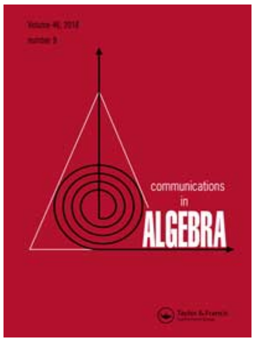

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The tensor product of Gorenstein-projective modules over category algebras

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ABSTRACT
For a finite free and projective EI category, we prove that Gorenstein-projective modules over its category algebra are closed under the tensor product if and only if each morphism in the given category is a monomorphism.

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1. Introduction

Let k be a field. Let $\mathscr{C}$ be a finite category, that is, it has only finitely many morphisms, and consequently it has only finitely many objects. Denote by k-mod the category of finite dimensional k-vector spaces and $(k\text{-mod})^{\mathscr{C}}$ the category of covariant functors from $\mathscr{C}$ to k-mod.

Recall that the category $k\mathscr{C}$-mod of left modules over the category algebra $k\mathscr{C}$ is identified with $(k\text{-mod})^{\mathscr{C}}$; see [7]. Hence $k\mathscr{C}$-mod is a symmetric monoidal category, whose tensor product is inherited from k-mod; see [8, 9].

Let $\mathscr{C}$ be a finite EI category. Here, the EI condition means that all endomorphisms in $\mathscr{C}$ are isomorphisms. In particular, $\mathrm{Hom}_{\mathscr{C}}(x,x) = \mathrm{Aut}_{\mathscr{C}}(x)$ is a finite group for each object x. Denote by $k\mathrm{Aut}_{\mathscr{C}}(x)$ the group algebra. Recall that a finite EI category $\mathscr{C}$ is *projective over* k if each $k\mathrm{Aut}_{\mathscr{C}}(y)$-$k\mathrm{Aut}_{\mathscr{C}}(x)$-bimodule $k\mathrm{Hom}_{\mathscr{C}}(x,y)$ is projective on both sides; see [5].

Denote by $k\mathscr{C}$-Gproj the full subcategory of $k\mathscr{C}$-mod consisting of Gorenstein-projective $k\mathscr{C}$-modules. We say that $\mathscr{C}$ is *GPT-closed*, if $X, Y \in k\mathscr{C}$-Gproj implies $X \hat{\otimes} Y \in k\mathscr{C}$-Gproj.

Let us explain the motivation to study GPT-closed categories. Recall from [6] that for a finite projective EI category $\mathscr{C}$, the stable category $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ modulo projective modules has a natural tensor triangulated structure such that it is tensor triangle equivalent to the singularity category of $k\mathscr{C}$. In general, its tensor product is not explicitly given. However, if $\mathscr{C}$ is GPT-closed, then the tensor product $-\hat{\otimes}-$ on Gorenstein-projective modules induces the one on $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$; see Proposition 3.4. In this case, we have a better understanding of the tensor triangulated category $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$.

Proposition 1.1. *Let $\mathscr{C}$ be a finite projective EI category. Assume that $\mathscr{C}$ is GPT-closed. Then each morphism in $\mathscr{C}$ is a monomorphism.*

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The concept of a finite *free* EI category is introduced in [3].

Theorem 1.2. *Let $\mathscr{C}$ be a finite projective and free EI category. Then the category $\mathscr{C}$ is GPT-closed if and only if each morphism in $\mathscr{C}$ is a monomorphism.*

2. Gorenstein triangular matrix algebras

In this section, we recall some necessary preliminaries on Gorenstein-projective modules and triangular matrix algebras.

Let A be a finite dimensional algebra over a field k. Denote by A-mod the category of finite dimensional left A-modules. The opposite algebra of A is denoted by A^{op}. We identify right A-modules with left A^{op}-modules.

Denote by $(-)^*$ the contravariant functor $\mathrm{Hom}_A(-,A)$ or $\mathrm{Hom}_{A^{\mathrm{op}}}(-,A)$. Let X be a left A-module. Then X^* is a right A-module and X^{**} is a left A-module. There is an evaluation map $\mathrm{ev}_X : X \to X^{**}$ given by $\mathrm{ev}_X(x)(f) = f(x)$ for $x \in X$ and $f \in X^*$. Recall that an A-module G is *Gorenstein-projective* provided that $\mathrm{Ext}^i_A(G,A) = 0 = \mathrm{Ext}^i_{A^{\mathrm{op}}}(G^*,A)$ for $i \geq 1$ and the evaluation map ev_G is bijective; see [1, Proposition 3.8].

We use pd and id to denote the projective dimension and the injective dimension of a module, respectively.

The algebra A is *Gorenstein* if $\mathrm{id}_A A < \infty$ and $\mathrm{id} A_A < \infty$. It is well known that for a Gorenstein algebra A we have $\mathrm{id}_A A = \mathrm{id} A_A$; see [10, Lemma A]. For $m \geq 0$, a Gorenstein algebra A is *m-Gorenstein* if $\mathrm{id}_A A = \mathrm{id} A_A \leq m$. Denote by A-Gproj the full subcategory of A-mod consisting of Gorenstein-projective A-modules, and A-proj the full subcategory of A-mod consisting of projective A-modules.

The following lemma is well known; see [1, Propositions 3.8 and 4.12 and Theorem 3.13].

