On extriangulated categories, exact infinity-categories and exact dg categories

In this lecture series, we will report on recent developments concerning the three generalizations of Quillen's notion of exact category mentioned in the title.

In the first lecture, we will introduce extriangulated categories following the work of Nakaoka-Palu (2019). This notion generalizes both, the notion of exact and that of triangulated category. It arises naturally in the categorification of cluster algebras with coefficients. Here the relevant categories are due to Pressland (for many examples) and Yilin Wu (in full generality).

The second lecture will be devoted to infinity-categories (modeled using quasi-categories) and more specifically to exact infinity-categories in the sense of Barwick (2015 and 2016). The link to extriangulated categories is given by Nakaoka-Palu's theorem (04/2020) stating that the homotopy category of an exact infinity-category carries a canonical extriangulated structure. Such extriangulated categories are called *topological* extriangulated categories.

The third lecture is motivated by the search for a suitable notion of *algebraic* extriangulated category, i.e. a class of differential graded (=dg) k-categories whose H^0 carries a natural extriangulated structure in analogy of with that of topological extriangulated categories. We will present the solution proposed by Xiaofa Chen in his ongoing Ph. D. thesis. It seems very likely that Lurie's dg nerve functor transforms an exact dg category in the sense of Chen into an exact infinity-category in the sense of Barwick and that the exact infinity-categories obtained in this way are precisely those admitting a k-linear structure.

2021-12-17 Lecture 1

On extriangulated categories, exact ∞-categories and exact dg categories

Plan: 0. Reminder on exact categories (Quillen 1972, Nobuo Yoneda 1960)

1. Extriangulated categories (Nakaoka-Palu 2019)

2. Exact ∞-categories (Barwick 2015 & 2016)

3. Exact dg categories (Xiaofa Chen 2021)

0. Reminder on exact categories

$\mathcal{C}$ a category.

"$\mathcal{C}$ additive": property or extra structure? Property!

$\mathcal{C}$ exact: additive with conflations satisfying $\mathrm{Ex}0 - (\mathrm{Ex}2)^{\mathrm{op}}$

Examples: extension closed subcategories of abelian categories, e.g. module categories.

Gabriel–Quillen embedding theorem: $\mathcal{C}$ small exact $\Rightarrow$

$\mathcal{C} \hookrightarrow \mathcal{A}$ fully faithful and fully exact into $\mathcal{A}$ abelian

Proof: $\mathcal{C} \xrightarrow[\text{Yoneda}]{} \mathcal{A} = \mathrm{Lex}(\mathcal{C}) \subset \mathrm{Mod}\,\mathcal{C}$,

$$\begin{tikzcd} \mathrm{Mod}\,\mathcal{C} \arrow[r] & \mathrm{Mod}\,\mathcal{C}/\mathrm{Eff}(\mathcal{C}) \\ \mathrm{Lex}(\mathcal{C}) \arrow[u, hook] \arrow[ur, "\sim"] & \end{tikzcd}$$

1. Extriangulated categories (Nakaoka–Palu, 2019)

1.1 Motivations

Extriangulated categories are a common generalization of exact categories (Quillen) and triangulated categories (Puppe, Grothendieck–Verdier). They allow to

1) Obtain unified proofs for theorems valid (with minor variations) in both, exact categories and triangulated categories (e.g. uniform treatment of AR-theory due to Iyama-Nakaoka-Palu 05/18).

2) Axiomatize the class of extension-closed subcategories of triangulated categories (in the same way that exact categories axiomatize the class of extension-closed subcat. of abelian categories).

3) Obtain the proper framework for the categorification of cluster alg. with coefficients generalizing Geiss-Leclerc-Schröer's approach to the categorification of cluster algebras.

1.2 Sketch of definition

An extriangulated category is the data of

- an additive category $\mathcal{C}$
- an biadditive bifunctor $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to Ab$
  (think: $\mathbb{E} = Ext^1$)
- an additive "realization map" s sending $\delta \in \mathbb{E}(Z,X)$ to
  an equivalence class of diagrams

$$s(\delta) : X \xrightarrow{i} Y \xrightarrow{p} Z$$

modulo the relation $(i,p) \sim (i',p')$ if $\exists$

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, equal] & Y \arrow[r, "p"] \arrow[d, "\wr"] & Z \arrow[d, equal] \\ X \arrow[r, "i'"] & Y' \arrow[r, "p'"] & Z \end{tikzcd}$$

satisfying certain axioms.

Essential new idea: In the def. of "exact cat." or "triang. cat.", we are given a class of sequences (i,p) whereas here we are given a class of sequences parametrized by a bifunctor.

1.3 Formal definition

Let $\mathcal{C}$ be an additive category and $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Ab$ a biadditive bifunctor.

Term.: Call the elements $\delta \in \mathbb{E}(Z,X)$ extensions of Z by X.

