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The finite EI categories of Cartan type

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ABSTRACT

To each symmetrizable Cartan matrix, we associate a finite free EI category. We prove that the corresponding category algebra is isomorphic to the algebra defined in Geiss et al. (2017) [4], which is associated to another symmetrizable Cartan matrix. In certain cases, the algebra isomorphism provides an algebraic enrichment of the well-known correspondence between symmetrizable Cartan matrices and graphs with automorphisms.

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1. Introduction

Recall that a finite category \mathcal{C} is EI, provided that each endomorphism is an isomorphism. Finite EI categories arise naturally in the modular representation theory of finite groups; see [9,16,10]. The notion of a free EI category is introduced in [8]. Each finite EI category is a quotient of a free EI category.

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For each symmetrizable Cartan matrix, there is a class of finite dimensional algebras [4], whose modules categorify the corresponding root system over an arbitrary field k . More precisely, let (C, D, Ω) be a Cartan triple, where C is a symmetrizable Cartan matrix, D is its symmetrizer and Ω is an acyclic orientation of C . A k -algebra $H(C, D, \Omega)$ is introduced in [4], where the rank vectors of certain $H(C, D, \Omega)$ -modules are some positive roots of the Lie algebra associated to C .

There are remarkable common features of the work [8] and [4]: the category algebra $k\mathcal{C}$ of a free EI category \mathcal{C} is a tensor algebra, and is 1-Gorenstein under mild assumptions [13]; the algebra $H(C, D, \Omega)$ is also a tensor algebra, and is always 1-Gorenstein; see also [5]; moreover, both algebras are closely related to the path algebras of finite acyclic quivers. We mention the work [7], which provides a categorical perspective for the algebra $H(C, D, \Omega)$, and the paper [2], inspired by [4], where Gorenstein tensor algebras are studied in general.

In view of these common features, the following question arises: when are the above two classes of algebras, $k\mathcal{C}$ and $H(C, D, \Omega)$, arising in different contexts isomorphic? Another reason to be interested in this question is that it might establish new links between the modular representation theory and Lie theory.

We give a partial answer to the above question. Indeed, inspired by [4], we associate to each Cartan triple (C, D, Ω) a free EI category $\mathcal{C}(C, D, \Omega)$, called a *free EI category of Cartan type*. We prove that the category algebra $k\mathcal{C}(C, D, \Omega)$ is isomorphic to the algebra $H(C', D', \Omega')$ associated to another, usually different, Cartan triple (C', D', Ω') ; see Theorem 4.3. In forthcoming work, we will apply the obtained isomorphism to study certain skew group algebras and their representations.

The construction of (C', D', Ω') from (C, D, Ω) depends on the characteristic p of the base field k ; see Subsection 4.2. The case where p is zero or coprime to each entry of D seems to be of particular interest; in this case, C' is symmetric and D' is the identity matrix. Then the algebra $H(C', D', \Omega')$ is isomorphic to the path algebra of an acyclic quiver of type C' . There is a well-known correspondence between symmetrizable Cartan matrices and graphs with automorphisms in [11, Section 14.1]. It turns out that our construction is compatible with this correspondence. More precisely, if p is zero or coprime to each entry of D , the Cartan matrix C corresponds exactly to the graph of C' with a certain automorphism by the mentioned correspondence. Roughly speaking, our algebra isomorphism between the category algebra $k\mathcal{C}(C, D, \Omega)$ of type C and the path algebra of type C' might be viewed as an algebraic enrichment of the correspondence in [11]; see Proposition 6.10. We mention that a different enrichment appears implicitly in [6, Section 6] using species and Galois extensions of the base field.

The paper is structured as follows. In Section 2, we recall basic facts on free EI categories. We recall from [4] the algebra $H(C, D, \Omega)$ in Section 3. We introduce the notion of a free EI category of Cartan type in Section 4. The construction of (C', D', Ω') from (C, D, Ω) is given in Subsection 4.2. The core in the proof of Theorem 4.3 is Proposition 5.4 in Section 5, which states an explicit decomposition of a tensor bimodule.

In Subsection 6.1, we prove that in the construction of Subsection 4.2, the valued graph of C is a disjoint union of Dynkin graphs (*resp.* Euclidean graphs, graphs of indefinite type) if and only if so is the valued graph of C' ; see Proposition 6.5. In Subsection 6.2, we compare the construction with the correspondence in [11, Section 14.1].

2. Finite free EI categories

In this section, we recall the construction of a finite free EI category from a finite EI quiver. We observe that the category algebra of the resulting EI category is isomorphic to a certain tensor algebra.

Let \mathcal{C} be a finite category, that is, a category with only finitely many morphisms. Consequently, the category \mathcal{C} contains only finitely many objects. Denote by $\text{Mor}(\mathcal{C})$ (*resp.* $\text{Obj}(\mathcal{C})$) the finite set of morphisms (*resp.* objects) in \mathcal{C} . The finite category \mathcal{C} is said to be EI provided that every endomorphism is an isomorphism. Therefore, for each object x , $\text{Hom}_{\mathcal{C}}(x, x) = \text{Aut}(x)$ is a finite group.

Let k be a field. The *category algebra* $k\mathcal{C}$ of \mathcal{C} over k is defined as follows: $k\mathcal{C} = \bigoplus_{\alpha \in \text{Mor}(\mathcal{C})} k\alpha$ as a k -vector space and the product is given by the rule

$$\alpha\beta = \begin{cases} \alpha \circ \beta, & \text{if } \alpha \text{ and } \beta \text{ can be composed in } \mathcal{C}; \\ 0, & \text{otherwise.} \end{cases}$$

The unit is given by $1_{k\mathcal{C}} = \sum_{x \in \text{Obj}(\mathcal{C})} \text{Id}_x$, where Id_x denotes the identity endomorphism of x .

Let G and H be two groups. Recall that a (G, H) -biset is a set X with a left G -action and a right H -action such that the two actions commute, that is, $(gx)h = g(xh)$ for all $g \in G, x \in X$ and $h \in H$.

Let X be a (G, H) -biset and Y be a (H, K) -biset. The *biset product* of X and Y , denoted by $X \times_H Y$, is the set $X \times Y / \sim$ of equivalence classes under the equivalence relation $(x, hy) \sim (xh, y)$ for $x \in X, h \in H$ and $y \in Y$. By abuse of notation, the elements in $X \times Y / \sim$ are still denoted by (x, y) for $x \in X$ and $y \in Y$. The set $X \times_H Y$ is naturally a (G, K) -biset; see [15, Section 8].

For a (G, H) -biset X , we denote by kX the k -vector space with a k -basis X . Then kX is naturally a kG - kH -bimodule.

The following fact is well known.

Lemma 2.1. *Let X be a (G, H) -biset and Y be a (H, K) -biset. Then there is an isomorphism of kG - kK -bimodules*

$$kX \otimes_{kH} kY \xrightarrow{\sim} k(X \times_H Y), \quad x \otimes y \mapsto (x, y),$$

for $x \in X$ and $y \in Y$. \square

Recall that a finite quiver $Q = (Q_0, Q_1; s, t)$ consists of a finite set Q_0 of vertices, a finite set Q_1 of arrows endowed with two maps $s, t: Q_1 \rightarrow Q_0$. For an arrow α , $s(\alpha)$ and $t(\alpha)$ are called its starting vertex and terminating vertex, respectively. Then the arrow α is visualized as $\alpha: s(\alpha) \rightarrow t(\alpha)$. For $n \geq 2$, a path $p = \alpha_n \cdots \alpha_2 \alpha_1$ of length n consists of arrows α_i such that $t(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i < n$; we set $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$. Here, we write the concatenation from the right to the left. We denote by Q_n the set of paths of length n . We observe that a path of length one is just an arrow. We identify each vertex i with the corresponding path e_i of length zero.