Lemma 2.1. *Let $m \geq 0$. Let A be an m-Gorenstein algebra. Then we have the following statements.*

(1) *An A-module M is Gorenstein-projective if and only if $\mathrm{Ext}^i_A(M,A) = 0$ for all $i > 0$.*
(2) *If $M \in A$-Gproj and L is a right A-module with finite projective dimension, then $\mathrm{Tor}^A_i(L,M) = 0$ for all $i > 0$.*
(3) *If there is an exact sequence $0 \to M \to G_0 \to G_1 \to \cdots \to G_m$ with G_i Gorenstein-projective, then $M \in A$-Gproj.*

Recall that an $n \times n$ upper triangular matrix algebra has the form $\Gamma = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$, where each R_i is an algebra for $1 \leq i \leq n$, each M_{ij} is an R_i-R_j-bimodule for $1 \leq i < j \leq n$, and the matrix algebra map is denoted by $\psi_{ilj} : M_{il} \otimes_{R_l} M_{lj} \to M_{ij}$ for $1 \leq i < l < j < t \leq n$; see [5].

Recall that a left Γ-module $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ is described by a column vector, where each X_i is a left R_i-module for $1 \leq i \leq n$, and the left Γ-module structure map is denoted by $\varphi_{jl} : M_{jl} \otimes_{R_l} X_l \to X_j$ for $1 \leq j < l \leq n$; see [5]. Dually, we have the description of right Γ-modules via row vectors.

Notation 2.2. *Let Γ_t be the algebra given by the $t \times t$ leading principal submatrix of Γ and Γ'_{n-t} be the algebra given by cutting the first t rows and the first t columns of Γ. Denote the left Γ_t-module $\begin{pmatrix} M_{1,t+1} \\ \vdots \\ M_{t,t+1} \end{pmatrix}$ by M^*_t and the right Γ'_{n-t}-module $\left(M_{t,t+1}, \cdots, M_{tn}\right)$ by M^{**}_t, for $1 \leq t \leq n-1$. Denote by*

$\Gamma_t^D = \mathrm{diag}(R_1, \cdots R_t)$ *the sub-algebra of* Γ_t *consisting of diagonal matrices, and* $\Gamma'_{D,n-t} = \mathrm{diag}(R_{t+1}, \cdots R_n)$ *the sub-algebra of* Γ'_{n-t} *consisting of diagonal matrices.*

Let $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$ be an upper triangular matrix algebra. Recall that the R_1-R_2-bimodule M_{12} is *compatible*, if the following two conditions hold; see [11, Definition 1.1]:
(C1) If $Q^\bullet$ is an exact sequence of projective R_2-modules, then $M_{12} \otimes_{R_2} Q^\bullet$ is exact;
(C2) If $P^\bullet$ is a complete R_1-projective resolution, then $\mathrm{Hom}_{R_1}(P^\bullet, M_{12})$ is exact.

Lemma 2.3. *Let* Γ *be an upper triangular matrix algebra such that all* R_i *are Gorenstein. If* Γ *is Gorenstein, then each* Γ_t-R_{t+1}*-bimodule* M_t^* *is compatible and each* R_t-Γ'_{n-t}*-bimodule* M_t^{**} *is compatible for* $1 \le t \le n-1$.

Proof. Let $\Lambda = \begin{pmatrix} S_1 & N_{12} \\ 0 & S_2 \end{pmatrix}$ be an upper triangular matrix algebra. Recall the fact that if $\mathrm{pd}_{S_1}(N_{12}) < \infty$ and $\mathrm{pd}(N_{12})_{S_2} < \infty$, then N_{12} is compatible; see [11, Proposition 1.3]. Recall that Γ is Gorenstein if and only if all bimodules M_{ij} are finitely generated and have finite projective dimension on both sides; see [5, Proposition 3.4]. Then we have $\mathrm{pd}_{R_t}(M_t^{**}) < \infty$ and $\mathrm{pd}(M_t^*)_{R_{t+1}} < \infty$ for $1 \le t \le n-1$. By [5, Lemma 3.1], we have $\mathrm{pd}(M_t^{**})_{\Gamma'_{n-t}} < \infty$ and $\mathrm{pd}_{\Gamma_t}(M_t^*) < \infty$ for $1 \le t \le n-1$. Then we are done. □

Lemma 2.4 ([11, Theorem 1.4]). *Let* M_{12} *be a compatible* R_1-R_2*-bimodule, and* $\Gamma = \begin{pmatrix} R_1 & M_{12} \\ 0 & R_2 \end{pmatrix}$. *Then* $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \Gamma\text{-Gproj}$ *if and only if* $\varphi_{12} : M_{12} \otimes_{R_2} X_2 \to X_1$ *is an injective* R_1*-map,* $\mathrm{Coker}\varphi_{12} \in R_1\text{-Gproj}$, *and* $X_2 \in R_2\text{-Gproj}$.

We have a slight generalization of Lemma 2.4 in the case R_i being group algebras.

Lemma 2.5. *Let* Γ *be a Gorenstein upper triangular matrix algebra with each* R_i *a group algebra. Then* $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \in \Gamma\text{-Gproj}$ *if and only if each* R_t*-map* $\varphi_t^{**} = \sum_{j=t+1}^n \varphi_{tj} : M_t^{**} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \to X_t$ *sending* $(m_{t,t+1}, \cdots, m_{tn}) \otimes \begin{pmatrix} x_{t+1} \\ \vdots \\ x_n \end{pmatrix}$ *to* $\sum_{j=t+1}^n \varphi_{tj}(m_{tj} \otimes x_j)$ *is injective for* $1 \le t \le n-1$.

