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The MCM-approximation of the trivial module over a category algebra

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For a finite free EI category, we construct an explicit module over its category algebra. If in addition the category is projective over the ground field, the constructed module is a maximal Cohen–Macaulay approximation of the trivial module and is the tensor identity of the stable category of Gorenstein-projective modules over the category algebra. We give conditions on when the trivial module is Gorenstein-projective.

Keywords: Finite EI category; category algebra; Gorenstein-projective module; MCM-approximation; tensor identity.

Mathematics Subject Classification: 16G10, 16G50, 16D90, 18G25

1. Introduction

Let k be a field and $\mathscr{C}$ be a finite EI category. Here, the EI condition means that all endomorphisms in $\mathscr{C}$ are isomorphisms. In particular, $\mathrm{Hom}_{\mathscr{C}}(x,x) = \mathrm{Aut}_{\mathscr{C}}(x)$ is a finite group for each object x. Denote by $k\mathrm{Aut}_{\mathscr{C}}(x)$ the group algebra. Recall that a finite EI category $\mathscr{C}$ is *projective over* k if each $k\mathrm{Aut}_{\mathscr{C}}(y)$-$k\mathrm{Aut}_{\mathscr{C}}(x)$-bimodule $k\mathrm{Hom}_{\mathscr{C}}(x,y)$ is projective on both sides; see [11, Definition 4.2].

The concept of a finite *free* EI category is introduced in [8, 9]. Let $\mathscr{C}$ be a finite free EI category. For any morphism $x \xrightarrow{\alpha} y$ in $\mathscr{C}$, set $V(\alpha)$ to be the set of objects w such that there are factorizations $x \xrightarrow{\alpha'} w \xrightarrow{\alpha''} y$ of α with α'' a non-isomorphism. If α is an isomorphism, we have $V(\alpha) = \emptyset$. For any $w \in V(\alpha)$, we set $t_w(\alpha) = \alpha'' \circ (\sum_{g\in \mathrm{Aut}_{\mathscr{C}}(w)} g)$, which is an element in $k\mathrm{Hom}_{\mathscr{C}}(w,y)$. The freeness of $\mathscr{C}$ implies that the element $t_w(\alpha)$ is independent of the choice of α''; see Notation 2.2.

In what follows, we assume that the category $\mathscr{C}$ is skeletal. Denote by k-mod the category of finite-dimensional k-vector spaces. By [10, Theorem 7.1], we identify covariant functors from $\mathscr{C}$ to k-mod with left modules over the category algebra. We define a functor $E : \mathscr{C} \to k$-mod as follows: for each object x, $E(x)$ is a k-vector space with basis $B_x = \bigsqcup_{w \neq x} \mathrm{Hom}_{\mathscr{C}}(w, x) \cup \{e_x\}$, and for each morphism $x \xrightarrow{\alpha} y$, the k-linear map $E(\alpha) : E(x) \to E(y)$ sends e_x to $e_y + \sum_{w \in V(\alpha)} t_w(\alpha)$, and γ to $\alpha \circ \gamma$ for $\gamma \in \mathrm{Hom}_{\mathscr{C}}(w, x)$ with $w \neq x$; see Proposition 2.7.

Recall that the *constant functor* $\underline{k} : \mathscr{C} \to k$-mod is defined as follows: $\underline{k}(x) = k$ for each object x, and $\underline{k}(\alpha) = \mathrm{Id}_k$ for each morphism α. The constant functor corresponds to the *trivial module* of the category algebra. We mention that the trivial module $\underline{k}$ plays an important role in the cohomological study of categories [14]. It is the tensor identity in the category of modules over the category algebra [15].

The notion of a *Gorenstein-projective module* is introduced in [1]; cf. [7]. They are generalizations of *maximal Cohen–Macaulay modules* (MCM-modules for short) over Gorenstein rings. Hence, in the literature, Gorenstein-projective modules are often called MCM-modules.

Let A be a finite-dimensional algebra over k. For an A-module X, an *MCM-approximation* of X is a map $\theta : G \to X$ with G Gorenstein-projective such that any map $G' \to X$ with G' Gorenstein-projective factors through θ. The study of such approximations goes back to [2]. In general, the MCM-approximation seems difficult to construct explicitly.

In this paper, we construct an explicit MCM-approximation of the trivial module, provided that the category $\mathscr{C}$ is free and projective over k. In this case, we observe that the category algebra is 1-Gorenstein; see [11, Theorem 5.3]. We observe a surjective natural transformation $E \xrightarrow{\pi} \underline{k}$ as follows: for each object x, $E(x) \xrightarrow{\pi_x} \underline{k}(x) = k$ sends e_x to 1_k, and γ to zero for $\gamma \in \mathrm{Hom}_{\mathscr{C}}(w, x)$ with $w \neq x$.

Theorem 1.1. *Let $\mathscr{C}$ be a finite free EI category. Assume that the category $\mathscr{C}$ is projective over k. Then the map $E \xrightarrow{\pi} \underline{k}$ is an MCM-approximation of the trivial module $\underline{k}$.*

Denote by $k\mathscr{C}$-Gproj the full subcategory of $k\mathscr{C}$-mod consisting of Gorenstein-projective $k\mathscr{C}$-modules and denote by $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ the corresponding stable category modulo projective $k\mathscr{C}$-modules. The category $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ has a natural tensor triangulated category structure, when $\mathscr{C}$ is a projective EI category; see Sec. 4.

Proposition 1.2. *Let $\mathscr{C}$ be a finite projective EI category. Assume that $\mathscr{C}$ is free. Then E is the tensor identity of $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$.*

This paper is organized as follows. In Sec. 2, we recall some notation on finite free EI categories and construct the functor E. In Sec. 3, we prove that E, viewed as a module over the category algebra, is Gorenstein-projective if in addition $\mathscr{C}$ is projective and prove Theorem 1.1. We mention that Propositions 2.1 and 3.3 are new characterizations of finite free EI categories, and that Proposition 3.5 describes when the trivial module is Gorenstein-projective. In Sec. 4, we prove Proposition 1.2.

2. Finite Free EI Categories and the Functor E

Let k be a field. Let $\mathscr{C}$ be a finite category, that is, it has only finitely many morphisms, and consequently it has only finitely many objects. Denote by $\mathrm{Mor}\,\mathscr{C}$ the finite set of all morphisms in $\mathscr{C}$. Recall that the *category algebra* $k\mathscr{C}$ of $\mathscr{C}$ is defined as follows: $k\mathscr{C} = \bigoplus_{\alpha\in\mathrm{Mor}\,\mathscr{C}} k\alpha$ as a k-vector space and the product $*$ is given by the rule

$$\alpha * \beta = \begin{cases} \alpha\circ\beta & \text{if } \alpha \text{ and } \beta \text{ can be composed in } \mathscr{C};\\ 0 & \text{otherwise.}\end{cases}$$

The unit is given by $1_{k\mathscr{C}} = \sum_{x\in\mathrm{Obj}\,\mathscr{C}} \mathrm{Id}_x$, where Id_x is the identity endomorphism of an object x in $\mathscr{C}$.