Denote by $kQ = \bigoplus_{n \geq 0} kQ_n$ the path algebra of Q , whose multiplication is given by the concatenation of paths. In particular, each path e_i of length zero is an idempotent. Then we have $1_{kQ} = \sum_{i \in Q_0} e_i$. The quiver Q is *acyclic* provided that it does not contain oriented cycles, in which case the path algebra kQ is finite dimensional.

Recall from [8, Definition 2.1] that a *finite EI quiver* (Q, X) consists of a finite acyclic quiver Q and an assignment $X = (X(i), X(\alpha))_{i \in Q_0, \alpha \in Q_1}$. More precisely, for each vertex $i \in Q_0$, $X(i)$ is a finite group, and for each arrow α , $X(\alpha)$ is a finite $(X(t(\alpha)), X(s(\alpha)))$ -biset. For a path $p = \alpha_n \cdots \alpha_2 \alpha_1$ of length n , it is natural to define

$$X(p) = X(\alpha_n) \times_{X(t(\alpha_{n-1}))} X(\alpha_{n-1}) \times_{X(t(\alpha_{n-2}))} \cdots \times_{X(t(\alpha_2))} X(\alpha_2) \times_{X(t(\alpha_1))} X(\alpha_1).$$

Then $X(p)$ is naturally a $(X(t(p)), X(s(p)))$ -biset. Indeed, for two paths p, q satisfying $s(p) = t(q)$, we have a natural isomorphism

$$X(p) \times_{X(t(q))} X(q) \xrightarrow{\sim} X(pq), \tag{2.1}$$

where pq denotes the concatenation. We identify $X(e_i)$ with $X(i)$.

Each finite EI quiver (Q, X) gives rise to a finite EI category $\mathcal{C} = \mathcal{C}(Q, X)$. The objects of \mathcal{C} coincide with the vertices of Q . For two objects x and y , we have

$$\text{Hom}_{\mathcal{C}}(x, y) = \bigsqcup_{\{p \text{ path in } Q \text{ with } s(p)=x \text{ and } t(p)=y\}} X(p).$$

The composition is induced by the concatenation of paths and the isomorphism (2.1). Since Q is acyclic, we infer that \mathcal{C} is a finite category; moreover, we have that $\text{Hom}_{\mathcal{C}}(x, x) = X(x)$ is a finite group. It follows that the category \mathcal{C} is indeed finite EI.

Following [8, Definition 2.2], we call the above category $\mathcal{C}(Q, X)$ the *free EI category* associated to the finite EI quiver (Q, X) . More generally, a finite EI category \mathcal{C} is called *free* provided that it is isomorphic to $\mathcal{C}(Q, X)$ for some finite EI quiver (Q, X) . For alternative characterizations of finite free EI categories, we refer to [8, Proposition 2.8], [13, Proposition 4.5] and [14, Proposition 2.1].

We have the following observation.

Proposition 2.2. *Let $\mathcal{C} = \mathcal{C}(Q, X)$ be the finite free EI category associated to a finite EI quiver (Q, X) . Set $A = \prod_{x \in Q_0} kX(x)$ and $V = \bigoplus_{\alpha \in Q_1} kX(\alpha)$, where V is naturally an A - A -bimodule. Then there is an isomorphism $T_A(V) \simeq k\mathcal{C}$ of algebras.*

Here, $T_A(V) = A \oplus V \oplus V^{\otimes 2} \oplus \dots \oplus V^{\otimes n} \oplus \dots$ denotes the tensor algebra, where $V^{\otimes n} = V \otimes_A \dots \otimes_A V$ is the n -fold tensor product. The tensor algebra $T_A(V)$ is naturally \mathbb{N} -graded.

Proof. We observe that $k\mathcal{C} = \bigoplus_{n \geq 0} (k\mathcal{C})_n$ is naturally \mathbb{N} -graded, where $(k\mathcal{C})_n = k(\bigsqcup_{p \in Q_n} X(p))$. Hence, we have $(k\mathcal{C})_0 = A$ and $(k\mathcal{C})_1 = V$. By the universal property of tensor algebras, there is a unique homomorphism $\phi: T_A(V) \rightarrow k\mathcal{C}$ between \mathbb{N} -graded algebras such that its restriction to $A \oplus V$ is the identity.

It suffices to prove that ϕ induces an isomorphism $V^{\otimes n} \simeq (k\mathcal{C})_n$ for each $n \geq 2$. For this, we observe $V = \bigoplus_{x,y \in Q_0} {}_yV_x$, where

$${}_yV_x = k\left(\bigsqcup_{\{\alpha \in Q_1 \mid s(\alpha)=x, t(\alpha)=y\}} X(\alpha)\right)$$

is naturally a $kX(y)$ - $kX(x)$ -bimodule. We have the following isomorphisms

$$\begin{aligned} V^{\otimes n} &\simeq \bigoplus_{x_0, x_1, \dots, x_n \in Q_0} {}_{x_n}V_{x_{n-1}} \otimes_{kX(x_{n-1})} \dots \otimes_{kX(x_2)} {}_{x_2}V_{x_1} \otimes_{kX(x_1)} {}_{x_1}V_{x_0} \\ &\simeq \bigoplus_{\{\alpha_i \in Q_1 \mid t(\alpha_i)=s(\alpha_{i+1})\}} k(X(\alpha_n) \times_{X(t(\alpha_{n-1}))} \dots \times_{X(t(\alpha_2))} X(\alpha_2) \times_{X(t(\alpha_1))} X(\alpha_1)) \\ &= \bigoplus_{p \in Q_n} kX(p) \simeq (k\mathcal{C})_n, \end{aligned}$$

where the second isomorphism follows by applying Lemma 2.1 repeatedly. Since the restriction of ϕ on $V^{\otimes n}$ coincides with this composite isomorphism, we are done. \square

Remark 2.3. Assume that the above assignment X is trivial, that is, all sets $X(x)$ and $X(\alpha)$ are consisting of single elements. Then the category algebra $k\mathcal{C}$ coincides with the path algebra kQ of the quiver Q . The isomorphism in Proposition 2.2 in this case is well known; see [1, Proposition III.1.3].

3. The algebras associated to Cartan matrices

In this section, we recall the algebras defined in [4], which are associated to symmetrizable generalized Cartan matrices. For two nonzero integers $l, m \in \mathbb{Z}$, we denote by $\gcd(l, m)$ their greatest common divisor, which is always assumed to be positive.

Let $n \geq 1$ be a positive integer. A matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$ is called a *symmetrizable generalized Cartan matrix* provided that the following conditions hold:

- (C1) $c_{ii} = 2$ for all i ;
- (C2) $c_{ij} \leq 0$ for all $i \neq j$, and $c_{ij} < 0$ if and only if $c_{ji} < 0$;
- (C3) There is a diagonal matrix $D = \text{diag}(c_1, \dots, c_n) \in M_n(\mathbb{Z})$ with $c_i \geq 1$ for all i such that the product matrix DC is symmetric.

The matrix D appearing in (C3) is called a *symmetrizer* of C . From now on, by a *Cartan matrix* we always mean a symmetrizable generalized Cartan matrix.

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix. An *orientation* of C is a subset $\Omega \subset \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ such that the following conditions hold:

- (1) $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;
- (2) For each sequence $((i_1, i_2), (i_2, i_3), \dots, (i_t, i_{t+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$, we have $i_1 \neq i_{t+1}$.