Proof. We have that each R_t-Γ'_{n-t}-bimodule M_t^{**} is compatible for $1 \le t \le n-1$ by Lemma 2.3.

For the "only if" part, we use induction on n. The case $n = 2$ is due to Lemma 2.4. Assume that $n > 2$. Write $\Gamma = \begin{pmatrix} R_1 & M_1^{**} \\ 0 & \Gamma'_{n-1} \end{pmatrix}$, and $X = \begin{pmatrix} X_1 \\ X' \end{pmatrix}$. Since $X \in \Gamma$-Gproj, by Lemma 2.4, we have that the R_1-map $\varphi_1^{**} : M_1^{**} \otimes_{\Gamma'_{n-1}} X' \to X_1$ is injective and $X' \in \Gamma'_{n-1}$-Gproj. By induction, we have that each R_t-map $\varphi_t^{**} : M_t^{**} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \to X_t$ is injective for $1 \le t \le n-1$.

For the "if" part, we use induction on n. The case $n = 2$ is due to Lemma 2.4. Assume that $n > 2$. Write $\Gamma = \begin{pmatrix} R_1 & M_1^{**} \\ 0 & \Gamma'_{n-1} \end{pmatrix}$, and $X = \begin{pmatrix} X_1 \\ X' \end{pmatrix}$. By induction, we have $X' \in \Gamma'_{n-1}$-Gproj. Since the R_1-map

$\varphi_1^{**} : M_1^{**} \otimes_{\Gamma'_{n-1}} X' \to X_1$ is injective and its cokernel belongs to R_1-Gproj as R_1 is a group algebra, we have $X \in \Gamma$-Gproj by Lemma 2.4. □

Corollary 2.6. *Let Γ be a Gorenstein upper triangular matrix algebra with each R_i a group algebra. Assume that* $X = \begin{pmatrix} X_1 \\ \vdots \\ X_s \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \Gamma$-Gproj. *Then each R_i-map $\varphi_{is} : M_{is} \otimes_{R_s} X_s \to X_i$ is injective for $1 \le i < s \le n$.*

Proof. We write $X = \begin{pmatrix} X' \\ X'' \end{pmatrix}$, where $X'' = \begin{pmatrix} X_{i+1} \\ \vdots \\ X_s \\ \vdots \\ 0 \end{pmatrix}$ for each $1 \le i < s \le n$. We claim that each R_i-map $f_{is} : M_{is} \otimes_{R_s} X_s \to M_i^{**} \otimes_{\Gamma'_{n-i}} X''$ sending $m_{is} \otimes x_s$ to $\left(0, \cdots, m_{is}, \cdots, 0\right) \otimes \left(0, \cdots, x_s, \cdots, 0\right)^t$ is injective, where $(-)^t$ is the transpose. Since $\varphi_{is} = \varphi_i^{**} \circ f_{is}$ for $1 \le i < s \le n$, then we are done by Lemma 2.5.

For the claim, we observe that for each $1 \le i < s \le n$, the R_i-map f_{is} is a composition of the following

$$M_{is} \otimes_{R_s} X_s \xrightarrow{g_{is}} \left(0, \cdots, 0, M_{is}, \cdots, M_{in}\right) \otimes_{\Gamma'_{n-i}} X'' \xrightarrow{\iota \otimes \mathrm{Id}} M_i^{**} \otimes_{\Gamma'_{n-i}} X'',$$

where the right Γ'_{n-i}-map $\left(0, \cdots, M_{is}, \cdots, M_{in}\right) \xrightarrow{\iota} M_i^{**}$ is the inclusion map and g_{is} sends $m_{is} \otimes x_s$ to $\left(0, \cdots, m_{is}, \cdots, 0\right) \otimes \left(0, \cdots, x_s, \cdots, 0\right)^t$. We observe an R_i-map $\left(0, \cdots, M_{is}, \cdots, M_{in}\right) \otimes_{\Gamma'_{n-i}} X'' \xrightarrow{g'_{is}} M_{is} \otimes_{R_s} X_s$, $\left(0, \cdots, m_{is}, \cdots, m_{in}\right) \otimes \left(0, \cdots, x_s, \cdots, 0\right)^t \mapsto m_{is} \otimes x_s$ satisfying $g'_{is} \circ g_{is} = \mathrm{Id}_{M_{is} \otimes_{R_s} X_s}$. Hence the R_i-map g_{is} is injective. We observe that the right Γ'_{n-i}-modules $\left(0, \cdots, M_{is}, \cdots, M_{in}\right)$ and M_i^{**} have finite projective dimensions; see [5, Lemma 3.1], and $X'' \in \Gamma'_{n-i}$-Gproj by Lemma 2.4. Then the R_i-map $\iota \otimes \mathrm{Id}$ is injective by Lemma 2.1 (2). □

3. Proof of Proposition 1.1

Let k be a field. Let $\mathscr{C}$ be a finite category, that is, it has only finitely many morphisms, and consequently it has only finitely many objects. Denote by $\mathrm{Mor}\mathscr{C}$ the finite set of all morphisms in $\mathscr{C}$. The *category algebra* $k\mathscr{C}$ of $\mathscr{C}$ is defined as follows: $k\mathscr{C} = \bigoplus_{\alpha \in \mathrm{Mor}\mathscr{C}} k\alpha$ as a k-vector space and the product $*$ is given by the rule

$$\alpha * \beta = \begin{cases} \alpha \circ \beta, & \text{if } \alpha \text{ and } \beta \text{ can be composed in } \mathscr{C}; \\ 0, & \text{otherwise.} \end{cases}$$

The unit is given by $1_{k\mathscr{C}} = \sum_{x \in \mathrm{Obj}\mathscr{C}} \mathrm{Id}_x$, where Id_x is the identity endomorphism of an object x in $\mathscr{C}$.