Examples: 1) If $\mathcal{C}$ is exact, take $\mathbb{E} = Ext^1_{\mathcal{E}}(?,-)$. Then each $\delta \in Ext^1_{\mathcal{C}}(Z,X)$ corresponds to an extension (up to equiv.)

$$s(\delta) : X \overset{i}{\rightarrowtail} Y \overset{p}{\twoheadrightarrow} Z.$$

A morphism $f : X \to X'$ yields

$$f_* = \mathrm{Ext}^1(Z,f) : \mathrm{Ext}^1(X,Z) \longrightarrow \mathrm{Ext}^1(X',Z),\ \delta \mapsto f_*\delta$$

and a morphism $g : Z' \to Z$ yields

$$g^* = \mathrm{Ext}^1(g,X) : \mathrm{Ext}^1(X,Z) \longrightarrow \mathrm{Ext}^1(Z',Z),\ \delta \mapsto g^*\delta.$$

These correspond to

$$\begin{tikzcd}
s(g^*\delta): & X \arrow[r] \arrow[d, equal] & Y'' \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & Z' \arrow[d, "g"] \\
s(\delta): & X \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "PO"] & Y \arrow[r, "p"] \arrow[d] & Z \arrow[d, equal] \\
s(f_*\delta): & X' \arrow[r] & Y' \arrow[r] & Z
\end{tikzcd}$$

2) If $\mathcal{C}$ is triangulated, take $\mathbb{E} = \mathrm{Hom}_{\mathcal{C}}(?, \Sigma -)$.

Then $f_* = \mathrm{Hom}(Z, \Sigma f)$ and $g^* = \mathrm{Hom}(g, \Sigma X)$ and we have similar diagrams.

Def.: A *morphism of extensions* from $\delta \in \mathbb{E}(Z,X)$ to $\delta' \in \mathbb{E}(Z',X')$ is a pair

$$(u, w) \in \mathcal{C}(X,X') \times \mathcal{C}(Z,Z')$$

s.th. $u_* \delta = w^* \delta'$. (*)

Examples: 1) $\mathcal{C}$ exact: (*) means (exercise!) that there is a diagram

$$\begin{tikzcd}
\delta: X \arrow[r, tail] \arrow[d, "u"'] \arrow[dr, phantom, "\circleddash"] & Y \arrow[r, two heads] \arrow[d, dashed, "\exists v"] \arrow[dr, phantom, "\circleddash"] & Z \arrow[d, "w"] \\
\delta': X' \arrow[r, tail] & Y' \arrow[r, two heads] & Z'
\end{tikzcd}$$

2) $\mathcal{C}$ triang.: $(*)$ means

$$\begin{tikzcd} Z \arrow[r, "\delta"] \arrow[d, "w"'] \arrow[dr, phantom, "\ominus"] & \Sigma X \arrow[d, "\Sigma u"] \\ Z' \arrow[r, "\delta'"'] & \Sigma X' \end{tikzcd} .$$

2021-12-24 Lecture 2

Def: A *realization* for $\mathbb{E}$ is a family s of maps

$$s_{Z,X} : \mathbb{E}(Z,X) \longrightarrow \{\text{seq. } X \xrightarrow{i} Y \xrightarrow{p} Z\}/\sim$$

The sequences $X \xrightarrow{i} Y \xrightarrow{p} Z$ in the image of s are called *extriangles* or *conflations*, the morphisms i *inflations* and the morphisms p *deflations*.

Write $X \xrightarrow{i} Y \xrightarrow{p} Z \overset{\delta}{\dashrightarrow}$ to indicate that $(i,p) = s(\delta)$.

Def.: A realization s is *additive* if

(1) for each morphism of extensions $(u,w):\delta\to\delta'$

(i.e. $u_*\delta = w^*\delta'$), there is a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\circleddash"] & Y \arrow[r] \arrow[d, dashed, "\exists v"] \arrow[dr, phantom, "\circleddash"] & Z \arrow[r, dashed, "\delta"'] \arrow[d, "w"] & {} \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r, dashed, "\delta'"'] & {} \end{tikzcd}.$$

Then (u,v,w) is called a morphism of extriangles.

(2) $\delta = 0 \in \mathbb{E}(Z,X)$ yields a split extriangle

$$X \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} X\oplus Z \xrightarrow{[0\ 1]} Z \overset{\delta=0}{\dashrightarrow}.$$

(3) For $\delta\in\mathbb{E}(Z,X)$ and $\delta'\in\mathbb{E}(Z',X')$, the realization

of $\delta\oplus\delta' \in \mathbb{E}(Z\oplus Z', X\oplus X') \simeq \underset{\delta}{\mathbb{E}(Z,X)}\oplus\underset{0}{\mathbb{E}(Z,X')}\oplus\underset{0}{\mathbb{E}(Z',X)}\oplus\underset{\delta'}{\mathbb{E}(Z',X')}$

is the direct sum of the extriangles

$$X \longrightarrow Y \longrightarrow Z \overset{\delta}{\dashrightarrow}$$

and $$X' \longrightarrow Y' \longrightarrow Z' \overset{\delta'}{\dashrightarrow} .$$

Def. (Nakaoka–Palu 2019): An *extriangulated category* is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, where

(ET0) $\mathcal{C}$ is an additive category

(ET1) $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow Ab$ is a biadditive bifunctor

(ET2) $\mathfrak{s}$ is an additive realization

(ET3) Given extriangles $X \xrightarrow{i} Y \xrightarrow{p} Z \overset{\delta}{\dashrightarrow}$

and $X' \xrightarrow[i']{} Y' \xrightarrow[p']{} Z' \overset{\delta'}{\dashrightarrow}$

and a com. square
$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "i'"'] & Y' \end{tikzcd}$$

there is a morphism $(u, w) : \delta \longrightarrow \delta'$ s.th.
$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[d, "w"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \end{tikzcd}$$

commutes.