For an orientation Ω of C , let $Q = Q(C, \Omega)$ be the finite quiver with the set of vertices $Q_0 := \{1, 2, \dots, n\}$ and with the set of arrows

$$Q_1 := \{\alpha_{ij}^{(g)} : j \rightarrow i \mid (i, j) \in \Omega, 1 \leq g \leq \gcd(c_{ij}, c_{ji})\} \cup \{\varepsilon_i : i \rightarrow i \mid 1 \leq i \leq n\}.$$

We call Q a *quiver of type C*. Let $Q^\circ = Q^\circ(C, \Omega)$ be the quiver obtained from Q by deleting all loops ε_i . Then Q° is a finite acyclic quiver.

We will call (C, D, Ω) a *Cartan triple*, where C is a Cartan matrix, D its symmetrizer and Ω an orientation of C .

Definition 3.1. ([4, Section 1.4]) Let k be a field, and (C, D, Ω) be a Cartan triple. Then we have the quiver $Q = Q(C, \Omega)$. Let

$$H = H(C, D, \Omega) = kQ/I,$$

where kQ is the path algebra of Q , and I is the two-sided ideal of kQ defined by the following relations:

- (H1) For each vertex i , we have the *nilpotency relation*

$$\varepsilon_i^{c_i} = 0.$$

- (H2) For each $(i, j) \in \Omega$ and each $1 \leq g \leq \gcd(c_{ij}, c_{ji})$, we have the *commutativity relation*

$$\varepsilon_i^{\frac{c_i}{\gcd(c_i, c_j)}} \alpha_{ij}^{(g)} = \alpha_{ij}^{(g)} \varepsilon_j^{\frac{c_j}{\gcd(c_i, c_j)}}.$$

Remark 3.2. Assume that the Cartan matrix C is symmetric. We take $D = cI_n$ to be a scalar matrix for $c \geq 1$. Then the algebra $H(C, D, \Omega)$ is isomorphic to $kQ^\circ \otimes_k k[\varepsilon]/(\varepsilon^c)$.

This algebra is studied in [3,12] from different perspectives. The main concerns of [4] are non-symmetric Cartan matrices.

Let us introduce some notation. Fix the Cartan triple (C, D, Ω) as above. For each $1 \leq i \leq n$, set $H_i = k[\epsilon_i]/(\epsilon_i^{c_i})$. For each $(i, j) \in \Omega$, set $H_{ij} = k[\epsilon_{ij}]/(\epsilon_{ij}^{\gcd(c_i, c_j)})$. Consider the following injective algebra homomorphisms

$$H_{ij} \rightarrow H_i, \quad \epsilon_{ij} \mapsto \epsilon_i^{\frac{c_i}{\gcd(c_i, c_j)}}$$

and

$$H_{ij} \rightarrow H_j, \quad \epsilon_{ij} \mapsto \epsilon_j^{\frac{c_j}{\gcd(c_i, c_j)}}$$

For each $1 \leq g \leq \gcd(c_{ij}, c_{ji})$, set ${}_iH_j^{(g)} := H_i \otimes_{H_{ij}} H_j$, which is naturally an H_i - H_j -bimodule. The element $1 \otimes 1$ in ${}_iH_j^{(g)}$ will be denoted by $\beta_{ij}^{(g)}$. Set $B = \prod_{i=1}^n H_i$ and $W = \bigoplus_{(i,j) \in \Omega} \bigoplus_{g=1}^{\gcd(c_{ij}, c_{ji})} {}_iH_j^{(g)}$. Then W is naturally a B - B -bimodule.

The following result is essentially due to [4, Proposition 6.4]. We make a slight modification and give a complete proof. Recall the notation $H = H(C, D, \Omega)$.

Proposition 3.3. *Keep the notation as above. Then there is an isomorphism of algebras*

$$T_B(W) \xrightarrow{\sim} H,$$

which induces an isomorphism of algebras

$$H_i \simeq e_i H e_i$$

for each $1 \leq i \leq n$, and an isomorphism of H_i - H_j -bimodules

$${}_iH_j^{(g)} \simeq \text{Span}_k \{ \epsilon_i^{n_i} \alpha_{ij}^{(g)} \epsilon_j^{n_j} \mid 0 \leq n_i \leq c_i - 1, 0 \leq n_j \leq c_j - 1 \}$$

for each $(i, j) \in \Omega$ and $1 \leq g \leq \gcd(c_{ij}, c_{ji})$.

Here, e_i denotes the idempotent of H corresponding to the vertex i . The notation “ Span_k ” means the subspace of H spanned by the mentioned elements.

Proof. There is an algebra homomorphism $\phi_0: B \rightarrow H$ sending ϵ_i to ε_i . Moreover, each $(i, j) \in \Omega$ and $1 \leq g \leq \gcd(c_{ij}, c_{ji})$, there is a bilinear map $H_i \times H_j \rightarrow e_i H e_j$ sending $(\epsilon_i^a, \epsilon_j^b)$ to $\varepsilon_i^a \alpha_{ij}^{(g)} \varepsilon_j^b$. By the commutativity relations in H , this map is H_{ij} -balanced. In particular, we have an induced H_i - H_j -bimodule homomorphism ${}_iH_j^{(g)} \rightarrow e_i H e_j$ sending

$\beta_{ij}^{(g)}$ to $\alpha_{ij}^{(g)}$. These bimodule homomorphisms give rise to a B - B -bimodule homomorphism $\phi_1: W \rightarrow H$. By the universal property of tensor algebras, we have an algebra homomorphism $\phi: T_B(W) \rightarrow H$ extending ϕ_0 and ϕ_1 .

On the other hand, there is a unique algebra homomorphism $\psi: H \rightarrow T_B(W)$ such that $\psi(e_i) = 1_{H_i}$, $\psi(\varepsilon_i) = \epsilon_i$ and $\psi(\alpha_{ij}^{(g)}) = \beta_{ij}^{(g)}$. It is direct to verify the defining relations of H in $T_B(W)$, using the identity

$$\epsilon_i^{\frac{c_i}{\gcd(c_i, c_j)}} \beta_{ij}^{(g)} = \beta_{ij}^{(g)} \epsilon_j^{\frac{c_j}{\gcd(c_i, c_j)}}$$

in ${}_i H_j^{(g)}$. Moreover, we have $\psi \circ \phi = \text{Id}_{T_B(W)}$ and $\phi \circ \psi = \text{Id}_H$, since they hold trivially on generators. Then we are done. \square

4. The category algebra of a free EI category of Cartan type

In this section, we introduce a free EI category, which is associated to a Cartan triple. The main result states that the corresponding category algebra is isomorphic to the algebra associated to another Cartan triple; see Theorem 4.3.

4.1. Free EI categories of Cartan type

Let $n \geq 1$ and $(C = (c_{ij}), D = \text{diag}(c_1, c_2, \dots, c_n), \Omega)$ be a Cartan triple. Recall that $Q = Q(C, \Omega)$ is the quiver of type C , and that Q° is the quiver obtained from Q by deleting all loops ε_i .