If $\mathscr{C}$ and $\mathscr{D}$ are two equivalent finite categories, then $k\mathscr{C}$ and $k\mathscr{D}$ are Morita equivalent; see [7, Proposition 2.2]. In particular, $k\mathscr{C}$ is Morita equivalent to $k\mathscr{C}_0$, where $\mathscr{C}_0$ is any skeleton of $\mathscr{C}$. So we may assume that $\mathscr{C}$ is *skeletal*, that is, for any two distinct objects x and y in $\mathscr{C}$, x is not isomorphic to y.

The category $\mathscr{C}$ is called a *finite EI category* provided that all endomorphisms in $\mathscr{C}$ are isomorphisms. In particular, $\mathrm{Hom}_{\mathscr{C}}(x, x) = \mathrm{Aut}_{\mathscr{C}}(x)$ is a finite group for any object x in $\mathscr{C}$. Denote by $k\mathrm{Aut}_{\mathscr{C}}(x)$ the group algebra.

For the rest of this paper, we assume that $\mathscr{C}$ is a finite EI category which is skeletal, and $\mathrm{Obj}\mathscr{C} = \{x_1, x_2, \cdots, x_n\}$, $n \geq 2$, satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$.

Let $M_{ij} = k\mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$. Write $R_i = M_{ii}$, which is a group algebra. Recall that the category algebra $k\mathscr{C}$ is isomorphic to the corresponding upper triangular matrix algebra $\Gamma_{\mathscr{C}} = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$; see [5, Section 4].

Recall from [7, Proposition 2.1] that the category $k\mathscr{C}$-mod of left modules over the category algebra $k\mathscr{C}$, is identified with $(k\text{-mod})^{\mathscr{C}}$. The category $k\mathscr{C}$-mod is a symmetric monoidal category. More precisely, the tensor product -$\hat{\otimes}$- is defined by

$$(M\hat{\otimes}N)(x) = M(x) \otimes_k N(x)$$

for any $M, N \in (k\text{-mod})^{\mathscr{C}}$ and $x \in \mathrm{Obj}\mathscr{C}$, and $\alpha.(m \otimes n) = \alpha.m \otimes \alpha.n$ for any $\alpha \in \mathrm{Mor}\mathscr{C}, m \in M(x), n \in N(x)$; see [8, 9].

In what follows, $\mathscr{C}$ is a finite EI category, and $\Gamma = \Gamma_{\mathscr{C}} = \begin{pmatrix} R_1 & M_{12} & \cdots & M_{1n} \\ & R_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & R_n \end{pmatrix}$ is the corresponding upper triangular matrix algebra.

Let $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ and $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$ be two Γ-modules, where the left Γ-module structure maps are denoted by φ_{ij}^X and φ_{ij}^Y, respectively. We observe that $X\hat{\otimes}Y = \begin{pmatrix} X_1 \otimes_k Y_1 \\ \vdots \\ X_n \otimes_k Y_n \end{pmatrix}$, where the module structure map $\varphi_{ij} : M_{ij} \otimes_{R_j} (X_j \otimes_k Y_j) \to X_i \otimes_k Y_i$ is induced by the following: $\varphi_{ij}(\alpha_{ij} \otimes (a_j \otimes b_j)) = \varphi_{ij}^X(\alpha_{ij} \otimes a_j) \otimes \varphi_{ij}^Y(\alpha_{ij} \otimes b_j)$, where $\alpha_{ij} \in \mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$, $a_j \in X_j$ and $b_j \in Y_j$.

Definition 3.1. *We say that $\mathscr{C}$ is* GPT-closed, *if $X, Y \in \Gamma$-Gproj implies $X\hat{\otimes}Y \in \Gamma$-Gproj.*

Recall from [5, Definition 4.2] that $\mathscr{C}$ is *projective over k* provided that each $k\mathrm{Aut}_{\mathscr{C}}(y)$-$k\mathrm{Aut}_{\mathscr{C}}(x)$-bimodule $k\mathrm{Hom}_{\mathscr{C}}(x, y)$ is projective on both sides. We recall the fact that the category algebra $k\mathscr{C}$ is Gorenstein if and only if $\mathscr{C}$ is projective over k; see [5, Proposition 5.1].

Denote by C_i the i-th column of Γ which is a Γ-R_i-bimodule and projective on both sides.

Proposition 3.2. *Assume that $\mathscr{C}$ is projective. Then the following statements are equivalent.*
(1) *The category $\mathscr{C}$ is GPT-closed.*
(2) *For any $1 \leq p \leq q \leq n$, $C_p\hat{\otimes}C_q \in \Gamma$-Gproj.*
(3) *For any $1 \leq p \leq q \leq n$, $C_p\hat{\otimes}C_q \in \Gamma$-proj.*

Proof.
"(1)⇒ (2)" and "(3)⇒ (2)" are obvious.