$(ET3)^{op}$ Given extriangles as in $(ET3)$ and a com. square
$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[d, "v"'] & Z \arrow[d, "w"] \\ Y' \arrow[r, "p'"'] & Z' \end{tikzcd}$$

there is a morphism $(u,w) : \delta \to \delta'$ s.th.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & Z \arrow[d, "w"] \\
X' \arrow[r] & Y' \arrow[r] & Z'
\end{tikzcd}$$

commutes.

(ET4) Given the black extriangles

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, equal] & Y \arrow[r, "p"] \arrow[d, "j"] & Z' \arrow[r, dashed, "\delta"] \arrow[d, "k"] & {} \\
X \arrow[r, "ji"'] & T \arrow[r, "t"'] \arrow[d, "q"] & Y' \arrow[r, dashed, "\varepsilon"'] \arrow[d, "r"] & \Sigma X \arrow[d, dashed, "\Sigma i"] \\
 & U \arrow[r, equal] \arrow[d, dashed, "\delta'"'] & U \arrow[r, dashed, "\delta'"] \arrow[d, dashed, "p_* \delta'"] & \Sigma Y \\
 & {} & {} &
\end{tikzcd}$$

we can complete the diagram so that

- $(1\!\!1_X, j, k)$ and $(p, t, 1\!\!1_U)$ are morphisms of extriangles
- $(r, i) : \varepsilon \to \delta'$ is a morphism of extensions.

$(ET4)^{op}$

Examples: 1) $\mathcal{E}$ exact $\rightsquigarrow$ $\mathcal{C} = \mathcal{E}$ with $\mathbb{E} = \mathrm{Ext}^1_{\mathcal{E}}$ and the classical realization.

2) $\mathcal{T}$ triangulated $\rightsquigarrow$ $\mathcal{C} = \mathcal{T}$ with $\mathbb{E} = \mathrm{Hom}_{\mathcal{T}}(?, \Sigma -)$ and the classical realization.

3) $\mathcal{T}$ triangulated, $\mathcal{C} \subseteq \mathcal{T}$ a full extension closed subcategory, $\mathbb{E} = \mathrm{Hom}_{\mathcal{T}}(?, \Sigma -)$,

$\mathfrak{s}$ induced by the realization of $\mathcal{T}$.

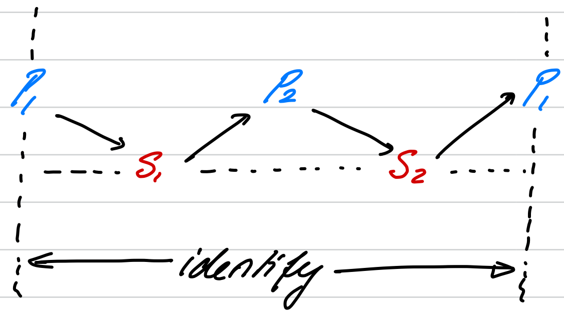
Prop. [NP]: Let $\mathcal{E}$ be an exact category and $\mathcal{P} \subseteq \mathcal{E}$ a full subcategory whose objects are projective-injective. Endow $\mathcal{C} = \mathcal{E}/(\mathcal{P})$ with the bifunctor $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to \mathrm{Ab}$ induced by $\mathrm{Ext}^1_{\mathcal{E}}(?,-)$ and with the realization $\mathfrak{s}$ induced by that of $\mathcal{E}$. Then $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is extriangulated.

Special case: Suppose $\mathcal{E}$ is Frobenius exact and $\mathcal{P}$ is the subcategory of *all* projective-injectives. Then we recover Happel's Thm.

Example: Let Λ be the preprojective algebra of type A_2:

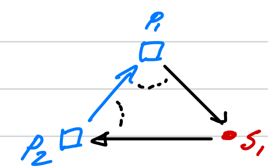
$$\Lambda : \; 1 \underset{\bar{a}}{\overset{a}{\rightleftarrows}} 2, \qquad a\bar{a} = 0, \; \bar{a}a = 0.$$

Then Λ is selfinjective and $\mathcal{E} = \mathrm{mod}\,\Lambda$ is a Frobenius abelian category which is stably 2-CY. The AR-quiver of Λ is



$\mathcal{E}$ contains two basic cluster-tilting objects $T = P_1 \oplus P_2 \oplus S_1$ and $T' = P_1 \oplus P_2 \oplus S_2$.

The algebra $\mathrm{End}(T)$ is given by the ice quiver with relations



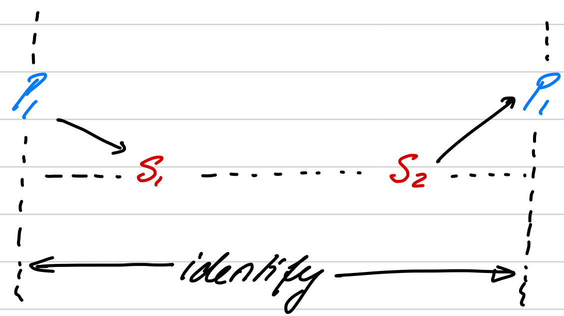
Thus, $\mathcal{E}$ with T yields an additive categorification of this ice quiver.