We will define a finite EI quiver (Q°, X) . Recall that $Q_0^\circ = \{1, 2, \dots, n\}$ and $Q_1^\circ = \{\alpha_{ij}^{(g)}: j \rightarrow i \mid (i, j) \in \Omega, 1 \leq g \leq \gcd(c_{ij}, c_{ji})\}$. The assignment X is given such that $X(i) = \langle \eta_i \mid \eta_i^{c_i} = 1 \rangle$ is a cyclic group of order c_i . For each $(i, j) \in \Omega$, we set $G_{ij} = \langle \eta_{ij} \mid \eta_{ij}^{\gcd(c_i, c_j)} = 1 \rangle$ to be a cyclic group of order $\gcd(c_i, c_j)$. There are injective group homomorphisms

$$G_{ij} \longrightarrow X(i), \quad \eta_{ij} \mapsto \eta_i^{\frac{c_i}{\gcd(c_i, c_j)}}$$

and

$$G_{ij} \longrightarrow X(j), \quad \eta_{ij} \mapsto \eta_j^{\frac{c_j}{\gcd(c_i, c_j)}}$$

Then we have the $(X(i), X(j))$ -biset $X(i) \times_{G_{ij}} X(j)$. We set

$$X(\alpha_{ij}^{(g)}) = X(i) \times_{G_{ij}} X(j)$$

for each $1 \leq g \leq \gcd(c_{ij}, c_{ji})$. We observe that both the left $X(i)$ -action and right $X(j)$ -action on $X(\alpha_{ij}^{(g)})$ are free.

Definition 4.1. For a Cartan triple (C, D, Ω) , the free EI category $\mathcal{C}(Q^\circ, X)$ associated to the above finite EI quiver (Q°, X) will be denoted by $\mathcal{C}(C, D, \Omega)$, which will be called the free EI category of type C .

We give an intrinsic characterization of free EI categories of Cartan type.

Let \mathcal{C} be a finite EI category. Recall that a non-isomorphism α is called *unfactorizable*, provided that α_1 or α_2 is an isomorphism in any factorization $\alpha = \alpha_1 \circ \alpha_2$. Denote by $\text{Hom}_{\mathcal{C}}^0(x, y)$ the set of unfactorizable morphisms between two objects x and y ; it is naturally an $(\text{Aut}(y), \text{Aut}(x))$ -biset. For example, if $\mathcal{C} = \mathcal{C}(C, D, \Omega)$, we have

$$\text{Hom}_{\mathcal{C}}^0(j, i) = \bigsqcup_{1 \leq g \leq \gcd(c_{ij}, c_{ji})} X(\alpha_{ij}^{(g)}).$$

In particular, the category $\mathcal{C}(C, D, \Omega)$ satisfies the conditions (EC1)-(EC3) below.

Proposition 4.2. *Let \mathcal{C} be a finite free EI category. Assume that the following conditions are satisfied:*

- (EC1) $\text{Obj}(\mathcal{C}) = \{x_1, x_2, \dots, x_n\}$ and $\text{Aut}(x_i) = \langle \eta_i \mid \eta_i^{c_i} = 1 \rangle$ is cyclic of order c_i for each i ;
- (EC2) The left $\text{Aut}(x_i)$ -action and right $\text{Aut}(x_j)$ -action on $\text{Hom}_{\mathcal{C}}^0(x_j, x_i)$ are both free for each i, j .
- (EC3) $\eta_i^{\frac{c_i}{\gcd(c_i, c_j)}} \circ \alpha = \alpha \circ \eta_j^{\frac{c_j}{\gcd(c_i, c_j)}}$ for each $\alpha \in \text{Hom}_{\mathcal{C}}^0(x_j, x_i)$.

Then \mathcal{C} is isomorphic to $\mathcal{C}(C, D, \Omega)$ for some Cartan triple (C, D, Ω) .

Proof. We write the $(\text{Aut}(x_i), \text{Aut}(x_j))$ -biset $\text{Hom}_{\mathcal{C}}^0(x_j, x_i)$ as a disjoint union of orbits

$$\text{Hom}_{\mathcal{C}}^0(x_j, x_i) = \bigsqcup_{l=1}^{m_{ij}} \text{Aut}(x_i)\alpha_{ij}^{(l)}\text{Aut}(x_j),$$

where $\{\alpha_{ij}^{(l)} \mid 1 \leq l \leq m_{ij}\}$ is a complete representative set of orbits. Set $m_{ij} = 0$ if $\text{Hom}_{\mathcal{C}}^0(x_j, x_i) = \emptyset$.

We define a Cartan matrix $C = (c_{ij}) \in M_n(\mathbb{Z})$ as follows: $c_{ii} = 2$ for $1 \leq i \leq n$; if both m_{ij} and m_{ji} are zero, we set $c_{ij} = 0 = c_{ji}$; if $m_{ij} > 0$, we set $c_{ij} = -\frac{c_j}{\gcd(c_i, c_j)} \cdot m_{ij}$ and $c_{ji} = -\frac{c_i}{\gcd(c_i, c_j)} \cdot m_{ij}$, in which case we have $m_{ij} = \gcd(c_{ij}, c_{ji})$. Set $D = \text{diag}(c_1, c_2, \dots, c_n)$, and $\Omega = \{(i, j) \mid \text{Hom}_{\mathcal{C}}^0(x_j, x_i) \neq \emptyset\}$. Then (C, D, Ω) is a Cartan triple.

We claim that $\mathcal{C} \simeq \mathcal{C}(C, D, \Omega)$. Recall from [8, Proposition 2.9] that free EI categories are completely determined by the automorphism groups of objects and the bisets of unfactorizable morphisms. It suffices to prove that for each $1 \leq l \leq m_{ij}$, the $(\text{Aut}(x_i), \text{Aut}(x_j))$ -bisets $X(\alpha_{ij}^{(l)})$ and $\text{Aut}(x_i)\alpha_{ij}^{(l)}\text{Aut}(x_j)$ are isomorphic. Indeed by (EC3), the following biset map

$$\phi: X(\alpha_{ij}^{(l)}) \longrightarrow \text{Aut}(x_i)\alpha_{ij}^{(l)}\text{Aut}(x_j), \quad (\eta_i^a, \eta_j^b) \mapsto \eta_i^a \circ \alpha_{ij}^{(l)} \circ \eta_j^b$$

is well-defined, which is surjective. The cardinality of $X(\alpha_{ij}^{(l)})$ equals $\frac{c_i c_j}{\gcd(c_i, c_j)}$. By (EC2), both c_i and c_j divide the cardinality of $\text{Aut}(x_i)\alpha_{ij}^{(l)}\text{Aut}(x_j)$. It follows that ϕ is bijective. This proves the claim, and completes the proof. \square

4.2. The construction of (C', D', Ω')

In what follows, we assume that the base field k has *enough roots of unity*, that is, any polynomial $t^a - 1$ in $k[t]$ splits for each $a \geq 2$.

We assume first that $\text{char} k = p > 0$. We fix a Cartan triple (C, D, Ω) as above. Assume that

$$c_i = p^{r_i} d_i$$

with $r_i \geq 0$ and $\gcd(p, d_i) = 1$.

Set $m = \sum_{i=1}^n d_i$, which is the cardinality of the set

$$M = \bigsqcup_{1 \leq i \leq n} \{(i, l_i) \mid 0 \leq l_i < d_i\}.$$

For each $1 \leq i, j \leq n$, we set

$$\Sigma_{ij} = \{(l_i, l_j) \mid 0 \leq l_i < d_i, 0 \leq l_j < d_j, l_i p^{r_i} \equiv l_j p^{r_j} \pmod{\gcd(d_i, d_j)}\}.$$

We will define a Cartan matrix $C' \in M_m(\mathbb{Z})$, whose rows and columns are indexed by M . The diagonal entries of C' are 2, while the off-diagonal entries are given as follows

$$c'_{(i,l_i),(j,l_j)} = \begin{cases} -\gcd(c_{ij}, c_{ji}) p^{r_j - \min(r_i, r_j)}, & \text{if } (l_i, l_j) \in \Sigma_{ij}; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $c'_{(i,l_i),(i,l'_i)} = 0$ for $l_i \neq l'_i$. Let D' be a diagonal matrix of rank m , whose (i, l_i) -th component is given by p^{r_i} . Since

$$p^{r_i} c'_{(i,l_i),(j,l_j)} = p^{r_j} c'_{(j,l_j),(i,l_i)},$$

the product matrix $D'C'$ is symmetric. Set

$$\Omega' = \{((i, l_i), (j, l_j)) \mid (i, j) \in \Omega, (l_i, l_j) \in \Sigma_{ij}\},$$

which is an orientation of C' . Then we have the required Cartan triple (C', D', Ω') . We point out that a similar construction is given in [11, Section 14.1] and [6, Lemma 21]. We refer to Section 6 for further study on this construction.