"(2)⇒ (3)" We only need to prove that the Γ-module $C_p\hat{\otimes}C_q$ has finite projective dimension, since a Gorenstein-projective module with finite projective dimension is projective. We have $C_p\hat{\otimes}C_q =$

$\begin{pmatrix} M_{1p} \otimes_k M_{1q} \\ \vdots \\ M_{p-1,p} \otimes_k M_{p-1,q} \\ R_p \otimes_k M_{pq} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Since $\mathscr{C}$ is projective, we have that each M_{ip} is a projective R_i-module for $1 \leq i \leq p$. Then each $M_{ip} \otimes_k M_{iq}$ is a projective R_i-module since R_i is a group algebra for $1 \leq i \leq p$. Hence the Γ-module $C_p \hat{\otimes} C_q$ has finite projective dimension by [5, Corollary 3.6]. Then we are done.

"(2)⇒ (1)" We have that Γ is a Gorenstein algebra by [5, Proposition 5.1]. Then there is $d \geq 0$ such that Γ is a d-Gorenstein algebra.

For any $M \in \Gamma$-Gproj, consider the following exact sequence

$$0 \to M \to P_0 \to P_1 \to \cdots \to P_d \to Y \to 0$$

with P_i projective, $0 \leq i \leq d$. Applying -$\hat{\otimes}N$ on the above exact sequence, we have an exact sequence

$$0 \to M\hat{\otimes}N \to P_0\hat{\otimes}N \to P_1\hat{\otimes}N \to \cdots \to P_d\hat{\otimes}N \to Y\hat{\otimes}N \to 0, \tag{3.1}$$

since the tensor product -$\hat{\otimes}$- is exact in both variables. If N is projective, we have that each $P_i\hat{\otimes}N$ is Gorenstein-projective for $0 \leq i \leq d$ by (2). Then we have $M\hat{\otimes}N \in \Gamma$-Gproj by Lemma 2.1 (3). If N is Gorenstein-projective, we have that each $P_i\hat{\otimes}N$ is Gorenstein-projective for $0 \leq i \leq d$ in exact sequence (3.1) by the above process. Then we have $M\hat{\otimes}N \in \Gamma$-Gproj by Lemma 2.1 (3). Then we are done. □

The argument in "(2)⇒ (3)" of Proposition 3.2 implies the following result. It follows that the tensor product -$\hat{\otimes}$- on Γ-Gproj *induces* the one on Γ-$\underline{\text{Gproj}}$, still denoted by -$\hat{\otimes}$-.

Lemma 3.3. *Assume that $\mathscr{C}$ is GPT-closed. Let $M \in \Gamma$-Gproj and $P \in \Gamma$-proj. Then $M\hat{\otimes}P \in \Gamma$-proj.*

Recall that a complex in $\mathrm{D}^b(\Gamma\text{-mod})$, the bounded derived category of finitely generated left Γ-modules, is called a *perfect complex* if it is isomorphic to a bounded complex of finitely generated projective modules. Recall from [2] that the *singularity category* of Γ, denoted by $\mathrm{D}_{sg}(\Gamma)$, is the Verdier quotient category $\mathrm{D}^b(\Gamma\text{-mod})/\mathrm{perf}(\Gamma)$, where $\mathrm{perf}(\Gamma)$ is a thick subcategory of $\mathrm{D}^b(\Gamma\text{-mod})$ consisting of all perfect complexes.

Assume that $\mathscr{C}$ is projective. Recall from [6] that there is a triangle equivalence

$$F : \Gamma\text{-}\underline{\text{Gproj}} \xrightarrow{\sim} \mathrm{D}_{sg}(\Gamma) \tag{3.2}$$

sending a Gorenstein-projective module to the corresponding stalk complex concentrated on degree zero. The functor F *transports* the tensor product on $\mathrm{D}_{sg}(\Gamma)$ to Γ-$\underline{\text{Gproj}}$ such that the category Γ-$\underline{\text{Gproj}}$ becomes a tensor triangulated category.

Proposition 3.4. *Assume that $\mathscr{C}$ is projective. If $\mathscr{C}$ is GPT-closed, then the tensor product -$\hat{\otimes}$- on Γ-$\underline{\text{Gproj}}$ induced by the tensor product on Γ-Gproj coincide with the one transported from $\mathrm{D}_{sg}(\Gamma)$, up to natural isomorphism.*

Proof. Consider the functor F in (3.2). Recall that the tensor product on $\mathrm{D}_{sg}(\Gamma)$ is induced by the tensor product -$\hat{\otimes}$- on $\mathrm{D}^b(\Gamma\text{-mod})$, where the later is given by -$\hat{\otimes}$- on Γ-mod. We have $F(M)\hat{\otimes}F(N) = F(M\hat{\otimes}N)$ in $\mathrm{D}_{sg}(\Gamma)$ for any $M, N \in \Gamma$-$\underline{\text{Gproj}}$. This implies that F is a tensor triangle equivalence. Then we are done. □

Let G be a finite group. Recall that a left (resp. right) G-set is a set with a left (resp. right) G-action. Let Y be a left G-set and X be a right G-set. Recall an equivalence relation "$\sim$" on the product $X \times Y$ as follows: $(x, y) \sim (x', y')$ if and only if there is an element $g \in G$ such that $x = x'g$ and $y = g^{-1}y'$ for $x, x' \in X$ and $y, y' \in Y$. Write the quotient set $X \times Y/\sim$ as $X \times_G Y$.

The following two lemmas are well known.

