Remark

- $K(\mathcal{V}_k)$ can be constructed using Γ-space.
- $\mathcal{V}_k$ is an assembler, i.e. a site with some conditions.
- It turns out that $X : S \to W(S \wedge \mathcal{C})$ is a Γ-category for any assembler $\mathcal{C}$, S is a pointed finite set.
- Now, $K(\mathcal{C}_k)$ is the spectrum obtained from this Γ-space, whose 0th space is $|NW(\mathbf{1} \wedge \mathcal{V}_k)|$.

Definition

Let $\mathcal{C}$ be an assembler. We define $W(\mathcal{C})$ to have objects $\{A_i\}_{i\in I}$, where I is a finite set and all $A_i \in \operatorname{Obj} \mathcal{C}^\circ$. A morphism $f : \{A_i\}_{i\in I} \to \{B_i\}_{j\in J}$ is a map $f : I \to J$ and morphisms $f_i : A_i \to B_{f(i)}$. Such that $\{f_i : A_i \to B_j\}_{i\in f^{-1}(j)}$ is a finite disjoint covering family in $\mathcal{C}$.

Why should we care about K-theory

Theorem

- *Let R be the ring of integers of an algebraic number field, then* $K_0(R) \simeq \mathbb{Z} \oplus Pic(R), K_1(R) \simeq R^\times$.
- $K_2(\mathbb{Q}) \simeq \mathbb{Z}/2 \oplus \bigoplus_{p \text{ odd prime}} (\mathbb{F}_p)^\times$. *And actually each factor corresponds to tame symbol on* $\mathbb{Q}_p$ *which is related to Hilbert symbol, and from this calculation we can reprove Hilbert reciprocity law.*

Theorem

For a quasi-projective smooth variety X over a field. We have following isomorphism

$$K_i(X) \otimes \mathbb{Q} \simeq \bigoplus_q CH^q(X, i) \otimes \mathbb{Q}$$

Why should we care about K-theory

Vandiver conjecture

Let p be an irregular prime, i.e. $p | \# Pic(\mathbb{Z}[\zeta_p])$. Then $Pic(\mathbb{Z}[\zeta_p + \zeta_p^{-1}])$ has no $p-$torsion. And it turns out, this conjecture related to $K_{4n}(\mathbb{Z})$. If up to $2-$torsion groups $K_{4n}(\mathbb{Z}) = 0, n \neq 0$, then the conjecture holds.

Theorem (Barratt-Priddy-Quillen-Segal)

If we consider the category of finite sets, the K-theory spectrum is exactly the sphere spectrum $\mathbb{S}$.

Why should we care about K-theory

Theorem (Bhatt, Morrow and Scholze)

For any quasiregular semiperfect $\mathbb{F}_p-$algebra S, the K-theory $K_*(S;\mathbb{Z}_p)$ vanishes in odd degrees, while

$$K_{2i}(S;\mathbb{Z}_p) \cong A_{\mathrm{crys}}(S)^{\varphi=p^i}.$$

Example

Let $S = \mathbb{F}_p[x^{1/p^\infty}]/(x)$, then $A_{\mathrm{crys}}(S)$ is the p-adic completion of the subring $\mathbb{Z}_p[x^{1/p^\infty}, \frac{x^i}{i!}]_{i\geq 0} \subset \mathbb{Q}_p[x^{1/p^\infty}]$.

Why should we care about K-theory

The study of Grothendieck ring of varieties is also important, it has many application in birational geometry. For example, Kontsevich use Hodge motivic measure to prove following fact.

Theorem

Let X_1, X_2 be two birational equivalent Calabi-Yau varieties(we only require canonical divisor trivial), then X_1 and X_2 have same Hodge numbers.

Definition (Hodge measure)

$$Hdg : K_0(\mathrm{Var}_k) \to \mathbb{Z}[u, v]$$
$$[X] \mapsto \sum_{p,q \geq 0, 0 \leq k \leq 2 \dim X} (-1)^k h^{p,q}(\mathrm{H}^k(X, \mathbb{Q})) u^p v^q,$$

How to understand K-theory

Quillen used the plus construction to compute the K-theory for finite fields.

Theorem (Quillen)

Let $q = p^k$, p is a prime number.

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z}, & \text{if } n = 0; \\ 0, & \text{if } n \neq 0, 2|n; \\ \mathbb{Z}/(q^i - 1) & \text{if } n = 2i - 1. \end{cases}$$

Quillen proved several basic properties using the $Q-$construction.

Theorem (localization)

Let A be a Dedekind domain, F be its fraction field. Then there is a long exact sequence

$$\ldots \to \bigoplus_{\mathfrak{m} \text{ is maximal}} K_n(A/\mathfrak{m}) \to K_n(A) \to K_n(F) \to \bigoplus_{\mathfrak{m} \text{ is maximal}} K_{n-1}(A/\mathfrak{m}) \to \ldots$$

How to understand K-theory

- And Quillen proved that for ring of algebraic integers of number filed R, $K_*(R)$ are finitely generated abelian groups.
- With above facts, using the group cohomology we can calculate the rank of $K_*(R)$.