The stable category $\underline{\mathcal{E}}$ only has 2 indecomposables: S_1 and $S_2 = \tau S_1 = \Sigma S_1$.

It is triangle equivalent to the cluster category $\mathcal{C}_{A_1}$ of type A_1. It is 2-CY.

It yields an additive categorification of the cluster algebra without coefficients associated with the A_1-quiver (which is just $\mathbb{Q}[x_1^{\pm 1}]$).

Now consider the quotient $\mathcal{C} = \mathcal{E}/(P_2)$. By the above Prop., it inherits an extriangulated structure from $\mathcal{E}$. Its quiver is



It also has a well-defined AR-theory (Iyama-Nakaoka-Palu). In particular, we have the AR-extriangle

$$S_2 \longrightarrow P_1 \longrightarrow S_1 \dashrightarrow$$

(which looks like a conflation) and the AR-extriangle

$$S_1 \longrightarrow 0 \longrightarrow S_2 \dashrightarrow$$

(which looks like a triangle). The category $\mathcal{C} = \mathcal{E}/(P_2)$ is in fact Frobenius extriangulated and stably 2-CY with stable category $\underline{\mathcal{C}} = \underline{\mathcal{E}} \cong \mathcal{C}_{A_1}$. It contains the basic cluster tilting objects $T = P_1 \oplus S_1$ and $T' = S_2 \oplus P_1$.
The quiver of $\mathrm{End}(T)$ is the ice quiver $\underset{P_1}{\square} \longrightarrow \underset{S_1}{\bullet}$. The category $\mathcal{C}$ with T yield a Frobenius extriangulated categorification of the corresponding cluster algebra

with coefficients. It is clear that this ice quiver does not admit a Frobenius exact categorification. This example has been generalized (a lot) in Yilin Wu's thesis.

Rk: More general notions than "extriangulated category" are due to:

- Baillargeon-Brüstle-Gorsky-Hassoun (09/20): weakly exact resp. extr. categories
- Herschend-Liu-Nakaoka (09/17): n-exangulated categories

1.4 Extriangulated functors

Recall: If $\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories a triangle functor $\mathcal{T} \to \mathcal{T}'$ is a pair (F, φ) where $F: \mathcal{T} \to \mathcal{T}'$ is additive and $\varphi: F\Sigma \xrightarrow{\sim} \Sigma F$ is an isomorphism s.th. for each triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ of $\mathcal{T}$, the sequence

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{(\varphi X)\circ(Fw)} \Sigma FX$$

with $FZ \xrightarrow{Fw} F\Sigma X \xrightarrow{\varphi X} \Sigma FX$

is a triangle of $\mathcal{T}'$.

Def. [Bennett-Tennenhaus – Shah 03/20]: Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{D}, \mathbb{F}, \mathfrak{t})$ be extriang. cat.

An extriangulated functor from $\mathcal{C}$ to $\mathcal{D}$ is given by

- an additive functor $F : \mathcal{C} \longrightarrow \mathcal{D}$

- a natural transformation $\Gamma : \mathbb{E}(?,-) \longrightarrow \mathbb{F}(F?, F-)$

such that for each extriangle $A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C \overset{\delta}{\dashrightarrow}$ of $\mathcal{C}$, we have the extriangle $FA \overset{Fi}{\rightarrowtail} FB \overset{Fp}{\twoheadrightarrow} FC \overset{\Gamma_{C,A}(\delta)}{\dashrightarrow}$ in $\mathcal{D}$.

Def. [Haugland 12/19]: If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is fully faithful and $\Gamma : \mathbb{E}(Z,X) \longrightarrow \mathbb{F}(FZ, FX)$ is injective, then the essential image of F endowed with the extriang. str. inherited from $\mathcal{C}$ is called an extriangulated subcategory of $\mathcal{D}$.

1.5 Basic results

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated.

Prop. [NP]: Given an extriangle $A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$ there are exact sequences

$$\mathcal{C}(?,A) \longrightarrow \mathcal{C}(?,B) \longrightarrow \mathcal{C}(?,C) \xrightarrow{\delta_\#} \mathbb{E}(?,A) \longrightarrow \mathbb{E}(?,B) \longrightarrow \mathbb{E}(?,C)$$

$$\mathcal{C}(C,?) \longrightarrow \mathcal{C}(B,?) \longrightarrow \mathcal{C}(A,?) \xrightarrow{\delta^\#} \mathbb{E}(C,?) \longrightarrow \mathbb{E}(B,?) \longrightarrow \mathbb{E}(A,?),$$

where for $g : C' \longrightarrow C$, we put $\delta_\# g = g^* \delta$,

and for $f : A \longrightarrow A'$, we put $\delta^\# f = f_* \delta$.