We observe that the above definitions work well if $\text{char}k = 0$, in which case we put $r_i = 0$ and $d_i = c_i$. Then the Cartan matrix C' is symmetric and the symmetrizer D' is the identity matrix.

The main result provides an explicit isomorphism between certain category algebras and the algebras studied in [4]. It motivates the notion of a free EI category of Cartan type.

Theorem 4.3. *Let (C, D, Ω) be a Cartan triple and $\mathcal{C}(C, D, \Omega)$ as in Definition 4.1. Assume that k has enough roots of unity. Consider the Cartan triple (C', D', Ω') as above and the algebra $H(C', D', \Omega')$ in Definition 3.1. Then there is an isomorphism of algebras*

$$k\mathcal{C}(C, D, \Omega) \simeq H(C', D', \Omega').$$

Proof. We assume that $\text{char}k = p > 0$. Set $\mathcal{C} = \mathcal{C}(C, D, \Omega)$ and $H = H(C', D', \Omega')$.

We observe that for $((i, l_i), (j, l_j)) \in \Omega'$, we have

$$\text{gcd}(c'_{(i,l_i),(j,l_j)}, c'_{(j,l_j),(i,l_i)}) = \text{gcd}(c_{ij}, c_{ji}). \tag{4.1}$$

By Proposition 2.2, there is an isomorphism of algebras

$$k\mathcal{C} \xrightarrow{\sim} T_A(V),$$

where $A = \prod_{i=1}^n kX(i)$ and

$$\begin{aligned} V &= \bigoplus_{(i,j) \in \Omega} \bigoplus_{g=1}^{\text{gcd}(c_{ij}, c_{ji})} k(X(i) \times_{G_{ij}} X(j)) \\ &\simeq \bigoplus_{(i,j) \in \Omega} \bigoplus_{g=1}^{\text{gcd}(c_{ij}, c_{ji})} kX(i) \otimes_{kG_{ij}} kX(j). \end{aligned}$$

Here, the last isomorphism follows from Lemma 2.1.

By Proposition 3.3, there is another isomorphism of algebras

$$H \xrightarrow{\sim} T_B(W),$$

where $B = \prod_{i=1}^n \prod_{l_i=0}^{d_i-1} H(i, l_i)$ and

$$\begin{aligned} W &= \bigoplus_{((i,l_i),(j,l_j)) \in \Omega'} \bigoplus_{g=1}^{\text{gcd}(c'_{(i,l_i),(j,l_j)}, c'_{(j,l_j),(i,l_i)})} H(i, l_i) \otimes_{H'_{ij}} H(j, l_j) \\ &= \bigoplus_{(i,j) \in \Omega} \bigoplus_{(l_i, l_j) \in \Sigma_{ij}} \bigoplus_{g=1}^{\text{gcd}(c_{ij}, c_{ji})} H(i, l_i) \otimes_{H'_{ij}} H(j, l_j) \end{aligned}$$

$$= \bigoplus_{(i,j) \in \Omega} \bigoplus_{g=1}^{\gcd(c_{ij}, c_{ji})} \bigoplus_{(l_i, l_j) \in \Sigma_{ij}} H_{(i, l_i)} \otimes_{H'_{l_j}} H_{(j, l_j)}.$$

The second equality uses (4.1). Here, we denote $H_{(i, l_i), (j, l_j)}$ by H'_{ij} for short, and we recall that $H_{(i, l_i)} = k[\epsilon_{(i, l_i)}] / (\epsilon_{(i, l_i)}^{r_i})$, and that $H'_{ij} = k[\epsilon_{ij}] / (\epsilon_{ij}^{\min(r_i, r_j)})$; see the notation of Section 3.

To obtain the required isomorphism, it suffices to have an isomorphism $A \simeq B$ of algebras and an isomorphism $V \simeq W$ of A - A -bimodules, where the A - A -bimodule structure on W is induced by the given algebra isomorphism.

For the two isomorphisms, we will show in the next section that there is an isomorphism of algebras

$$\Theta_i : kX(i) \xrightarrow{\sim} \prod_{l_i=0}^{d_i-1} H_{(i, l_i)}$$

and an isomorphism of $kX(i)$ - $kX(j)$ -bimodules

$$kX(i) \otimes_{kG_{ij}} kX(j) \xrightarrow{\sim} \bigoplus_{(l_i, l_j) \in \Sigma_{ij}} H_{(i, l_i)} \otimes_{H'_{l_j}} H_{(j, l_j)}, \tag{4.2}$$

for any $1 \leq i, j \leq n$. We emphasize that the $kX(i)$ - $kX(j)$ -bimodule structure on $H_{(i, l_i)} \otimes_{H'_{l_j}} H_{(j, l_j)}$ is induced by the isomorphisms Θ_i and Θ_j .

Since $X(i) = \langle \eta_i \mid \eta_i^{c_i} = 1 \rangle$, we identify the group algebra $kX(i)$ with $k[\eta_i] / (\eta_i^{c_i} - 1)$. Then the above isomorphisms follow immediately from Proposition 5.4 below.

The proof in the characteristic zero case is very similar. We omit the details. \square

5. An explicit decomposition of a bimodule

In this section, we give an explicit decomposition of a tensor bimodule needed at the end of the proof of Theorem 4.3; see Proposition 5.4. Here, the tensor bimodule means the tensor product considered as a bimodule. The decomposition seems to be elementary, but somehow technical.

Let k be a field having enough roots of unity with $\text{char} k = p > 0$. Let a, b be two positive integers. Write $a = p^r a'$ and $b = p^s b'$ with $r, s \geq 0$ and $\gcd(p, a') = 1 = \gcd(p, b')$. Then we have $\gcd(a, b) = p^{\min(r, s)} \gcd(a', b')$.

Let $A = k[x] / (x^a - 1)$, $B = k[y] / (y^b - 1)$ and $C = k[z] / (z^{\gcd(a, b)} - 1)$. There are injective algebra homomorphisms $\iota_1 : C \rightarrow A$ and $\iota_2 : C \rightarrow B$ such that $\iota_1(z) = x^{\frac{a}{\gcd(a, b)}}$ and $\iota_2(z) = y^{\frac{b}{\gcd(a, b)}}$. We are mainly concerned with the tensor A - B -bimodule $A \otimes_C B$.

We fix an a' -th primitive root ζ_1 of unity and a b' -th primitive root ζ_2 of unity such that $\zeta_1^{\frac{a'}{\gcd(a', b')}} = \zeta_2^{\frac{b'}{\gcd(a', b')}}$; the common value will be denoted by ζ , which is a $\gcd(a', b')$ -th primitive root of unity.

We have the following well-known isomorphisms of algebras

$$\pi_A: A \xrightarrow{\sim} \prod_{i=0}^{a'-1} k[x]/(x - \zeta_1^i)^{p^r} = \prod_{i=0}^{a'-1} A_i,$$

$$\pi_B: B \xrightarrow{\sim} \prod_{j=0}^{b'-1} k[y]/(y - \zeta_2^j)^{p^s} = \prod_{j=0}^{b'-1} B_j,$$

and

$$\pi_C: C \xrightarrow{\sim} \prod_{l=0}^{\gcd(a',b')-1} k[z]/(z - \zeta^l)^{p^{\min(r,s)}} = \prod_{l=0}^{\gcd(a',b')-1} C_l,$$

each of which sends \bar{f} to $(\bar{f}, \dots, \bar{f})$ for any polynomial f .