Theorem (Motivic spectral sequence)

For any coefficient group A, and any smooth scheme X over a field k, there is a spectral sequence, natural in X and A :

$$E_2^{p,q} = H^{p-q}(X, A(-q)) \Rightarrow K_{-p-q}(X; A).$$

If $X = \mathrm{Spec}(k)$ and $A = \mathbb{Z}/m$, where $1/m \in k$, then the E_2 terms are just the étale cohomology groups of k, truncated to lie in the octant $q \leq p \leq 0$.

How to understand K-theory

Definition

If A is an E_∞-ring spectrum, then $\mathrm{THH}(A)$ is a $\mathbb{T}$-equivariant E_∞-ring spectrum with a non-equivariant map $A \to \mathrm{THH}(A)$ of E_∞-ring spectra, and $\mathrm{THH}(A)$ is initial with these properties.

Moreover, if $C_p \subset \mathbb{T}$ is the cyclic subgroup of order p, then there is a natural Frobenius map

$$\varphi_p : \mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}$$

that is a map of E_∞-ring spectra which is equivariant for the $\mathbb{T}$-actions.

$$\mathrm{TC}^-(A) = \mathrm{THH}(A)^{h\mathbb{T}}, \mathrm{TP}(A) = \mathrm{THH}(A)^{t\mathbb{T}} = \mathsf{cofib}\,(\mathsf{Nm})\,.$$

$$\mathsf{Nm} : -_{h\mathbb{T}} \to -^{h\mathbb{T}}.$$

How to understand K-theory

If we consider p-adic case, then

$$\mathrm{TC}(A;\mathbb{Z}_p) = \mathrm{fib}(can_p - \varphi_p : \mathrm{TC}^-(A;\mathbb{Z}_p) \to \mathrm{TP}(A;\mathbb{Z}_p))$$

There is a map from K-theory to TC, called cyclotomic trace map, we denote the fiber of cyclotomic trace as K^{inv}.

Theorem (Dundas-Goodwillie-McCarthy)

Let $R \to R'$ be a ring map which is surjection with nilpotent kernel. Then $K^{inv}(R) \to K^{inv}(R')$ is an equivalence.

Theorem (Clausen, Mathew and Morrow)

Let S be a ring which is henselian along pS, and S/pS is semiperfect. Then the trace map induce isomorphism $K_*(S;\mathbb{Z}_p) \simeq \mathrm{TC}_*(S;\mathbb{Z}_p)$, for $* \geq 0$.

How to understand K-theory

Theorem (Zakharevich)

There is a spectral sequence for $\pi_* K(\mathcal{V}) = K_*(\mathcal{V})$, the first page of the spectral sequence is

$$E^1_{p,q} = \pi_p(K(\mathcal{V}^{q,q-1})) \simeq \bigoplus_{\alpha \in B_q} \pi_p(\Sigma^\infty_+ B \operatorname{Aut} \alpha) \Longrightarrow K_p(\mathcal{V}).$$

We can define the "filtration" for $K_0(\mathrm{Var}_k)$.

$$F_n := \mathbb{Z}[X| \dim X \leq n] / < [X] - [X \setminus Y] - [Y] >,$$

where the generators are isomorphism classes of varieties with dimension $\leq n$, Y is a closed subvariety of X. There is a map, namely

$$\psi_n : F_n \to K_0(\mathrm{Var}_k)$$
$$[X] \to [X].$$

The above theorem were used to prove that $\exists N \geq 0$, ψ_N is not injective.

Question

- Is it possible to define THH for assembler of varieties?
- Is there any chance that tilting functor preserves K-theory?
- Let $\mathcal{P}$ be the set of prime numbers. Fix an ultrafilter $\mathcal{U}$ for $\mathcal{P}$, Ax-Kochen theorem says

$$(\prod_{p\in\mathcal{P}} \mathbb{Q}_p)/\mathcal{U} \equiv (\prod_{p\in\mathcal{P}} \mathbb{F}_p(t))/\mathcal{U}.$$

  What about their K-theories?
- $\mathcal{U} \subset 2^{\mathcal{P}}$ such that,
  1. If $X, Y \in \mathcal{U}$ then $X \cap Y \in \mathcal{U}$.
  2. If $X \subseteq Y \subseteq \mathcal{P}$, and $X \in \mathcal{U}$ then $Y \in \mathcal{U}$.
  3. $\emptyset \notin \mathcal{U}$.
  4. For any $X \subseteq \mathcal{P}$, $X \in \mathcal{U}$ or $\mathcal{P} \setminus X \in \mathcal{U}$.