If $\mathscr{C}$ and $\mathscr{D}$ are two equivalent finite categories, then $k\mathscr{C}$ and $k\mathscr{D}$ are Morita equivalent; see [13, Proposition 2.2]. In particular, $k\mathscr{C}$ is Morita equivalent to $k\mathscr{C}_0$, where $\mathscr{C}_0$ is any skeleton of $\mathscr{C}$.

In this paper, we assume that the finite category $\mathscr{C}$ is *skeletal*, that is, any two distinct objects in $\mathscr{C}$ are not isomorphic.

2.1. *Finite free EI categories*

The category $\mathscr{C}$ is called a *finite EI category* provided that all endomorphisms in $\mathscr{C}$ are isomorphisms. In particular, $\mathrm{Hom}_{\mathscr{C}}(x,x) = \mathrm{Aut}_{\mathscr{C}}(x)$ is a finite group for each object x in $\mathscr{C}$. Denote by $k\mathrm{Aut}_{\mathscr{C}}(x)$ the group algebra.

Let $\mathscr{C}$ be a finite EI category. Recall from [8, Definition 2.3] that a morphism $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ is *unfactorizable* if α is a non-isomorphism and whenever it has a factorization $x \xrightarrow{\beta} z \xrightarrow{\gamma} y$, then either β or γ is an isomorphism. Let $x \xrightarrow{\alpha} y$ be an unfactorizable morphism. Then $h\circ\alpha\circ g$ is also unfactorizable for every $h \in \mathrm{Aut}_{\mathscr{C}}(y)$ and every $g \in \mathrm{Aut}_{\mathscr{C}}(x)$; see [8, Proposition 2.5]. Any non-isomorphism $x \xrightarrow{\alpha} y$ in $\mathscr{C}$ has a decomposition $x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} x_n = y$ with each α_i unfactorizable; see [8, Proposition 2.6].

Following [8, Definition 2.7], we say that a finite EI category $\mathscr{C}$ satisfies the Unique Factorization Property (UFP), if whenever a non-isomorphism α has two decompositions into unfactorizable morphisms:

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} x_m = y$$

and

$$x = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} y_n = y,$$

then $m = n$, $x_i = y_i$, and there are $h_i \in \mathrm{Aut}_{\mathscr{C}}(x_i)$, $1 \le i \le m-1$, such that the following diagram commutes:

$$\begin{array}{ccccccccccccc} x = x_0 & \xrightarrow{\alpha_1} & x_1 & \xrightarrow{\alpha_2} & x_2 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{m-1}} & x_{m-1} & \xrightarrow{\alpha_m} & x_m = y \\ \| & & \downarrow{h_1} & & \downarrow{h_2} & & & & \downarrow{h_{m-1}} & & \| \\ x = x_0 & \xrightarrow{\beta_1} & x_1 & \xrightarrow{\beta_2} & x_2 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{m-1}} & x_{m-1} & \xrightarrow{\beta_m} & x_m = y \end{array}$$

Let $\mathscr{C}$ be a finite EI category. Following [9, Sec. 6], we say that $\mathscr{C}$ is a finite *free* EI category if it satisfies the UFP. By [8, Proposition 2.8], this is equivalent to the original definition [8, Definition 2.2].

The following result is another characterization of a finite free EI category.

Proposition 2.1. *Let $\mathscr{C}$ be a finite EI category. Then $\mathscr{C}$ is free if and only if for any two decompositions $x \xrightarrow{\alpha'} z \xrightarrow{\alpha''} y$ and $x \xrightarrow{\beta'} w \xrightarrow{\beta''} y$ of a non-isomorphism α with α'' and β'' non-isomorphisms, there is $z \xrightarrow{\gamma} w$ in $\mathscr{C}$ satisfying $\alpha'' = \beta'' \circ \gamma$ and $\beta' = \gamma \circ \alpha'$, or there is $w \xrightarrow{\delta} z$ in $\mathscr{C}$ satisfying $\beta'' = \alpha'' \circ \delta$ and $\alpha' = \delta \circ \beta'$.*

Proof. For the "only if" part, we assume that $\mathscr{C}$ is a finite free EI category. Let α be a non-isomorphism with two decompositions $x \xrightarrow{\alpha'} z \xrightarrow{\alpha''} y$ and $x \xrightarrow{\beta'} w \xrightarrow{\beta''} y$ for α'' and β'' non-isomorphisms.

If $w = x$, we have that β' is an isomorphism. Then we take $\delta = \alpha' \circ \beta'^{-1}$. If $z = x$, we have that α' is an isomorphism. Then we take $\gamma = \beta' \circ \alpha'^{-1}$.

If $w, z \neq x$, we have that both α' and β' are non-isomorphisms. Write the non-isomorphisms $\alpha', \alpha'', \beta', \beta''$ as compositions of unfactorizable morphisms. Then we have two decompositions of α as follows:

$$\begin{array}{ccccccccccccc} x & \xrightarrow{\alpha'_1} & x_1 & \xrightarrow{\alpha'_2} & \cdots & \xrightarrow{\alpha'_m} & x_m = z & \xrightarrow{\alpha''_1} & z_1 & \xrightarrow{\alpha''_2} & \cdots & \xrightarrow{\alpha''_s} & z_s = y \\ \| & & & & & & & & & & & & \| \\ x & \xrightarrow{\beta'_1} & w_1 & \xrightarrow{\beta'_2} & \cdots & \xrightarrow{\beta'_n} & w_n = w & \xrightarrow{\beta''_1} & y_1 & \xrightarrow{\beta''_2} & \cdots & \xrightarrow{\beta''_t} & y_t = y, \end{array}$$

where all $\alpha'_i, \alpha''_j, \beta'_l, \beta''_k$ are unfactorizable morphisms and $\alpha' = \alpha'_m \circ \cdots \circ \alpha'_1$, $\alpha'' = \alpha''_s \circ \cdots \circ \alpha''_1$, $\beta' = \beta'_n \circ \cdots \circ \beta'_1$, $\beta'' = \beta''_t \circ \cdots \circ \beta''_1$. We apply the UFP for α. If $m < n$, then $x_i = w_i$ for $1 \leq i \leq m$, $z_j = w_{m+j}$ for $1 \leq j \leq n-m$, $y_l = z_{n-m+l}$ for $1 \leq l \leq t$, and there is $g : x_m \to x_m = w_m$ such that $\alpha'' = \beta'' \circ \beta'_n \circ \cdots \circ \beta'_{m+1} \circ g$ and $g \circ \alpha' = \beta'_m \circ \cdots \circ \beta'_1$. We take $\gamma = \beta'_n \circ \cdots \circ \beta'_{m+1} \circ g$. The cases $m = n$ and $m > n$ are similar.