Lemma 3.5. *Let Y be a left G-set and X be a right G-set. Then there is an isomorphism of k-vector spaces*

$$\varphi : kX \otimes_{kG} kY \xrightarrow{\sim} k(X \times_G Y), \quad x \otimes y \mapsto (x, y),$$

where $x \in X$ and $y \in Y$.

Lemma 3.6. *Let Y_1 and Y_2 be two left G-sets. Then we have an isomorphism of left kG-modules*

$$\varphi : kY_1 \otimes_k kY_2 \xrightarrow{\sim} k(Y_1 \times Y_2), \quad y_1 \otimes y_2 \mapsto (y_1, y_2),$$

where $y_1 \in Y_1, y_2 \in Y_2$.

Lemma 3.7. *Assume that $\mathscr{C}$ is projective, and $1 \le p \le q \le n$. Then $C_p \hat{\otimes} C_q \in \Gamma$-proj implies that each morphism in $\bigsqcup_{y \in \mathrm{Obj}\mathscr{C}} \mathrm{Hom}_{\mathscr{C}}(x_p, y)$ is a monomorphism.*

Proof. We have $C_p \hat{\otimes} C_q = \begin{pmatrix} M_{1p} \otimes_k M_{1q} \\ \vdots \\ M_{p-1,p} \otimes_k M_{p-1,q} \\ R_p \otimes_k M_{pq} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then each R_i-map

$$\varphi_{ip} : M_{ip} \otimes_{R_p} (R_p \otimes_k M_{pq}) \to M_{ip} \otimes_k M_{iq}$$

sending $\alpha \otimes (g \otimes \beta)$ to $\alpha \circ g \otimes \alpha \circ \beta$, where $\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_p, x_i), g \in \mathrm{Aut}_{\mathscr{C}}(x_p), \beta \in \mathrm{Hom}_{\mathscr{C}}(x_q, x_p)$, is injective for $1 \le i < p \le q \le n$ by Corollary 2.6. We have that the sets $\mathrm{Hom}_{\mathscr{C}}(x_p, x_i) \times_{\mathrm{Aut}_{\mathscr{C}}(x_p)} (\mathrm{Aut}_{\mathscr{C}}(x_p) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_p))$ and $\mathrm{Hom}_{\mathscr{C}}(x_p, x_i) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_i)$ are k-basis of $M_{ip} \otimes_{R_p} (R_p \otimes_k M_{pq})$ and $M_{ip} \otimes_k M_{iq}$, respectively by Lemma 3.5 and Lemma 3.6. For each $1 \le i < p$, since φ_{ip} is injective, we have an injective map

$$\varphi : \mathrm{Hom}_{\mathscr{C}}(x_p, x_i) \times_{\mathrm{Aut}_{\mathscr{C}}(x_p)} (\mathrm{Aut}_{\mathscr{C}}(x_p) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_p)) \to \mathrm{Hom}_{\mathscr{C}}(x_p, x_i) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_i)$$

sending $(\alpha, (g, \beta))$ to $(\alpha \circ g, \alpha \circ \beta)$, for $\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_p, x_i), g \in \mathrm{Aut}_{\mathscr{C}}(x_p), \beta \in \mathrm{Hom}_{\mathscr{C}}(x_q, x_p)$.

For each $1 \le i < p$, and $\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_p, x_i)$, let $\beta, \beta' \in \mathrm{Hom}_{\mathscr{C}}(x_q, x_p)$ satisfy $\alpha \circ \beta = \alpha \circ \beta'$. Then we have $(\alpha, \alpha \circ \beta) = (\alpha, \alpha \circ \beta')$, that is, $\varphi(\alpha, (\mathrm{Id}_{x_p}, \beta)) = \varphi(\alpha, (\mathrm{Id}_{x_p}, \beta'))$. Since φ is injective, we have $(\alpha, (\mathrm{Id}_{x_p}, \beta)) = (\alpha, (\mathrm{Id}_{x_p}, \beta'))$ in $\mathrm{Hom}_{\mathscr{C}}(x_p, x_i) \times_{\mathrm{Aut}_{\mathscr{C}}(x_p)} (\mathrm{Aut}_{\mathscr{C}}(x_p) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_p))$. Hence $\beta = \beta'$. Then we have that α is a monomorphism. □

Proposition 3.8. *Assume that $\mathscr{C}$ is projective. If $\mathscr{C}$ is GPT-closed, then each morphism in $\mathscr{C}$ is a monomorphism.*

Proof. It follows from Proposition 3.2 and Lemma 3.7. □

Let $\mathcal{P}$ be a finite poset. We assume that $\mathrm{Obj}\mathcal{P} = \{x_1, \cdots, x_n\}$ satisfying $x_i \nleq x_j$ if $i < j$, and Γ is the corresponding upper triangular matrix algebra. We observe that each entry of Γ is 0 or k, and each projective Γ-module is a direct sum of some C_i, where C_i is the i-th column of Γ for $1 \le i \le n$. For any $a, b \in \mathrm{Obj}\mathcal{P}$ satisfying $a \nleq b$ and $b \nleq a$, denote by $L_{a,b} = \{x \in \mathrm{Obj}\mathcal{P} \mid a < x, b < x\}$.

Example 3.9. Let $\mathcal{P}$ be a finite poset. Then $\mathcal{P}$ is *GPT*-closed if and only if any two distinct minimal elements in $L_{a,b}$ has no common upper bound for $a, b \in \mathrm{Obj}\mathcal{P}$ satisfying $a \nleq b$ and $b \nleq a$.