Cor.: a) For an extriangle $A \overset{i}{\rightarrowtail} B \overset{p}{\longrightarrow} C \overset{\delta}{\dashrightarrow}$, we have

i is a section $\Longleftrightarrow \delta = 0 \Longleftrightarrow p$ is a retraction.

b) In a morphism of extriangles

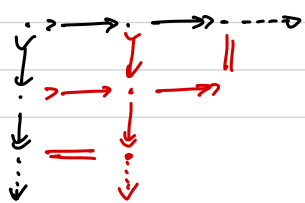
$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "u"'] & B \arrow[r, two heads, "p"] \arrow[d, "v"] \arrow[dl, dashed, bend right] & C \arrow[r, dashed, "\delta"] \arrow[d, "w"] \arrow[dl, dashed, bend right] & {} \\ A' \arrow[r, tail, "i'"'] & B' \arrow[r, two heads, "p'"'] & C' \arrow[r, dashed, "\delta'"] & {} \end{tikzcd}$$

we have

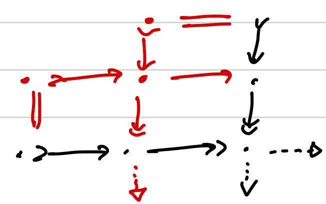
$$\underline{u \text{ extends along } i} \iff u_*\delta = 0 \iff w^*\delta' = 0 \iff \underline{w \text{ lifts along } p'}.$$

c) In a morphism of extriangles (u, v, w), if any two among u, v, w are invertible, so is the third.

Rk: The axioms imply "shifted versions" of ET4, where we start from the black part in



resp.



Cor.: If $X \xrightarrow{f} Y$ is a morphism and $X \rightarrowtail^{i} Y'$ an inflation, then $X \xrightarrow{\begin{bmatrix} f \\ i \end{bmatrix}} Y \oplus Y'$ is an inflation.

Prop. [NP]: a) An extriangulated category is exact (with the same conflations) iff each inflation is mono and each deflation is epi.

b) Let $\mathcal{C}$ be additive and $\Sigma : \mathcal{C} \to \mathcal{C}$ an autoequivalence. Put $\mathbb{E} = \operatorname{Hom}(?, \Sigma -)$. Suppose $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is extriangulated. Define the triangles $X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{\delta} \Sigma X$ to correspond to the extriangles $X \xrightarrow{i} Y \xrightarrow{p} Z \overset{\delta}{\dashrightarrow}$. Then $\mathcal{C}$ becomes triangulated and $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is the extriang. cat. assoc. with this triangulated category.

2022-01-06 Lecture 3

1.6 Some constructions of extriangulated categories

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be extriangulated.

Def: An object $P \in \mathcal{C}$ is *projective* if

$$\begin{tikzcd} & & P \arrow[d, "\forall"] \arrow[dll, dashed, "\exists"'] \\ \forall\ Y \arrow[rr, two heads] & & Z \end{tikzcd}$$

An object $I \in \mathcal{C}$ is *injective* if

$$\begin{tikzcd} \forall\ X \arrow[rr, tail] \arrow[d, "\forall"'] & & Y \arrow[dll, dashed, "\exists"] \\ I & & \end{tikzcd}$$

Quotients

Prop. [NP]: Let $\mathcal{P} \subseteq \mathcal{C}$ be a full subcategory formed by projective-injectives.

Then $\mathcal{C}/(\mathcal{P})$ inherits an extriangulated structure from $\mathcal{C}$.

Rk: If $\mathcal{C}$ is Frobenius exact and $\mathcal{P}$ the subcategory of <u>all</u> projective-injectives,

we recover Heller-Happel's thm.

Subcategories

Prop. [NP]: Let $\mathcal{D} \subseteq \mathcal{C}$ be a full extension-closed subcategory (i.e. if

$X \rightarrowtail Y \twoheadrightarrow Z \dashrightarrow$ is an extriangle and $X, Z \in \mathcal{D}$, then $Y \in \mathcal{D}$).

Then $(\mathcal{D}, \mathbb{E}|_{\mathcal{D}^{op} \times \mathcal{D}}, \mathfrak{s}|)$ is extriangulated.

Subbifunctors

Prop. [Herschend-Liu-Nakaoka, 0917 v3]: Let $\mathbb{F} \subseteq \mathbb{E}$ be a biadditive subbifunctor.

If the class of $\mathbb{F}$-inflations (resp. $\mathbb{F}$-deflations) is stable under

composition, then $(\mathcal{C}, \mathbb{F}, \mathfrak{s}|_{\mathbb{F}})$ is extriangulated.

2. Exact ∞-categories

2.1 ∞-categories

Rk: The term "∞-category" is short for "$(\infty,1)$-category", i.e. a "lax ∞-category" where all n-morphisms for $n \geq 2$ are equivalences.

Q: What is an ∞-category?

A: There are many (Quillen equivalent) answers, e.g.

1) a category $\mathcal{C}$ enriched in the category Top of topological spaces (i.e. $\mathcal{C}(X,Y) \in \mathrm{Top}$, $\forall X,Y$ and composition and unit maps are continuous).