For the "if" part, let α be a non-isomorphism with the following two decompositions into unfactorizable morphisms:

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} x_m = y$$

and

$$x = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} y_n = y.$$

Since $\alpha'' = \alpha_m \circ \cdots \circ \alpha_2$ and $\beta'' = \beta_n \circ \cdots \circ \beta_2$ are non-isomorphisms, by the hypothesis we may assume that there is $x_1 \xrightarrow{g_1} y_1$ in $\mathscr{C}$ such that $\beta_1 = g_1 \circ \alpha_1$ and $\alpha'' = \beta'' \circ g_1$. Since both α_1 and β_1 are unfactorizable, we infer that g_1 is an isomorphism and thus $x_1 = y_1$. Then we have the following two decompositions of α'' into unfactorizable morphisms:

$$x_1 \xrightarrow{\alpha_2} x_2 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_m} x_m = y$$

and

$$x_1 \overset{\beta_2 \circ g_1}{\longrightarrow} y_2 \overset{\beta_3}{\to} \cdots \overset{\beta_n}{\to} y_n = y.$$

Then we are done by repeating the above argument. □

2.2. *The functor E*

Let $\mathscr{C}$ be a finite free EI category. We will construct the functor $E : \mathscr{C} \longrightarrow k$-mod in the introduction, where k-mod is the category of finite-dimensional k-vector spaces.

Denote by $k\mathscr{C}$-mod the category of finite-dimensional left modules over the category algebra $k\mathscr{C}$, and by $(k\text{-mod})^{\mathscr{C}}$ the category of covariant functors from $\mathscr{C}$ to k-mod. There is a well-known equivalence $k\mathscr{C}\text{-mod} \simeq (k\text{-mod})^{\mathscr{C}}$; see [10, Theorem 7.1]. We identify a $k\mathscr{C}$-module with a functor from $\mathscr{C}$ to k-mod.

Notation 2.2. Let $\mathscr{C}$ be a finite free EI category. For any $x \overset{\alpha}{\to} y$ in $\mathscr{C}$, set

$$V(\alpha) = \{w \in \mathrm{Obj}\mathscr{C} \mid \exists x \overset{\alpha'}{\to} w \overset{\alpha''}{\to} y \text{ such that } \alpha = \alpha'' \circ \alpha' \text{ with } \alpha'' \text{ a non-isomorphism}\}.$$

If α is an isomorphism, $V(\alpha) = \emptyset$. For any $w \in V(\alpha)$, we define

$$t_w(\alpha) = \alpha'' \circ \left(\sum_{g \in \mathrm{Aut}_{\mathscr{C}}(w)} g \right) \in k\mathrm{Hom}_{\mathscr{C}}(w, y),$$

where the non-isomorphism $\alpha'' : w \to y$ is given by a factorization $x \overset{\alpha'}{\to} w \overset{\alpha''}{\to} y$ of α. By Proposition 2.1, such a morphism α'' is unique up to an automorphism of w. It follows that the element $t_w(\alpha)$ is independent of the choice of α''.

Remark 2.3. Assume that $x \overset{\alpha}{\to} y$ is a non-isomorphism with a decomposition into unfactorizable morphisms:

$$x = x_0 \overset{\alpha_1}{\to} x_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_m}{\to} x_m = y.$$

By the UFP, we observe that $V(\alpha) = \{x_0, x_1, \ldots, x_{m-1}\}$, where these x_i's are pairwise distinct. We have $V(\alpha \circ g) = V(\alpha) = V(h \circ \alpha)$ for any $g \in \mathrm{Aut}_{\mathscr{C}}(x)$ and $h \in \mathrm{Aut}_{\mathscr{C}}(y)$.

Lemma 2.4. *Let $\mathscr{C}$ be a finite free EI category. For any $x \overset{\alpha}{\to} y$ and $y \overset{\beta}{\to} z$ in $\mathscr{C}$, we have*

$$V(\beta \circ \alpha) = V(\beta) \sqcup V(\alpha),$$

where the right-hand side is a disjoint union.

Proof. If α or β is an isomorphism, we have $V(\beta \circ \alpha) = V(\beta) \sqcup V(\alpha)$ by Remark 2.3. Assume that α and β are non-isomorphisms. Let

$$x = x_0 \overset{\alpha_1}{\to} x_1 \overset{\alpha_2}{\to} \cdots \overset{\alpha_m}{\to} x_m = y$$

and

$$y = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} y_n = z$$

be decompositions of α and β into unfactorizable morphisms, respectively. Then we have $V(\alpha) = \{x, x_1, \ldots, x_{m-1}\}$ and $V(\beta) = \{y, y_1, \ldots, y_{n-1}\}$ by Remark 2.3. Since

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} x_m = y \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_n} y_n = z$$

is a decomposition of $\beta \circ \alpha$ into unfactorizable morphisms, by Remark 2.3, we have $V(\beta \circ \alpha) = \{x, x_1, \ldots, x_{m-1}, y, y_1, \ldots, y_{n-1}\} = V(\alpha) \sqcup V(\beta)$. □

Lemma 2.5. *Let $\mathscr{C}$ be a finite free EI category and $x \xrightarrow{\alpha} y$, $y \xrightarrow{\beta} z$ be two morphisms in $\mathscr{C}$. For $w \in V(\beta) \subseteq V(\beta\circ\alpha)$, we have $t_w(\beta\circ\alpha) = t_w(\beta)$. For $w \in V(\alpha) \subseteq V(\beta\circ\alpha)$, we have $t_w(\beta \circ \alpha) = \beta \circ t_w(\alpha)$.*

Proof. For an object w in $\mathscr{C}$, we set $G_w = \mathrm{Aut}_{\mathscr{C}}(w)$. If $w \in V(\beta)$, then there exist $y \xrightarrow{\beta'} w \xrightarrow{\beta''} z$ in $\mathscr{C}$ such that $\beta = \beta'' \circ \beta'$ with β'' a non-isomorphism. By definition we have $t_w(\beta) = \beta'' \circ (\sum_{g\in G_w} g)$. By $\beta \circ \alpha = \beta'' \circ (\beta' \circ \alpha)$, we have $t_w(\beta \circ \alpha) = \beta'' \circ (\sum_{g\in G_w} g) = t_w(\beta)$.