For the "if" part, assume that any two distinct minimal elements in $L_{a,b}$ has no common upper bound. By Proposition 3.2, we only need to prove that $C_t\hat{\otimes}C_n$ is projective for $1 \le t \le n$, since the general case of $C_t\hat{\otimes}C_j$ can be considered in $\Gamma_{max\{t,j\}}$.

For each $1 \le t \le n$, if $(C_n)_t = k$, that is, $x_n \le x_t$, then $(C_t)_i = k$ implies $(C_n)_i = k$ for $1 \le i \le t$. Hence we have $C_t\hat{\otimes}C_n \simeq C_t$. Assume that $(C_n)_t = 0$, that is, $x_n \nleq x_t$. Let $L'_{x_t,x_n} = \{x_{s_1}, \cdots, x_{s_r}\}$ be all distinct minimal elements in L_{x_t,x_n}. For each $1 \le i < t$, if $(C_t)_i = k = (C_n)_i$, that is, $x_n \le x_i, x_t \le x_i$, then there is a unique $x_{s_l} \in L'_{x_t,x_n}$ satisfying $x_{s_l} \le x_i$, that is, there is a unique $x_{s_l} \in L'_{x_t,x_n}$ satisfying $(C_{s_l})_i = k$, since any two distinct elements in L'_{x_t,x_n} has no common upper bound. Then we have $C_t\hat{\otimes}C_n \simeq \bigoplus_{l=1}^r C_{s_l}$.

For the "only if" part, assume that $x_t, x_j \in \mathrm{Obj}\mathcal{P}$ satisfying $x_t \nleq x_j$ and $x_j \nleq x_t$ and $C_t\hat{\otimes}C_j \simeq \bigoplus_{l=1}^r C_{s_l}$. Then each $x_{s_l} \in L_{x_t,x_j}$. Assume that x_{s_1} and x_{s_2} be two distinct minimal elements in L_{x_t,x_j} having a common upper bound x_i. Then $(C_t\hat{\otimes}C_j)_i = k$ and $(C_{s_1} \oplus C_{s_2})_i = k \oplus k$, which is a contradiction.

4. Proof of Theorem 1.2

Recall from [3, Definition 2.3] that a morphism $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ is *unfactorizable* if α is not an isomorphism and whenever it has a factorization as a composite $x \xrightarrow{\beta} z \xrightarrow{\gamma} y$, then either β or γ is an isomorphism. Let $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ be an unfactorizable morphism. Then $h \circ \alpha \circ g$ is also unfactorizable for every $h \in \mathrm{Aut}_{\mathscr{C}}(y)$ and every $g \in \mathrm{Aut}_{\mathscr{C}}(x)$; see [3, Proposition 2.5]. Let $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ be a morphism with $x \ne y$. Then it has a decomposition $x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n = y$ with all α_i unfactorizable; see [3, Proposition 2.6].

Following [3, Definition 2.7], we say that $\mathscr{C}$ satisfies the Unique Factorization Property (UFP), if whenever a non-isomorphism α has two decompositions into unfactorizable morphisms:

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} x_m = y$$

and

$$x = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} y_n = y,$$

then $m = n$, $x_i = y_i$, and there are $h_i \in \mathrm{Aut}_{\mathscr{C}}(x_i)$, $1 \le i \le m-1$, such that the following diagram commutes :

$$\begin{array}{ccccccccccc}
x = x_0 & \xrightarrow{\alpha_1} & x_1 & \xrightarrow{\alpha_2} & x_2 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{m-1}} & x_{m-1} & \xrightarrow{\alpha_m} & x_m = y \\
\Big\| & & \Big\downarrow{\scriptstyle h_1} & & \Big\downarrow{\scriptstyle h_2} & & & & \Big\downarrow{\scriptstyle h_{m-1}} & & \Big\| \\
x = x_0 & \xrightarrow{\beta_1} & x_1 & \xrightarrow{\beta_2} & x_2 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{m-1}} & x_{m-1} & \xrightarrow{\beta_m} & x_m = y
\end{array}$$

Following [4, Section 6], we say that $\mathscr{C}$ is a finite *free* EI category if it satisfies the UFP. By [3, Proposition 2.8], this is equivalent to the original definition [3, Definition 2.2].

Assume that $\mathscr{C}$ is projective and free. Then Γ is 1-Gorenstein; see [5, Theorem 5.3].

Set $\mathrm{Hom}^0_{\mathscr{C}}(x_j, x_i) = \{\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_j, x_i) \mid \alpha \text{ is unfactorizable}\}$. Denote by $M^0_{ij} = k\mathrm{Hom}^0_{\mathscr{C}}(x_j, x_i)$, which is an R_i-R_j-sub-bimodule of M_{ij}; see [5, Notation 4.8]. Recall the left Γ_t-module M^*_t and the right

Γ'_{n-t}-module M_t^{**} in Notation 2.2, for $1 \le t \le n-1$. Observe that $M_t^{**} \simeq (M^0_{t,t+1}, M^0_{t,t+2}, \ldots, M^0_{tn}) \otimes_{\Gamma'_{D,n-t}} \Gamma'_{n-t}$; compare [5, Lemmas 4.10 and 4.11], which implies that $M_t^{*} \simeq \Gamma_t \otimes_{\Gamma_t^D} \begin{pmatrix} M^0_{1,t+1} \\ \vdots \\ M^0_{t,t+1} \end{pmatrix}$.