For each $0 \leq l < \gcd(a', b')$, we set

$$\Sigma(l) = \{m \mid 0 \leq m < a', mp^r \equiv lp^{\min(r,s)} \pmod{\gcd(a', b')}\}.$$

Then we have a disjoint union

$$\{0, 1, \dots, a' - 1\} = \bigsqcup_{l=0}^{\gcd(a',b')-1} \Sigma(l). \tag{5.1}$$

Here, we use the fact that \bar{p} is invertible in $\mathbb{Z}_{\gcd(a',b')}$. Similarly, we set

$$\Sigma'(l) = \{m \mid 0 \leq m < b', mp^s \equiv lp^{\min(r,s)} \pmod{\gcd(a', b')}\},$$

and have the disjoint union

$$\{0, 1, \dots, b' - 1\} = \bigsqcup_{l=0}^{\gcd(a',b')-1} \Sigma'(l).$$

Lemma 5.1. *Assume that $m \in \Sigma(l)$. Then the following identity*

$$\left(x^{\frac{a}{\gcd(a,b)}} - \zeta^l\right)^{p^{\min(r,s)}} = 0$$

holds in the algebra A_m .

Proof. We observe that $\frac{a}{\gcd(a,b)}p^{\min(r,s)} = \frac{a'}{\gcd(a',b')}p^r$ and $\zeta^{lp^{\min(r,s)}} = \zeta^{mp^r}$ by the fact that $m \in \Sigma(l)$. Then we have the first equality in the following identity

$$\begin{aligned}
 (x^{\frac{a}{\gcd(a,b)}} - \zeta^l)^{p^{\min(r,s)}} &= x^{\frac{a'}{\gcd(a',b')}} p^r - \zeta^m p^r \\
 &= (x^{\frac{a'}{\gcd(a',b')}} - \zeta^m)^{p^r} \\
 &= (x^{\frac{a'}{\gcd(a',b')}} - (\zeta_1^m)^{\frac{a'}{\gcd(a',b')}})^{p^r}.
 \end{aligned}$$

We observe that $(x - \zeta_1^m)^{p^r}$ is a factor. Therefore, it is zero in A_m . \square

The above lemma implies that the following algebra homomorphism

$$\phi_l: C_l \longrightarrow \prod_{i \in \Sigma(l)} A_i, \quad z \mapsto (x^{\frac{a}{\gcd(a,b)}}, \dots, x^{\frac{a}{\gcd(a,b)}})$$

is well defined for each $0 \leq l < \gcd(a', b')$. By the disjoint union (5.1), we have the following algebra homomorphism

$$(\phi_l): \prod_{l=0}^{\gcd(a',b')-1} C_l \longrightarrow \prod_{l=0}^{\gcd(a',b')-1} \left(\prod_{i \in \Sigma(l)} A_i \right) = \prod_{i=0}^{a'-1} A_i.$$

By a similar argument, we have a well-defined algebra homomorphism

$$\psi_l: C_l \longrightarrow \prod_{j \in \Sigma'(l)} B_j, \quad z \mapsto (y^{\frac{b}{\gcd(a,b)}}, \dots, y^{\frac{b}{\gcd(a,b)}}),$$

for each $0 \leq l < \gcd(a', b')$. Summing them up, we have an algebra homomorphism

$$(\psi_l): \prod_{l=0}^{\gcd(a',b')-1} C_l \longrightarrow \prod_{l=0}^{\gcd(a',b')-1} \left(\prod_{j \in \Sigma'(l)} B_j \right) = \prod_{j=0}^{b'-1} B_j.$$

Lemma 5.2. *Keep the notation as above. Then the following diagram commutes*

$$\begin{array}{ccccc}
 A & \xleftarrow{\iota_1} & C & \xrightarrow{\iota_2} & B \\
 \pi_A \downarrow & & \downarrow \pi_C & & \downarrow \pi_B \\
 \prod_{i=0}^{a'-1} A_i & \xleftarrow{(\phi_l)} & \prod_{l=0}^{\gcd(a',b')-1} C_l & \xrightarrow{(\psi_l)} & \prod_{j=0}^{b'-1} B_j.
 \end{array}$$

Proof. It suffices to observe that the commutativity holds clearly for the generator z of C . \square

For each $0 \leq i < a'$ and $0 \leq j < b'$, we set $A'_i := k[\epsilon_i]/(\epsilon_i^{p^r})$, $B'_j = k[\eta_j]/(\eta_j^{p^s})$ and $C' = k[\omega]/(\omega^{p^{\min(r,s)}})$. There are injective algebra homomorphisms $\phi'_i: C' \rightarrow A'_i$ and $\psi'_j: C' \rightarrow B'_j$ given by $\phi'_i(\omega) = \epsilon_i^{p^{r-\min(r,s)}}$ and $\psi'_j(\omega) = \eta_j^{p^{s-\min(r,s)}}$. Consequently, we have the tensor A'_i - B'_j -bimodule $A'_i \otimes_{C'} B'_j$.

We observe an isomorphism

$$\theta_i: A'_i \longrightarrow A_i$$

of algebras sending ϵ_i to $x^{a'} - 1$. We remark that there is another obvious isomorphism $A'_i \rightarrow A_i$ sending ϵ_i to $x - \zeta_1^i$. However, we really have to choose the isomorphism θ_i making Lemma 5.3 work.

Similarly, we have isomorphisms $\rho_j: B'_j \rightarrow B_j$ and $\gamma_l: C' \rightarrow C_l$ given by $\rho_j(\eta_j) = y^{b'} - 1$ and $\gamma_l(\omega) = z^{\gcd(a', b')} - 1$.

Lemma 5.3. *Keep the notation as above. We assume that $i \in \Sigma(l)$ and $j \in \Sigma'(l)$ for some l . Then the following diagram commutes*

$$\begin{array}{ccccc} A'_i & \xleftarrow{\phi'_i} & C' & \xrightarrow{\psi'_j} & B'_j \\ \theta_i \downarrow & & \downarrow \gamma_l & & \downarrow \rho_j \\ A_i & \xleftarrow{p_i \circ \phi_l} & C_l & \xrightarrow{p'_j \circ \psi_l} & B_j, \end{array}$$

where $p_i: \prod_{m \in \Sigma(l)} A_m \twoheadrightarrow A_i$ and $p'_j: \prod_{m \in \Sigma'(l)} B_m \twoheadrightarrow B_j$ are the canonical projections.

Proof. We just observe that the commutativity holds clearly for the generator ω of C' . Here, we use the specific choice of θ_i , γ_l and ρ_j . \square

The main result in this section claims a decomposition of the tensor A - B -bimodule $A \otimes_C B$. We set

$$\Sigma = \{(i, j) \mid 0 \leq i < a', 0 \leq j < b', ip^r \equiv jp^s \pmod{\gcd(a', b')}\}.$$

Proposition 5.4. *Keep the notation as above. Then the following statements hold.*

(1) *There are isomorphisms of algebras*

$$\left(\prod_{i=0}^{a'-1} \theta_i\right)^{-1} \circ \pi_A: A \longrightarrow \prod_{i=0}^{a'-1} A'_i \quad \text{and} \quad \left(\prod_{j=0}^{b'-1} \rho_j\right)^{-1} \circ \pi_B: B \longrightarrow \prod_{j=0}^{b'-1} B'_j.$$

(2) *There is an isomorphism of A - B -bimodules*

$$A \otimes_C B \xrightarrow{\sim} \bigoplus_{(i,j) \in \Sigma} A'_i \otimes_{C'} B'_j$$

where the A - B -bimodule structure on $A'_i \otimes_{C'} B'_j$ is induced by the algebra isomorphisms in (1).