If $w \in V(\alpha)$, then there exists $x \xrightarrow{\alpha'} w \xrightarrow{\alpha''} y$ in $\mathscr{C}$ such that $\alpha = \alpha'' \circ \alpha'$ with α'' a non-isomorphism. By definition, we have $t_w(\alpha) = \alpha'' \circ (\sum_{g\in G_w} g)$. We observe that $\beta \circ \alpha = (\beta \circ \alpha'') \circ \alpha'$ with $\beta \circ \alpha''$ a non-isomorphism, since α'' is a non-isomorphism. Then we have $t_w(\beta \circ \alpha) = (\beta \circ \alpha'') \circ (\sum_{g\in G_w} g) = \beta \circ t_w(\alpha)$. □

Definition 2.6. Let $\mathscr{C}$ be a finite free EI category. We define

$$E : \mathscr{C} \longrightarrow k\text{-mod}$$

as follows: for any $x \in \mathrm{Obj}\mathscr{C}$, $E(x)$ is a k-vector space with basis

$$B_x = \bigsqcup_{w\neq x} \mathrm{Hom}_{\mathscr{C}}(w, x) \cup \{e_x\};$$

for any $x \xrightarrow{\alpha} y$ in $\mathscr{C}$, the k-linear map $E(\alpha) : E(x) \to E(y)$ is given by

$$E(\alpha)(e_x) = e_y + \sum_{w\in V(\alpha)} t_w(\alpha)$$

and

$$E(\alpha)(\gamma) = \alpha \circ \gamma$$

for $\gamma \in \mathrm{Hom}_{\mathscr{C}}(w, x)$ with $w \neq x$.

Proposition 2.7. *Let $\mathscr{C}$ be a finite free EI category. The above $E : \mathscr{C} \longrightarrow k$-mod is a functor.*

Proof. For any $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ in $\mathscr{C}$, since $E(\beta\circ\alpha)(\gamma) = \beta\circ\alpha\circ\gamma = E(\beta)(E(\alpha)(\gamma))$ for $\gamma \in \mathrm{Hom}_{\mathscr{C}}(w, x)$ with $w \neq x$, it suffices to prove that $E(\beta\circ\alpha)(e_x) = E(\beta)(E(\alpha)(e_x))$.

We have

$$E(\beta\circ\alpha)(e_x) = e_z + \sum_{w\in V(\beta\circ\alpha)} t_w(\beta\circ\alpha)$$

and

$$E(\beta)(E(\alpha)(e_x)) = e_z + \sum_{w\in V(\beta)} t_w(\beta) + \beta\circ\left(\sum_{w\in V(\alpha)} t_w(\alpha)\right).$$

Then we are done by Lemmas 2.4 and 2.5. □

We now introduce subfunctors of E for later use. Let $\mathscr{C}$ be a finite free EI category. By the EI property, we may assume that $\mathrm{Obj}\,\mathscr{C} = \{x_1, x_2, \ldots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$.

Notation 2.8. For each $1 \le t \le n$, we define a functor

$$Y^t : \mathscr{C} \to k\text{-mod}$$

as follows: $Y^t(x_i)$ is a k-vector space with basis

$$B_i^t = \bigsqcup_{l=i+1}^{t} \mathrm{Hom}_{\mathscr{C}}(x_l, x_i) \cup \{e_{x_i}\}$$

for $1 \le i \le t$ and $Y^t(x_i) = 0$ for $i > t$; for any $x_j \xrightarrow{\alpha} x_i$ with $j \le t$, the k-linear map $Y^t(\alpha) : Y^t(x_j) \to Y^t(x_i)$ is given by

$$Y^t(\alpha)(e_{x_j}) = e_{x_i} + \sum_{x_l\in V(\alpha)} t_{x_l}(\alpha)$$

and

$$Y^t(\alpha)(\gamma) = \alpha\circ\gamma$$

for $\gamma \in B_j^t \backslash \{e_{x_j}\}$; $Y^t(\alpha) = 0$ if $j > t$.

We observe that $Y^n = E$ and that Y^t is a subfunctor of Y^{t+1} for each $1 \le t \le n-1$.

2.3. *An exact sequence*

We will describe a subfunctor of the functor E such that the quotient functor is isomorphic to the constant functor.

Recall that the *constant functor* $\underline{k} : \mathscr{C} \to k$-mod is defined by $\underline{k}(x) = k$ for all $x \in \mathrm{Obj}\,\mathscr{C}$ and $\underline{k}(\alpha) = \mathrm{Id}_k$ for all $\alpha \in \mathrm{Mor}\,\mathscr{C}$. The corresponding $k\mathscr{C}$-module is called the *trivial module*.

Notation 2.9. Let $\mathscr{C}$ be a finite EI category. We define a functor

$$K : \mathscr{C} \to k\text{-mod}$$

as follows: for any $x \in \operatorname{Obj} \mathscr{C}$, $K(x)$ is a k-vector space with basis

$$B'_x = \bigsqcup_{w \neq x} \operatorname{Hom}_{\mathscr{C}}(w, x);$$

for any $x \xrightarrow{\alpha} y$ in $\mathscr{C}$, the k-linear map $K(\alpha) : K(x) \to K(y)$ is given by

$$K(\alpha)(\gamma) = \alpha \circ \gamma$$

for $\gamma \in \operatorname{Hom}_{\mathscr{C}}(w, x)$ with $w \neq x$.

Let $\mathscr{C}$ be a finite free EI category. We observe that K is a subfunctor of E.

We have a surjective natural transformation $E \xrightarrow{\pi} \underline{k}$ as follows: for any $x \in \operatorname{Obj} \mathscr{C}$, $E(x) \xrightarrow{\pi_x} \underline{k}(x) = k$ sends e_x to 1_k, and γ to zero for $\gamma \in \operatorname{Hom}_{\mathscr{C}}(w, x)$ with $w \neq x$. Then we have an exact sequence of functors

$$0 \to K \xrightarrow{\text{inc}} E \xrightarrow{\pi} \underline{k} \to 0. \tag{2.1}$$

In what follows, we study when the above exact sequence splits, or equivalently, the epimorphism π splits.

Recall that a finite category $\mathscr{C}$ is *connected* if for any two distinct objects x and y, there is a sequence of objects $x = x_0, x_1, \ldots, x_m = y$ such that either $\operatorname{Hom}_{\mathscr{C}}(x_i, x_{i+1})$ or $\operatorname{Hom}_{\mathscr{C}}(x_{i+1}, x_i)$ is not empty, $0 \leq i \leq m-1$. We say that the category $\mathscr{C}$ has a *smallest object* z, if $\operatorname{Hom}_{\mathscr{C}}(z, x) \neq \emptyset$ for each object x.

The following lemma is well known. For completeness, we give a proof.

Lemma 2.10. *Let $(I, \leq)$ be a connected finite poset without a smallest element. Then there are two distinct minimal elements with a common upper bound.*

Proof. For each $z \in I$, set $S_z = \{a \in I \mid a \geq z\}$. Then $I = \bigcup_{\{z \in I \mid z \text{ is minimal}\}} S_z$. Assume on the contrary that any two distinct minimal elements do not have a common upper bound. Then $I = \bigcup_{\{z \in I \mid z \text{ is minimal}\}} S_z$ is a disjoint union. Let $x \in S_z$ with z minimal, and $y \in I$. We claim that if x and y are comparable, then $y \in S_z$. Indeed, if $x \leq y$, then $z \leq x \leq y$ and thus $y \in S_z$. If $y \leq x$, we assume that $y \in S_{z'}$ with z' minimal. Then $z' \leq y \leq x$, and thus $x \in S_{z'}$. Hence $z' = z$.