Let $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ be a left Γ-module. For each $1 \le t \le n-1$, we have

$$\begin{aligned} M_t^{**} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} &\simeq (M^0_{t,t+1}, M^0_{t,t+2}, \cdots, M^0_{tn}) \otimes_{\Gamma'_{D,n-t}} \Gamma'_{n-t} \otimes_{\Gamma'_{n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \\ &\simeq (M^0_{t,t+1}, M^0_{t,t+2}, \cdots, M^0_{tn}) \otimes_{\Gamma'_{D,n-t}} \begin{pmatrix} X_{t+1} \\ \vdots \\ X_n \end{pmatrix} \\ &\simeq \bigoplus_{j=t+1}^{n} M^0_{tj} \otimes_{R_j} X_j. \end{aligned}$$

Recall the R_t-map φ_t^{**} in Lemma 2.5. Here, we observe that

$$\varphi_t^{**} : \bigoplus_{j=t+1}^{n} M^0_{tj} \otimes_{R_j} X_j \to X_t, \quad \sum_{j=t+1}^{n} (m_j \otimes x_j) \mapsto \sum_{j=t+1}^{n} \varphi_{tj}(m_j \otimes x_j).$$

Lemma 4.1. *Assume that $\mathscr{C}$ is projective and free, and $1 \le p \le q \le n$. If each morphism in $\bigsqcup_{y \in \mathrm{Obj}\mathscr{C}} \bigsqcup_{j=1}^{p} \mathrm{Hom}_{\mathscr{C}}(x_j, y)$ is a monomorphism, then $C_p \hat{\otimes} C_q \in \Gamma$-proj.*

Proof. We only need to prove that each R_t-map

$$\varphi_t^{**} : \bigoplus_{j=t+1}^{p} M^0_{tj} \otimes_{R_j} (M_{jp} \otimes_k M_{jq}) \to M_{tp} \otimes_k M_{tq}$$

is injective for $1 \le t < p$ by Lemma 2.5 and Proposition 3.2.

By Lemmas 3.5 and 3.6, we have that the set $\mathrm{Hom}_{\mathscr{C}}(x_p, x_t) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_t)$ is a k-basis of $M_{tp} \otimes_k M_{tq}$, and the set

$$\bigsqcup_{j=t+1}^{p} \mathrm{Hom}^0_{\mathscr{C}}(x_j, x_t) \times_{\mathrm{Aut}_{\mathscr{C}}(x_j)} \big(\mathrm{Hom}_{\mathscr{C}}(x_p, x_j) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_j)\big) =: B$$

is a k-basis of $\bigoplus_{j=t+1}^{p} M^0_{tj} \otimes_{R_j} (M_{jp} \otimes_k M_{jq})$.

We have the following commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\subseteq} & \bigoplus_{j=t+1}^{p} M^0_{tj} \otimes_{R_j} (M_{jp} \otimes_k M_{jq}) \\ \big\downarrow{\scriptstyle \varphi_t^{**}|_B} & & \big\downarrow{\scriptstyle \varphi_t^{**}} \\ \mathrm{Hom}_{\mathscr{C}}(x_p, x_t) \times \mathrm{Hom}_{\mathscr{C}}(x_q, x_t) & \xrightarrow{\subseteq} & M_{tp} \otimes_k M_{tq} \end{array}$$

Observe that φ_t^{**} is injective if and only if $\varphi_t^{**}\mid_B$ is injective for each $1 \leq t < p$.

Assume that $\varphi_t^{**}(\alpha,(\beta,\theta)) = \varphi_t^{**}(\alpha',(\beta',\theta'))$, where $\alpha \in \mathrm{Hom}^0_{\mathscr{C}}(x_j,x_t)$, $\beta \in \mathrm{Hom}_{\mathscr{C}}(x_p,x_j)$, $\theta \in \mathrm{Hom}_{\mathscr{C}}(x_q,x_j)$ and $\alpha' \in \mathrm{Hom}^0_{\mathscr{C}}(x_{j'},x_t)$, $\beta' \in \mathrm{Hom}_{\mathscr{C}}(x_p,x_{j'})$, $\theta' \in \mathrm{Hom}_{\mathscr{C}}(x_q,x_{j'})$. Then we have $\alpha\beta = \alpha'\beta'$ in $\mathrm{Hom}_{\mathscr{C}}(x_p,x_t)$ and $\alpha\theta = \alpha'\theta'$ in $\mathrm{Hom}_{\mathscr{C}}(x_q,x_t)$. Since $\mathscr{C}$ is free and α,α' are unfactorizable, we have that $j = j'$ and there is $g \in \mathrm{Aut}_{\mathscr{C}}(x_j)$ such that $\alpha = \alpha' g$ and $\beta = g^{-1}\beta'$. Since $\alpha\theta = \alpha'\theta' = \alpha g^{-1}\theta'$ and α is a monomorphism, we have that $\theta = g^{-1}\theta'$. Then we have that $(\alpha,(\beta,\theta)) = (\alpha' g,(g^{-1}\beta',g^{-1}\theta')) = (\alpha',(\beta',\theta'))$, which implies that the map $\varphi_t^{**}\mid_B$ is injective. □

Theorem 4.2. *Let $\mathscr{C}$ be a finite projective and free EI category. Then the category $\mathscr{C}$ is GPT-closed if and only if each morphism in $\mathscr{C}$ is a monomorphism.*

Proof. The "if" part follows from Proposition 3.2 and Lemma 4.1. The "only if" part is justified by Proposition 3.8. □

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