Proof. It suffices to prove (2). The vertical morphisms in the commutative diagrams of the above two lemmas are all isomorphisms. By Lemma 5.2, we have the first isomorphism in the following identity

$$\begin{aligned}
 A \otimes_C B &\simeq \bigoplus_{l=0}^{\gcd(a',b')-1} \left(\prod_{i \in \Sigma(l)} A_i \right) \otimes_{C_l} \left(\prod_{j \in \Sigma'(l)} B_j \right) \\
 &\simeq \bigoplus_{(i,j) \in \Sigma} A_i \otimes_{C_l} B_j \\
 &\simeq \bigoplus_{(i,j) \in \Sigma} A'_i \otimes_{C'} B'_j.
 \end{aligned}$$

The last isomorphism follows from Lemma 5.3. \square

Choose A, B and C to be $kX(i)$, $kX(j)$ and kG_{ij} , respectively. Then we get the isomorphism (4.2) in the proof of Theorem 4.3.

6. The construction in Subsection 4.2 and examples

In this section, we study the construction from (C, D, Ω) to (C', D', Ω') in Subsection 4.2. We prove that the valued graph Γ of C is a disjoint union of Dynkin graphs (*resp.* Euclidean graphs, graphs of indefinite type) if and only if so is the valued graph Γ' of C' ; see Proposition 6.5.

We observe that our construction is related to the well-known correspondence in [11, Section 14.1] between Cartan matrices and graphs with automorphisms. Consequently, in the hereditary case, the isomorphism in Theorem 4.3 yields an algebraic enrichment of the mentioned correspondence; see Proposition 6.10.

6.1. The types of Cartan matrices

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix. Denote by Γ the associated *valued graph*. Recall that the vertices of Γ are given by $\{1, 2, \dots, n\}$ and that there is a unique edge between i and j if and only if $c_{ij} < 0$; moreover, we put a valuation $(|c_{ij}|, |c_{ji}|)$ on the edge. We emphasize that the valuation is not assigned to the edge, but really to the ordered pair (i, j) of vertices. We say that C is *connected* provided that Γ is a connected graph. In what follows, we will identify the Cartan matrix C with its valued graph Γ . In particular, symmetric Cartan matrices correspond to usual graphs, which do not have loops, but possibly have parallel edges.

Let $D = \text{diag}(c_1, c_2, \dots, c_n)$ be a symmetrizer of C . Denote by \mathbb{Z}^n the root lattice of C , which has a canonical basis $\{e_1, e_2, \dots, e_n\}$. Recall that the quadratic form $q_C: \mathbb{Z}^n \rightarrow \mathbb{Z}$ is given by $q_C(\mathbf{x}) = \sum_{i=1}^n c_i x_i^2 + \sum_{i < j} c_i c_{ij} x_i x_j$ for $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

We fix a prime number $p > 0$. Recall from Subsection 4.2 that (C', D', Ω') is constructed from (C, D, Ω) . We denote by Γ' its valued graph. The vertices of Γ' are indexed by the following set

$$M = \bigsqcup_{1 \leq i \leq n} \{(i, l_i) \mid 0 \leq l_i < d_i\}.$$

Here, we recall that $c_i = p^{r_i} d_i$. Its root lattice is denoted by \mathbb{Z}^M with a canonical basis $\{e_{(i, l_i)} \mid (i, l_i) \in M\}$ and its quadratic form $q_{C'}$.

The following result is inspired by [11, Subsection 13.2.9].

Lemma 6.1. *The natural injection $\theta: \mathbb{Z}^n \rightarrow \mathbb{Z}^M$ sending each e_i to $\sum_{0 \leq l_i < d_i} e_{(i, l_i)}$ is compatible with the quadratic form, that is, $q_C(\mathbf{x}) = q_{C'}(\theta(\mathbf{x}))$.*

Proof. Denote by $(-, -)_C$ and $(-, -)_{C'}$ the symmetric bilinear form on \mathbb{Z}^n and \mathbb{Z}^M corresponding to q_C and $q_{C'}$, respectively. It suffices to show that $(e_i, e_j)_C = (\theta(e_i), \theta(e_j))_{C'}$.

Recall that $(e_i, e_i)_C = 2c_i = 2p^{r_i} d_i$ and $(e_i, e_j)_C = c_i c_{ij}$ for $i \neq j$. Similarly, $(e_{(i, l_i)}, e_{(i, l_i)})_{C'} = 2p^{r_i}$ and $(e_{(i, l_i)}, e_{(j, l_j)})_{C'} = p^{r_i} c'_{(i, l_i), (j, l_j)}$. In particular, $(e_{(i, l_i)}, e_{(i, l'_i)})_{C'} = 0$ for $l_i \neq l'_i$. Then the required identity is easily verified for the case $i = j$.

We assume that $i \neq j$. Then we have

$$\begin{aligned} (\theta(e_i), \theta(e_j))_{C'} &= \sum_{0 \leq l_i < d_i, 0 \leq l_j < d_j} p^{r_i} c'_{(i, l_i), (j, l_j)} \\ &= - \sum_{(l_i, l_j) \in \Sigma_{ij}} p^{r_i + r_j - \min(r_i, r_j)} \gcd(c_{ij}, c_{ji}) \\ &= -p^{r_i + r_j - \min(r_i, r_j)} \frac{d_i d_j}{\gcd(d_i, d_j)} \gcd(c_{ij}, c_{ji}), \end{aligned}$$

where the last equality uses the fact that the cardinality of Σ_{ij} is $\frac{d_i d_j}{\gcd(d_i, d_j)}$. We assume without loss of generality that $r_i \geq r_j$. Since $c_i c_{ij} = c_j c_{ji}$, we have

$$p^{r_i - r_j} \frac{d_i}{\gcd(d_i, d_j)} \frac{c_{ij}}{\gcd(c_{ij}, c_{ji})} = \frac{d_j}{\gcd(d_i, d_j)} \frac{c_{ji}}{\gcd(c_{ij}, c_{ji})}.$$

Since d_i and d_j are coprime to p , we infer that

$$\frac{-c_{ij}}{\gcd(c_{ij}, c_{ji})} = \frac{d_j}{\gcd(d_i, d_j)}.$$

Then the required equality in the claim follows immediately. \square

Recall that the symmetrizer D is *minimal*, provided that $\gcd(c_1, c_2, \dots, c_n) = 1$. For a connected Cartan matrix, the minimal symmetrizer is unique and any symmetrizer is a multiple of the minimal one.

Lemma 6.2. *Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a connected Cartan matrix with its minimal symmetrizer D . Then the constructed Cartan matrix C' is also connected.*

Proof. For two distinct vertices (i, l) and (j, l') in Γ' , we will construct a path connecting them. By the connectedness of Γ , there is a path in Γ

$$i = x(1) - x(2) - \dots - x(m) - x(m + 1) = j$$

such that for each $1 \leq s \leq n$, there exists at least one a satisfying $x(a) = s$. For each $1 \leq a \leq m$, we set $g_a = \gcd(d_{x(a)}, d_{x(a+1)})$. By the minimality of D , we have $\gcd(g_1, g_2, \dots, g_m) = 1$. There exist $s_a \in \mathbb{Z}$ such that $\sum_{a=1}^m s_a g_a = 1$.