The above claim implies that each S_z is a connected component of I. This contradicts to the connectedness of I. □

Lemma 2.11. *Let $\mathscr{C}$ be a finite connected free EI category. Then we have the following two statements.*

(1) *If the category $\mathscr{C}$ has a smallest object z such that* $\operatorname{Hom}_{\mathscr{C}}(z, x)$ *has only one* $\operatorname{Aut}_{\mathscr{C}}(z)$*-orbit for each object x, then the epimorphism π in* (2.1) *splits.*

(2) *Assume that* $\mathrm{Aut}_{\mathscr{C}}(x)$ *acts freely on* $\mathrm{Hom}_{\mathscr{C}}(x,y)$ *for any objects* x *and* y. *If* π *splits, then the category* $\mathscr{C}$ *has a smallest object* z *such that* $\mathrm{Hom}_{\mathscr{C}}(z,x)$ *has only one* $\mathrm{Aut}_{\mathscr{C}}(z)$*-orbit for each object* x.

Proof. For (1), let z be the smallest object. We have that $\mathrm{Hom}_{\mathscr{C}}(z,x) = \alpha_x \circ \mathrm{Aut}_{\mathscr{C}}(z)$, where $\alpha_x \in \mathrm{Hom}_{\mathscr{C}}(z,x)$ is a chosen morphism for any object x. We define a natural transformation $s : \underline{k} \to E$ as follows: for any $x \in \mathrm{Obj}\ \mathscr{C}$, $\underline{k}(x) = k \overset{s_x}{\to} E(x)$ sends 1_k to $E(\alpha_x)(e_z) = e_x + \sum_{w\in V(\alpha_x)} t_w(\alpha_x)$. We observe that $\pi \circ s = \mathrm{Id}_{\underline{k}}$.

To prove that s is a natural transformation, it suffices to prove that $s_y(1_k) = E(\alpha)(s_x(1_k))$ for any morphism $x \overset{\alpha}{\to} y$. We observe that $s_y(1_k) = E(\alpha_y)(e_z)$ and $E(\alpha)(s_x(1_k)) = E(\alpha)(E(\alpha_x)(e_z)) = E(\alpha \circ \alpha_x)(e_z)$. Since $\mathrm{Hom}_{\mathscr{C}}(z,y) = \alpha_y \circ \mathrm{Aut}_{\mathscr{C}}(z)$, we have that $\alpha \circ \alpha_x = \alpha_y \circ g$ for some $g \in \mathrm{Aut}_{\mathscr{C}}(z)$. Then we have that $E(\alpha_y)(e_z) = E(\alpha \circ \alpha_x \circ g^{-1})(e_z) = E(\alpha \circ \alpha_x)(E(g^{-1})(e_z)) = E(\alpha \circ \alpha_x)(e_z)$, since $E(g^{-1})(e_z) = e_z$ by the construction of E.

For (2), assume that π splits. Then there is a natural transformation $s : \underline{k} \to E$ with $\pi \circ s = \mathrm{Id}_{\underline{k}}$. It follows that $\underline{k}(x) = k \overset{s_x}{\to} E(x)$ sends 1_k to $e_x + \sum_{\{\gamma: w\to x \mid w\neq x\}} c_\gamma \gamma$, $c_\gamma \in k$ for each $x \in \mathrm{Obj}\ \mathscr{C}$.

We recall that $\mathrm{Obj}\ \mathscr{C}$ can be viewed as a poset, where $x \leq y$ if and only if $\mathrm{Hom}_{\mathscr{C}}(x,y) \neq \emptyset$. We claim that $\mathscr{C}$ has a smallest object. Otherwise, there are two distinct minimal objects z and z' such that there is an object x satisfying $\mathrm{Hom}_{\mathscr{C}}(z,x) \neq \emptyset$ and $\mathrm{Hom}_{\mathscr{C}}(z',x) \neq \emptyset$; see Lemma 2.10. Since z and z' are minimal, we have that $s_z(1_k) = e_z$ and $s_{z'}(1_k) = e_{z'}$. Let $\alpha \in \mathrm{Hom}_{\mathscr{C}}(z,x)$ and $\beta \in \mathrm{Hom}_{\mathscr{C}}(z',x)$. We have that $E(\alpha)(s_z(1_k)) = s_x(1_k) = E(\beta)(s_{z'}(1_k))$, since s is a natural transformation. We observe that $E(\alpha)(s_z(1_k)) = E(\alpha)(e_z) = e_x + \sum_{w\in V(\alpha)} t_w(\alpha)$, and $E(\beta)(s_{z'}(1_k)) = E(\beta)(e_{z'}) = e_x + \sum_{w\in V(\beta)} t_w(\beta)$. Then we have $\sum_{w\in V(\alpha)} t_w(\alpha) = \sum_{w\in V(\beta)} t_w(\beta)$. Since $\mathrm{Aut}_{\mathscr{C}}(w)$ acts freely on $\mathrm{Hom}_{\mathscr{C}}(w,x)$, then $t_w(\alpha) \neq 0$ and $t_w(\beta) \neq 0$. Then we infer that $V(\alpha) = V(\beta)$ and $t_w(\alpha) = t_w(\beta)$ for each $w \in V(\alpha) = V(\beta)$. Then $z \in V(\alpha) = V(\beta)$, and thus $\mathrm{Hom}_{\mathscr{C}}(z',z) \neq \emptyset$. This is a contradiction.

Denote by z the smallest element in $\mathscr{C}$. Let α and β be two morphisms in $\mathrm{Hom}_{\mathscr{C}}(z,x)$ for any object x. It suffices to prove that α and β are in the same $\mathrm{Aut}_{\mathscr{C}}(z)$-orbit. Since s is a natural transformation, we have that $E(\alpha)(s_z(1_k)) = s_x(1_k) = E(\beta)(s_z(1_k))$. We observe that $E(\alpha)(s_z(1_k)) = E(\alpha)(e_z) = e_x + \sum_{w\in V(\alpha)} t_w(\alpha)$, and $E(\beta)(s_z(1_k)) = E(\beta)(e_z) = e_x + \sum_{w\in V(\beta)} t_w(\beta)$. Then we have that $t_z(\alpha) = t_z(\beta)$, that is, $\alpha \circ (\sum_{g\in \mathrm{Aut}_{\mathscr{C}}(z)} g) = \beta \circ (\sum_{g\in \mathrm{Aut}_{\mathscr{C}}(z)} g)$. Since $\mathrm{Aut}_{\mathscr{C}}(z)$ acts freely on $\mathrm{Hom}_{\mathscr{C}}(z,x)$, we have $\alpha = \beta \circ h$ for some $h \in \mathrm{Aut}_{\mathscr{C}}(z)$. Then we are done. □

3. The Gorenstein-Projective Module E

In this section, we prove that E, viewed as a module over the category algebra, is Gorenstein-projective if in addition $\mathscr{C}$ is projective and prove Theorem 1.1. For the proof, we study the subfunctor K of E. We observe that the projectivity of K is equivalent to the freeness of the category $\mathscr{C}$; see Proposition 3.3.