2) a category $\mathcal{C}$ enriched in the category sSet of simplicial sets (cf. below for sSet)

3) a relative category, i.e. a category $\mathcal{C}$ endowed with a class S of morphisms containing the identities and stable under composition (idea: refine $\mathcal{C}[S^{-1}]$)

4) a quasicategory (Joyal/Lurie)

⋮

There is now consensus that 4) is technically most advantageous and Lurie's work (several thousand pages developing "mathematics up to homotopy", e.g. derived alg. geometry) is (mostly) written in the framework of 4).

Simplicial sets and quasicategories

Ref.: Kerodon.net

the category of small categories

Def.: The simplex category Δ is the full subcategory of cat whose objects are the posets (viewed as categories)

$$[n] = \{0 < 1 < \dots < n\}, \quad n \in \mathbb{N}.$$

A simplicial set is a functor $M : \Delta^{op} \to \mathrm{Set}$. We write $M_n := M[n]$ and call it the set of n-simplices of M.

Denote by sSet the category of simplicial sets (think: "right Δ-moduli")

Example: If $\mathcal{A}$ is a small category its nerve is the simplicial set

$$N_\bullet \mathcal{A} : [n] \mapsto N_n = \mathrm{Fun}([n], \mathcal{A})$$

$$= \{\text{diagrams } A_0 \to \dots \to A_n \text{ in } \mathcal{A}\}$$

In particular, $N_0\mathcal{A} = \mathrm{obj}(\mathcal{A})$ and $N_1\mathcal{A} = \{\text{morphisms } A_0 \to A_1 \text{ of } \mathcal{A}\}$ and we can recover composition of $\mathcal{A}$ using the three "face maps"

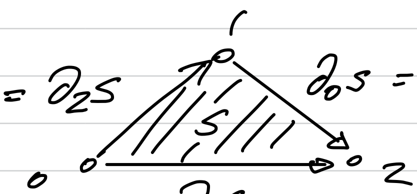
$$
\begin{array}{ccccc}
 & \overset{\partial_2}{\nearrow} & N_1\mathcal{A} & \ni & A_0 \xrightarrow{f} A_1 \\
s = (A_0 \xrightarrow{f} A_1 \xrightarrow{g} A_2) \in N_2\mathcal{A} & \xrightarrow[\partial_1]{} & N_1\mathcal{A} & \ni & A_0 \xrightarrow{g\circ f} A_2 \\
 & \underset{\partial_0}{\searrow} & N_1\mathcal{A} & \ni & A_1 \xrightarrow{g} A_2
\end{array}
$$

which are images under $N.$ of the three functors

$$\{0 \to 1\} \rightrightarrows\!\!\!\to \{0 \to 1 \to 2\}$$

with images $\{0,1\}$, $\{0,2\}$ and $\{1,2\}$.

Idea: $s = (A_0 \to A_1 \to A_2) \implies (A_0 \to A_1) = \partial_2 s$ $\quad$ $\partial_0 s = (A_1 \to A_2)$



$\partial_1 s$
$= (A_0 \to A_2)$

We deduce: $N_\circ : \mathrm{cat} \longrightarrow \mathrm{sSet}$ is fully faithful.

Idea: Define quasicategories as simplicial sets which are "a bit more general" than nerves of categories.

Def.: For $n \in \mathbb{N}$, the **standard n-simplex** is the representable functor

$$\Delta^n = \mathrm{Hom}_\Delta(?, [n]) : \Delta^{op} \longrightarrow \mathrm{Set}$$

Rk: Thus, by the Yoneda lemma, for each simplicial set X, we have $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \xrightarrow{\sim} X_n$.

Def.: The **boundary $\partial\Delta^n \subseteq \Delta^n$** is the simplicial subset given by

$$(\partial\Delta^n)_m = \{ f : [m] \longrightarrow [n] \mid f \text{ not surjective} \}$$

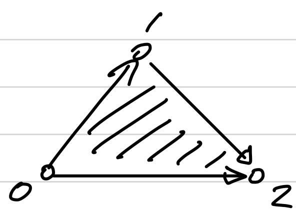
For $0 \le i \le n$, the **horn** $\Lambda^n_i \subseteq \partial\Delta^n$ is the simplicial subset given by

$$(\Lambda^n_i)_m = \{ f:[m]\to[n] \mid \{i\}\cup \operatorname{Im} f \neq [n] \}$$

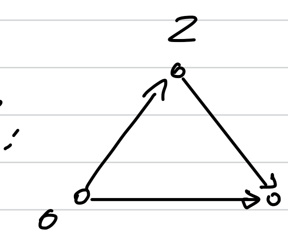
The horn Λ^n_i is **inner** if $0<i<n$.

Example: $n=2$:

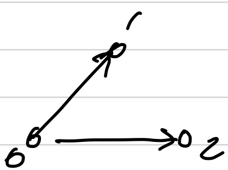
Δ^2:



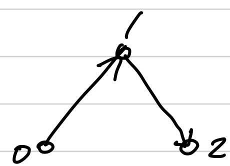
boundary: $\partial\Delta^2$:



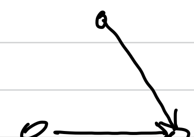
Λ^0_2:



, Λ^1_2:



, Λ^2_2:



not inner

inner horn

not inner.