3.1. *Gorenstein-projective modules*

Let A be a finite-dimensional algebra over k. Denote by A-mod the category of finite-dimensional left A-modules. The opposite algebra of A is denoted by $A^{\rm op}$. We identify right A-modules with left $A^{\rm op}$-modules.

Denote by $(-)^*$ the contravariant functor $\mathrm{Hom}_A(-, A)$ or $\mathrm{Hom}_{A^{\rm op}}(-, A)$. Let X be an A-module. Then X^* is a right A-module and X^{**} is a left A-module. There is an evaluation map $\mathrm{ev}_X : X \to X^{**}$ given by $\mathrm{ev}_X(x)(f) = f(x)$ for $x \in X$ and $f \in X^*$. Recall that an A-module G is *Gorenstein-projective* provided that $\mathrm{Ext}^i_A(G, A) = 0 = \mathrm{Ext}^i_{A^{\rm op}}(G^*, A)$ for $i \geq 1$ and the evaluation map ev_G is bijective; see [1, Proposition 3.8].

Denote by A-Gproj the full subcategory of A-mod consisting of Gorenstein-projective A-modules. Recall that A-Gproj is closed under extensions, that is, for a short exact sequence $0 \to X \to Y \to Z \to 0$ of A-modules, $X, Z \in A$-Gproj implies $Y \in A$-Gproj. It is well known that $\mathrm{Ext}^1_A(G, X) = 0$ for any A-module X with finite projective dimension and $G \in A$-Gproj; see [7, Sec. 10.2].

Recall that $k\mathscr{C}$-Gproj is a Frobenius category whose projective-injective objects are precisely finitely generated projective $k\mathscr{C}$-modules. Denote by $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ the corresponding stable category modulo projective $k\mathscr{C}$-modules. Then $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ has a natural triangulated structure: the translation functor is a quasi-inverse of the syzygy functor and the triangles are induced by short exact sequences with terms in $k\mathscr{C}$-Gproj.

In the literature, Gorenstein-projective modules are also called *maximal Cohen–Macaulay* (MCM for short) modules. For an A-module X, an *MCM-approximation* of X is a map $\theta : G \to X$ with G Gorenstein-projective such that any map $G' \to X$ with G' Gorenstein-projective factors through θ. We observe that such an MCM-approximation is necessarily surjective.

By a *special MCM-approximation*, we mean an epimorphism $\theta : G \to X$ with G Gorenstein-projective such that the kernel of θ, denoted by $\mathrm{Ker}\ \theta$, has finite projective dimension; cf. [7, Definition 7.1.6]. Since $\mathrm{Ext}^1_A(G', \mathrm{Ker}\ \theta) = 0$ for any Gorenstein-projective G', we have that a special MCM-approximation is an MCM-approximation.

The algebra A is *Gorenstein* if $\mathrm{id}_A\ A < \infty$ and $\mathrm{id}\ A_A < \infty$, where id denotes the injective dimension of a module. It is well known that for a Gorenstein algebra A we have $\mathrm{id}_A\ A = \mathrm{id}\ A_A$; see [17, Lemma A]. Let $m \geq 0$. A Gorenstein algebra A is *m-Gorenstein* if $\mathrm{id}_A\ A = \mathrm{id}\ A_A \leq m$.

The following result is well known; see [6, Lemma 4.2].

Lemma 3.1. *Let A be a 1-Gorenstein algebra. Then an A-module G is Gorenstein-projective if and only if there is a monomorphism $G \to P$ with P projective.*

Let $\mathscr{C}$ be a finite EI category which is skeletal. Recall from [11, Definition 4.2] that the category $\mathscr{C}$ is *projective over* k if each $k\mathrm{Aut}_{\mathscr{C}}(y)$-$k\mathrm{Aut}_{\mathscr{C}}(x)$-bimodule $k\mathrm{Hom}_{\mathscr{C}}(x, y)$ is projective on both sides. By [11, Proposition 5.1] the category algebra $k\mathscr{C}$ is Gorenstein if and only if $\mathscr{C}$ is projective over k, in which case, $k\mathscr{C}$ is 1-Gorenstein if and only if the category $\mathscr{C}$ is free; see [11, Theorem 5.3].

3.2. *Functors as modules*

Let us recall from [3, III.2] some notation on matrix algebras. Let $n \geq 2$. Recall that an $n \times n$ upper triangular matrix algebra

$$\Gamma = \begin{pmatrix} A_1 & M_{12} & \cdots & M_{1n} \\ & A_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & A_n \end{pmatrix}$$

is given by the following data: each A_i is a finite-dimensional algebra over k, and each M_{ij} is an A_i-A_j-bimodule on which k acts centrally for $1 \leq i < j \leq n$. The multiplication of the matrix algebra is induced by A_i-A_j-bimodule morphisms $\psi_{ilj} : M_{il} \otimes_{R_l} M_{lj} \to M_{ij}$, which satisfy the following identities

$$\psi_{ijt}(\psi_{ilj}(m_{il} \otimes m_{lj}) \otimes m_{jt}) = \psi_{ilt}(m_{il} \otimes \psi_{ljt}(m_{lj} \otimes m_{jt})),$$

for $1 \leq i < l < j < t \leq n$.

Recall that a left Γ-module

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

is described by a column vector: each X_i is a left A_i-module, the left Γ-module structure is induced by left A_j-module morphisms $\varphi_{jl} : M_{jl} \otimes_{R_l} X_l \to X_j$, which satisfy the following identities

$$\varphi_{ij} \circ (\mathrm{Id}_{M_{ij}} \otimes \varphi_{jl}) = \varphi_{il} \circ (\psi_{ijl} \otimes \mathrm{Id}_{X_l}),$$

for $1 \leq i < j < l \leq n$.

Let $\mathscr{C}$ be a skeletal finite EI category with $\mathrm{Obj}\,\mathscr{C} = \{x_1, x_2, \ldots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$. Set $M_{ij} = k\mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$. Write $A_i = M_{ii}$, which is the group algebra of $\mathrm{Aut}_{\mathscr{C}}(x_i)$. Recall that the category algebra $k\mathscr{C}$ is isomorphic to the corresponding upper triangular matrix algebra

$$\Gamma_{\mathscr{C}} = \begin{pmatrix} A_1 & M_{12} & \cdots & M_{1n} \\ & A_2 & \cdots & M_{2n} \\ & & \ddots & \vdots \\ & & & A_n \end{pmatrix};$$

see [11, Sec. 4]. The corresponding maps ψ_{ilj} are induced by the composition of morphisms in $\mathscr{C}$.