Lemma: $X \in s\mathrm{Set}$ is the nerve of a category iff "each inner horn of X admits a <u>unique</u> filling", i.e.

$$\forall\, 0<i<n: \quad \begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

Def.: A *quasicategory* is a simplicial set where each inner horn admits a (not necessarily unique!) filling.

Rk: The definition goes back to Boardman-Vogt (1973) who call such simplicial sets "weak Kan complexes". Joyal (2001) calls them "quasicategories".

Let C be a quasicategory.

Terminology: The **objects** of C are its 0-simplices: C_0

The **1-morphisms** of C are its 1-simplices: C_1

Rk: If $g \in C_1 = \mathrm{Hom}(\Delta^1, C)$ is a morphism, its **source** is defined as $X_0 = \partial_1 g \in C_0$ and its **target** as $X_1 = \partial_0 g \in C_0$.

If we have two 1-morphisms $X_0 \xrightarrow{g} X_1 \xrightarrow{f} X_2$,

they define an inner horn $\Lambda^2_1 \xrightarrow[(f,g)]{} C$ and each filling $\Delta^2 \xrightarrow[h]{} C$ yields a (non-unique!) "composition" $X_0 \xrightarrow[\partial_1 h]{} X_2$.

$= \text{"} g \circ f \text{"}$

$$
\begin{tikzcd}
 & X_1 \arrow[dr, "f"] \arrow[d, phantom, "h"] & \\
X_0 \arrow[ur, "g"] \arrow[rr, dashed, "\text{"}g\circ f\text{"}"'] & {} & X_2
\end{tikzcd}
$$

Def.: The **homotopy category** $h\mathcal{C}$ is the (ordinary) category with

- object set $\mathrm{obj}(h\mathcal{C}) = \mathcal{C}_0$
- $(\text{morphisms } X_0 \to X_1) = \dfrac{(\text{1-morphisms } X_0 \to X_1)}{\text{suitable htpy relation}}$
- composition defined by filling inner horns as above.

Example: If $\mathcal{C} = N_\bullet \mathcal{A}$, then $h\mathcal{C} = \mathcal{A}$.

Rks: 1) For dg categories, the analog of $\mathcal{C} \mapsto h\mathcal{C}$ is $\mathcal{A} \mapsto H^0\mathcal{A}$.
But there is an important difference: all 1-morphisms of $\mathcal{C}$ yield morphisms in $h\mathcal{C}$ but only 0-cycles $f \in Z^0\mathcal{A}(X_0, X_1)$ yield morphisms in $H^0\mathcal{A}$. A better analogy is obtained if

We only consider (strictly) connective dg categories, i.e. $\mathcal{A}(X,Y)^p = 0$ for all $p > 0$. Then each $f \in \mathcal{A}(X,Y)^0$ yields a morphism in $H^0\mathcal{A}$.

2) Joyal (and later Lurie) have extended classical category theory to quasicategories. In particular, we have notions of htpy initial object, htpy cartesian/cocartesian/bicartesian squares ... which specialize to the classical notions when applied to nerves of categories. In the sequel, we freely use these notions.

3) To each dg category $\mathcal{A}$, Lurie assigns a quasicategory $N^{dg}_{\cdot}\mathcal{A}$ called its dg nerve (cf. section 2.5 of the Kerodon) and such that the homotopy category $h\, N^{dg}_{\cdot}\mathcal{A}$ is canonically equivalent to the nerve of $H^0\mathcal{A}$.

4) Let $\mathcal{A}$ be a (strictly) pretriangulated dg category. Let

$$\begin{tikzcd} X_0 \arrow[r, "b"] \arrow[d, "e"'] \arrow[dr, dashed, "h"] & X_1 \arrow[d, "a"] \\ X_2 \arrow[r, "c"'] & X_3 \end{tikzcd} \qquad (*)$$

be a homotopy commutative square of $\mathcal{A}$ (i.e. the $\rightarrow$ are morphisms of $Z^0\mathcal{A}$ and $ab - ce = d(h)$). Then $(*)$ is a htpy bicartesian square of $N^{dg}_{\bullet}\mathcal{A}$ iff the canonical morphism

$$\operatorname{cone}(X_0 \longrightarrow X_1 \oplus X_2) \longrightarrow X_3$$

is invertible in $H^0\mathcal{A}$, i.e. iff the square becomes htpy bicartesian in the triangulated category $H^0\mathcal{A}$.

5) Let $\mathcal{A}$ be a pretriangulated dg category and $\mathcal{B} \subseteq \mathcal{A}$ a pretriangulated full dg subcategory. So $H^0\mathcal{B} \subseteq H^0\mathcal{A}$ is a full triang. subcategory. Let $Y \in H^0\mathcal{A}$.

The comma category $H^0\mathcal{B} \setminus Y$ has obj

- objects: $f: X \to Y$, f morph. of $H^0\mathcal{A}$, $X \in H^0\mathcal{B}$.
- morphisms:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ominus"] & Y \\ X' \arrow[ur, "f'"'] & \end{tikzcd}$$

com. in $H^0\mathcal{A}$.

The comma category is important for studying the existence of a right adj. to $H^0\mathcal{B} \hookrightarrow H^0\mathcal{A}$.

The comma category is not of the form $H^0\mathcal{C}$ for a dg category $\mathcal{C}$.

But it is of the form $h\mathcal{C}$ for an ∞-category $\mathcal{C}$, namely the comma ∞-category $\mathcal{C} = N^{dg}_{\bullet}\mathcal{B} \setminus X$ in $N^{dg}_{\bullet}\mathcal{A}$.

2.2 Exact ∞-categories

Def. (Lurie): An ∞-cat. (= quasicat.) $\mathcal{A}$ is **additive** if

(1) it admits a (htpy) zero object $0 \in \mathcal{A}$

(2) it admits all finite (htpy) products and coproducts

(3) its htpy category $h\mathcal{A}$ is additive.

Rk: "Being additive" is a property of an ∞-category

Examples: 1) $\mathcal{A} = N.\mathcal{B}$, $\mathcal{B}$ an ordinary add. cat.

2) $\mathcal{A} \subseteq \mathcal{B}$ a full ∞-subcat. stable under finite (htpy) coproducts in an add. ∞-cat. $\mathcal{B}$.

Def. (Barwick 2012, reformulated by G. Jasso): Let $\mathcal{E}$ be an additive ∞-cat. An **exact structure** on $\mathcal{E}$ is a class $\mathcal{S}$ of (htpy) bicartesian squares

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \arrow[d, "p"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

in $\mathcal{E}$ ($\mathcal{S}$ yields the class of **inflations** $\overset{i}{\rightarrowtail}$ and **deflations** $\overset{p}{\twoheadrightarrow}$) which is closed under isomorphisms in $h\,\mathrm{Fun}_\infty(\square, \mathcal{E})$ and s.th.

(E0): 1_0 is a deflation and an inflation.

(E1): inflations are closed under compositions and so are deflations

(E2): $\forall\ X \overset{i}{\rightarrowtail} Y$ and $\forall\ f: X \to X'$, there is a htpy pushout square

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ X' \arrow[r, "i'"] & Y' \end{tikzcd}$$

and i' is an inflation

(EZ^{op}) $\forall\ Y \xrightarrow{p} Z$ and $\forall\ Z' \xrightarrow{g} Z$, there is a htpy pullback square

$$\begin{tikzcd} Y' \arrow[r, "p'"] \arrow[d, "g'"'] & Z' \arrow[d, "g"] \\ Y \arrow[r, two heads, "p"'] & Z \end{tikzcd}$$

and p' is a deflation.

The pair $(\mathcal{E}, \mathcal{S})$ is called an exact ∞-category.

Examples: 1) $\mathcal{E}$ the nerve of of a Quillen exact 1-category with the class $\mathcal{S}$ given by its conflations.

2) $\mathcal{E}$ an additive ∞-category, $\mathcal{S}$ all htpy bicartesian squares

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \arrow[d, "p"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

s.th. $0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ is split short exact in $h\mathcal{E}$.

3) $\mathcal{E}$ a stable ∞-category (the ∞-version of a pretriang. dg category, in particular, $h\mathcal{E}$ is canonically triangulated) and $\mathcal{S}$ the class of

all htpy bicartesian squares $\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$ in $\mathcal{E}$.

Lemma: Let $(\mathcal{E}, \mathcal{S})$ be an exact ∞-cat. and $\mathcal{X} \subseteq \mathcal{E}$ extension closed. Then $(\mathcal{X}, \mathcal{S}|_{\mathcal{X}})$ is an exact ∞-cat.

Example: If $\mathcal{E}$ is a stable ∞-cat. (with all htpy bicart. $\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$), and $\overline{\mathcal{U}} \subseteq h\mathcal{E}$ is an aisle/co-aisle of a t-structure or co-t-structure, then $\mathcal{U} \subseteq \mathcal{E}$, the full ∞-subcat. with objects $\mathrm{obj}(\overline{\mathcal{U}})$, becomes an exact ∞-category.

Thm [NP]: Suppose $(\mathcal{E}, \mathcal{S})$ is an exact ∞-category. Then $h\mathcal{E}$ carries a canonical extriangulated structure.
04/20

Rks: 1) Extriangulated categories of the form $h\mathcal{E}$ are called topological extr. cat.

2) Nakaoka-Palu give a direct (but involved) proof. Alternatively, one can use the following analog of the Gabriel-Quillen embedding theorem:

diploma thesis under the supervision of Gustavo Jasso

Thm (Klemenc, 10/20): For each exact ∞-cat. $(\mathcal{E}, \mathcal{S})$, there is a (universal) exact functor $F: \mathcal{E} \longrightarrow \mathcal{T}$, where $\mathcal{T}$ is a stable ∞-category, which induces an equivalence of exact ∞-cat. onto an extension-closed subcat. of $\mathcal{T}$ endowed with the exact structure inherited from $\mathcal{T}$.

Rks: 1) Thus, all topological extriang. cat. are extension closed subcat. of triang. cat.

2) If $\mathcal{E}$ is the nerve of a Quillen exact cat. $\mathcal{A}$, then $\mathcal{T}$ is the dg nerve of the dg-derived category $\mathcal{D}^b_{dg}(\mathcal{E})$.

Open problem (under investigation by Xiaofa Chen): What is an algebraic extri. cat.?

In other words: What is an exact dg category?