Then we have the following composition of equivalences

$$(k\text{-mod})^{\mathscr{C}} \xrightarrow{\sim} k\mathscr{C}\text{-mod} \xrightarrow{\sim} \Gamma_{\mathscr{C}}\text{-mod}, \tag{3.1}$$

which sends a functor $X : \mathscr{C} \to k\text{-mod}$ to a $\Gamma_{\mathscr{C}}$-module

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

as follows: $X_i = X(x_i)$, and $\varphi_{ij} : M_{ij} \otimes_{A_j} X_j \to X_i$ sends $\alpha \otimes a_j$ to $X(\alpha)(a_j)$ for $\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$ and $a_j \in X_j$ for $1 \leq i < j \leq n$. In what follows, we identify these three categories.

Proposition 3.2. *Let $\mathscr{C}$ be a finite free EI category and $E : \mathscr{C} \to k$-mod be the functor in Definition* 2.6. *Assume that $\mathscr{C}$ is projective. Then E, viewed as a $k\mathscr{C}$-module, is Gorenstein-projective.*

Proof. For each $1 \leq t \leq n$, let Y^t be the functor in Notation 2.8. Then we have a filtration $0 = Y^0 \subseteq Y^1 \subseteq \cdots \subseteq Y^{n-1} \subseteq Y^n = E$ of subfunctors. Since Gorenstein-projective modules are closed under extensions, it suffices to prove that each Y^t/Y^{t-1} is a Gorenstein-projective $\Gamma_{\mathscr{C}}$-module for $1 \leq t \leq n$.

Recall from [11, Theorem 5.3] that $\Gamma_{\mathscr{C}}$ is 1-Gorenstein. We observe that

$$Y^1 \simeq \begin{pmatrix} ke_{x_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{and} \quad Y^t/Y^{t-1} \simeq \begin{pmatrix} M_{1t} \\ \vdots \\ M_{t-1,t} \\ ke_{x_t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

as $\Gamma_{\mathscr{C}}$-modules, where the structure maps φ_{ij} of Y^t/Y^{t-1} are described as follows: $\varphi_{ij} = \psi_{ijt}$ if $i < j < t$, $\varphi_{it}(\alpha \otimes e_{x_t}) = \alpha \circ (\sum_{g \in \mathrm{Aut}_{\mathscr{C}}(x_t)} g)$ for $\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_t, x_i)$, and $\varphi_{ij} = 0$ if $j > t$.

Denote by C_t the tth column of $\Gamma_{\mathscr{C}}$. It is a projective $\Gamma_{\mathscr{C}}$-module. We observe that each Y^t/Y^{t-1} is embedded in C_t, by sending e_{x_t} to $\sum_{g \in \mathrm{Aut}_{\mathscr{C}}(x_t)} g$. By Lemma 3.1, each Y^t/Y^{t-1} is Gorenstein-projective. □

Let $\mathscr{C}$ be a skeletal finite EI category with $\mathrm{Obj}\,\mathscr{C} = \{x_1, x_2, \ldots, x_n\}$ satisfying $\mathrm{Hom}_{\mathscr{C}}(x_i, x_j) = \emptyset$ if $i < j$ and let $\Gamma_{\mathscr{C}}$ be the corresponding upper triangular matrix algebra. For each $1 \leq t \leq n$, denote by the $\Gamma_{\mathscr{C}}$-module

$$i_t(R_t)^* = \begin{pmatrix} M_{1t} \\ \vdots \\ M_{t-1,t} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

whose structure map is given by $\varphi_{ij} = \psi_{ijt}$ if $i < j < t$, and $\varphi_{ij} = 0$ otherwise. Denote by Γ_t the algebra given by the $t \times t$ leading principal submatrix of $\Gamma_{\mathscr{C}}$. Denote by M_t^* the left Γ_t-module

$$\begin{pmatrix} M_{1,t+1} \\ \vdots \\ M_{t,t+1} \end{pmatrix}$$

for $1 \leq t \leq n-1$. We use the same notation in [11].

The following result is implicitly contained in [11, Proposition 4.5].

Proposition 3.3. *Let $\mathscr{C}$ be a finite projective EI category and $K : \mathscr{C} \to k$-mod be the functor in Notation 2.9. Then K, viewed as a $k\mathscr{C}$-module, is projective if and only if the category $\mathscr{C}$ is free.*

Proof. We observe that the functor $K : \mathscr{C} \to k$-mod in Notation 2.9 is isomorphic to $\bigoplus_{t=2}^{n} i_t(R_t)^*$ as $\Gamma_{\mathscr{C}}$-modules.

We have the fact that $i_t(R_t)^*$ is a projective $\Gamma_{\mathscr{C}}$-module if and only if M_{t-1}^* is a projective Γ_{t-1}-module for each $1 \leq t \leq n$; see [11, Lemma 2.1]. By [11, Proposition 4.5], the category $\mathscr{C}$ is free if and only if each M_t^* is a projective Γ_t-module for $1 \leq t \leq n-1$. Then we are done by the above observation. □

3.3. *Proof of Theorem* 1.1

We now are in a position to prove Theorem 1.1. Recall the map $E \xrightarrow{\pi} \underline{k}$ in (2.1). The constant functor $\underline{k}$ corresponds to the trivial module of $k\mathscr{C}$.

Theorem 3.4. *Let $\mathscr{C}$ be a finite free EI category. Assume that the category $\mathscr{C}$ is projective over k. Then the map $E \xrightarrow{\pi} \underline{k}$ is a special MCM-approximation of the trivial module $\underline{k}$.*

Proof. By Proposition 3.2, the $k\mathscr{C}$-module E is Gorenstein-projective. By Proposition 3.3, the $k\mathscr{C}$-module K is projective. Hence the exact sequence (2.1) is a special MCM-approximation of $\underline{k}$. □

Proposition 3.5. *Let $\mathscr{C}$ be a finite connected EI category, which is free and projective over k. Then we have the following statements.*

(1) *If the category $\mathscr{C}$ has a smallest object z such that $\mathrm{Hom}_{\mathscr{C}}(z,x)$ has only one $\mathrm{Aut}_{\mathscr{C}}(z)$-orbit for each object x, then the trivial module $\underline{k}$ is Gorenstein-projective.*

(2) *Assume that $\mathrm{Aut}_{\mathscr{C}}(x)$ acts freely on $\mathrm{Hom}_{\mathscr{C}}(x,y)$ for any objects x and y. If $\underline{k}$ is Gorenstein-projective, then the category $\mathscr{C}$ has a smallest object z such that $\mathrm{Hom}_{\mathscr{C}}(z,x)$ has only one $\mathrm{Aut}_{\mathscr{C}}(z)$-orbit for each object x.*

Proof. We observe that $\underline{k}$ is Gorenstein-projective if and only if the short exact sequence (2.1) splits. Here, we use the fact that Gorenstein-projective modules are closed under direct summands. Consequently, the statement follows immediately from Lemma 2.11. □

4. The Tensor Identity *E*

In this section, let $\mathscr{C}$ be a finite projective EI category. Then the category algebra $k\mathscr{C}$ is Gorenstein by [11, Proposition 5.1]. We observe that the stable category $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ is a *tensor triangulated category* in the sense of [4, Definition 1.1]. We will prove that E is the tensor identity of $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ if in addition $\mathscr{C}$ is a finite free EI category.

Recall that the category $k\mathscr{C}$-mod is identified with $(k\text{-mod})^{\mathscr{C}}$, the category of covariant functors from $\mathscr{C}$ to k-mod, by (3.1). The category $k\mathscr{C}$-mod is a symmetric monoidal category, written as ($k\mathscr{C}$-mod, $\hat{\otimes}$, $\underline{k}$), in which the trivial module $\underline{k}$ is the tensor identity. More precisely, the tensor product $\hat{\otimes}$ is defined by

$$(M\hat{\otimes}N)(x) = M(x) \otimes_k N(x)$$

for any $M, N \in (k\text{-mod})^{\mathscr{C}}$ and $x \in \mathrm{Obj}\,\mathscr{C}$, and $\alpha.(m \otimes n) = \alpha.m \otimes \alpha.n$ for any $\alpha \in \mathrm{Mor}\,\mathscr{C}, m \in M(x), n \in N(x)$; see [15, 16]. Since the tensor product $-\hat{\otimes}-$ is exact in both variables, it gives rise to a tensor product on $\mathrm{D}^b(k\mathscr{C}\text{-mod})$, the bounded derived category of finitely generated left $k\mathscr{C}$-modules. Hence, $\mathrm{D}^b(k\mathscr{C}\text{-mod})$ is naturally a tensor triangulated category. We shall still write $\hat{\otimes}$ and $\underline{k}$ for the tensor product and tensor identity in $\mathrm{D}^b(k\mathscr{C}\text{-mod})$, respectively.

Recall that a complex in $\mathrm{D}^b(k\mathscr{C}\text{-mod})$ is called a *perfect complex* if it is isomorphic to a bounded complex of finitely generated projective modules. Recall from [5] that the *singularity category* of $k\mathscr{C}$, denoted by $\mathrm{D}_{sg}(k\mathscr{C})$, is the Verdier quotient category $\mathrm{D}^b(k\mathscr{C}\text{-mod})/\mathrm{perf}(k\mathscr{C})$, where $\mathrm{perf}(k\mathscr{C})$ is a thick subcategory of $\mathrm{D}^b(k\mathscr{C}\text{-mod})$ consisting of all perfect complexes.

Lemma 4.1. *Let $\mathscr{C}$ be a finite projective EI category. Then the thick subcategory $\mathrm{perf}(k\mathscr{C})$ is a tensor ideal of $\mathrm{D}^b(k\mathscr{C}\text{-mod})$. In particular, the category $\mathrm{D}_{sg}(k\mathscr{C})$ is naturally a tensor triangulated category.*

We also write $\hat{\otimes}$ and $\underline{k}$ for the tensor product and tensor identity in $\mathrm{D}_{sg}(k\mathscr{C})$, respectively.

Proof. The final statement follows from [4, Remark 3.10]. We only need to prove that for any $k\mathscr{C}$-module X and any projective $k\mathscr{C}$-module P, the $k\mathscr{C}$-module $X\hat{\otimes}P$ has finite projective dimension. By the equivalences (3.1), X and P can be viewed as $\Gamma_{\mathscr{C}}$-modules

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \text{ and } \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}.$$

We observe that

$$X \hat{\otimes} P = \begin{pmatrix} X_1 \otimes_k P_1 \\ \vdots \\ X_n \otimes_k P_n \end{pmatrix},$$

where $\varphi_{ij} : M_{ij} \otimes_{A_j} (X_j \otimes_k P_j) \to X_i \otimes_k P_i$ sends $\alpha \otimes (x_j \otimes p_j)$ to $\varphi^X_{ij}(\alpha \otimes x_j) \otimes \varphi^P_{ij}(\alpha \otimes p_j)$ for $\alpha \in \mathrm{Hom}_{\mathscr{C}}(x_j, x_i)$ and $x_j \in X_j, p_j \in P_j$ for $1 \le i < j \le n$. Since P is a projective $\Gamma_{\mathscr{C}}$-module, we have that each A_i-module P_i has finite projective dimension by [11, Corollary 3.6]. Hence each A_i-module P_i is projective since A_i is a group algebra. Then we have that each $X_i \otimes_k P_i$ is a projective A_i-module for $1 \le i \le n$. By [11, Corollary 3.6], the $\Gamma_{\mathscr{C}}$-module $X\hat{\otimes}P$ has finite projective dimension. Then we are done. □

Let $\mathscr{C}$ be a projective EI category. The category algebra $k\mathscr{C}$ is Gorenstein. Then we have a triangle equivalence

$$F : k\mathscr{C}\text{-}\underline{\mathrm{Gproj}} \xrightarrow{\sim} \mathrm{D}_{sg}(k\mathscr{C}) \tag{4.1}$$

which sends a Gorenstein-projective module to the corresponding stalk complex concentrated on degree zero; see [5, Theorem 4.4.1]. The tensor product in $\mathrm{D}_{sg}(k\mathscr{C})$ can be transported to $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$. Consequently, the category $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ becomes a tensor triangulated category. The spectrum of $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ is studied in [16], when $\mathscr{C}$ is a finite transporter category. We mention that the intrinsic description of the tensor product in $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$ will be studied in [12].

Lemma 4.2. *Let $\mathscr{C}$ be a finite projective EI category. Assume that the trivial $k\mathscr{C}$-module $\underline{k}$ has a special MCM-approximation $G \xrightarrow{\theta} \underline{k}$. Then G is the tensor identity of $k\mathscr{C}$-$\underline{\mathrm{Gproj}}$.*

Proof. Let F be the functor in (4.1). We only need to prove that $F(G) \simeq \underline{k}$, the tensor identity of $\mathrm{D}_{sg}(k\mathscr{C})$. The exact sequence $0 \to \mathrm{Ker}\ \theta \to G \to \underline{k} \to 0$ induces a triangle $F(\mathrm{Ker}\ \theta) \to F(G) \to \underline{k} \to F(\mathrm{Ker}\ \theta)\ [1]$ in $\mathrm{D}_{sg}(k\mathscr{C})$. Since $\mathrm{Ker}\ \theta$ has finite projective dimension, we have $F(\mathrm{Ker}\ \theta) \simeq 0$. It follows that $F(G)$ is isomorphic to $\underline{k}$. □

We have the following result by Lemma 4.2 and Theorem 3.4.

Proposition 4.3. *Let $\mathscr{C}$ be a finite projective EI category. Assume that $\mathscr{C}$ is free. Then E is the tensor identity of $k\mathscr{C}$-$\underline{\text{Gproj}}$.* □

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