Set $l_1 = l$. For $2 \leq a \leq m + 1$, we define l_a to be the unique integer satisfying $0 \leq l_a < d_{x(a)}$ and

$$l_a p^{r_{x(a)}} \equiv l p^{r_i} + \sum_{b=1}^{a-1} s_b g_b (l' p^{r_j} - l p^{r_i}) \pmod{d_{x(a)}}.$$

We observe $l_{m+1} = l'$. Moreover, it is clear that $l_a p^{r_{x(a)}} \equiv l_{a+1} p^{r_{x(a+1)}} \pmod{g_a}$ for each $1 \leq a \leq m$. Therefore, there is an edge from $(x(a), l_a)$ to $(x(a + 1), l_{a+1})$ in Γ' . This yields the required construction. \square

The following result implies that in the construction of Subsection 4.2, we may assume that D is minimal.

Lemma 6.3. *Let $c = p^r d$ be a positive integer with $r \geq 0$ and $\gcd(p, d) = 1$. Denote by (C'', D'', Ω'') the Cartan triple constructed from (C, cD, Ω) as in Subsection 4.2, and by Γ'' its valued graph. Then Γ'' is isomorphic to the disjoint union of d copies of Γ .*

Proof. The vertices of Γ'' are indexed by $M' = \bigsqcup_{1 \leq i \leq n} \{(i, m_i) \mid 0 \leq m_i < dd_i\}$. The following set

$$\{(m_i, m_j) \mid 0 \leq m_i < dd_i, 0 \leq m_j < dd_j, m_i p^{r_i+r} \equiv m_j p^{r_j+r} \pmod{d \gcd(d_i, d_j)}\}$$

will be denoted by Σ'_{ij} . Then the off-diagonal entries of C'' are given as follows:

$$c''_{(i,m_i),(j,m_j)} = \begin{cases} -\gcd(c_{ij}, c_{ji}) p^{(r_j+r) - \min(r_i+r, r_j+r)}, & \text{if } (m_i, m_j) \in \Sigma'_{ij}; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that an off-diagonal entry $c''_{(i,m_i),(j,m_j)} \neq 0$ implies that $i \neq j$ and $m_i p^{r_i} \equiv m_j p^{r_j} \pmod{d}$, in which case it is equal to $c'_{(i,l_i),(j,l_j)}$ for $(l_i, l_j) \in \Sigma_{ij}$.

For each $0 \leq a < d$, we denote by Γ''_a the full subgraph of Γ'' formed by these vertices $\{(i, m_i) \in M' \mid m_i p^{r_i} \equiv a \pmod{d}\}$. It follows that Γ'' is a disjoint union of these

subgraphs Γ'_a 's. We observe that each Γ'_a is isomorphic to Γ' by sending (i, m_i) to (i, l_i) , where l_i is uniquely determined by the constraint

$$\frac{m_i p^{r_i} - a}{d} \equiv l_i p^{r_i} \pmod{d_i}.$$

Here for the isomorphism, we use the fact that (m_i, m_j) lies in Σ'_{ij} if and only if (l_i, l_j) lies in Σ_{ij} . \square

The following fact follows immediately from the above two lemmas and the fact that any symmetrizer of a connected Cartan matrix is a multiple of the minimal symmetrizer.

Corollary 6.4. *Keep the notation as above. Assume that C is connected. Then the connected components of the valued graph Γ' are isomorphic to each other. \square*

Recall that a connected Cartan matrix C is a *Dynkin graph* (resp. an *Euclidean graph*, a *graph of indefinite type*), provided that its quadratic form q_C is positive definite (resp. positive semi-definite, indefinite).

Proposition 6.5. *Keep the notation as above. Assume further that C and thus Γ are connected. Then Γ is a Dynkin graph (resp. an Euclidean graph, a graph of indefinite type) if and only if Γ' is a disjoint union of Dynkin graphs (resp. Euclidean graphs, graphs of indefinite type), whose connected components are of the same type.*

Proof. We first prove the statement on Dynkin graphs. The “if” part follows from Lemma 6.1. For the “only if” part, we recall that the symmetrizer D is a multiple of the minimal one. Then by Lemma 6.3 we may assume that the symmetrizer D is minimal. If C is symmetric and then D is the identity matrix, then $C' = C$. For non-symmetric Cartan matrices of Dynkin type and minimal symmetrizers, the description of Γ' is given in the third column of Table 1 below. We observe that Γ' is necessarily a Dynkin graph.

Using the proved statement on Dynkin graphs and Table 2, we infer the statement on Euclidean graphs with the same argument. Then the remaining statement on indefinite types follows immediately. \square

Remark 6.6. (1) We observe that usually Γ and the connected components of Γ' are of different types; see Tables 1 and 2 below.

(2) Proposition 6.5 also hold for $p = 0$ along the same line of reasoning. In the above notation, we have $d_i = c_i$, $r_i = 0 = r$ and $d = c$.

We mention that Tables 1 and 2 are essentially contained in [11, Section 14.1] with a completely different terminology. For the lists of Dynkin graphs and Euclidean graphs, we refer to [1, VII.3].

The following example unifies the actual computation in establishing Tables 1 and 2.

Table 1
Non-simply-laced Dynkin graphs Γ .

Γ	Minimal symmetrizer D	Γ'	(Γ', σ) in [11]
B_n : $(n \geq 3)$	$\text{diag}(2, 1, \dots, 1)$	If $p = 2, B_n$ If $p \neq 2, D_{n+1}$	[11, 14.1.6 (b)]
C_n : $(n \geq 2)$	$\text{diag}(1, 2, \dots, 2)$	If $p = 2, C_n$ If $p \neq 2, A_{2n-1}$	[11, 14.1.6 (a)]
F_4 :	$\text{diag}(2, 2, 1, 1)$	If $p = 2, F_4$ If $p \neq 2, E_6$	[11, 14.1.6 (d)]
G_2 :	$\text{diag}(3, 1)$	If $p = 3, G_2$ If $p \neq 3, D_4$	[11, 14.1.6 (c)]

Example 6.7. Let C be a connected Cartan matrix such that its minimal symmetrizer D equals $\text{diag}(q^{a_1}, \dots, q^{a_n})$, where q is a prime number and $a_i \geq 0$. We observe that

$$c_{ij} = -\text{gcd}(c_{ij}, c_{ji})q^{a_j - \min(a_i, a_j)}, i \neq j.$$

Consider the construction of (C', D', Ω') , where the characteristic of the field is p . It follows that $(C, D, \Omega) = (C', D', \Omega')$ and $\Gamma' = \Gamma$ provided that $p = q$.

We assume that $p \neq q$. Then C' is symmetric and the symmetrizer D' is the identity matrix. In other words, the valued graph Γ' of C' is a usual graph. Moreover, as we will see in Subsection 6.2 below, the graph Γ' has an admissible automorphism σ , listed in the rightmost column in the tables. We observe that, the graphs (Γ', σ) with automorphisms, listed in Tables 1 and 2, are due to [11, 14.1.5 (a)-(d) and 14.1.6 (a)-(i)].

6.2. Comparison with the correspondence in [11, Section 14.1]

In this subsection, we compare the construction in Subsection 4.2 to the well-known correspondence in [11, Section 14.1], which is between Cartan matrices and graphs with admissible automorphisms.

Denote by Γ a finite graph without loops. An automorphism $\sigma : \Gamma \rightarrow \Gamma$ is *admissible*, provided that there is no edge joining two vertices in the same σ -orbit. Since we will not distinguish parallel edges, two automorphisms are identified if their actions on vertices are the same. Similarly, we have the notion of *admissible automorphisms* for finite acyclic quivers.

Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a Cartan matrix with $D = \text{diag}(c_1, \dots, c_n)$ its symmetrizer. We associate a graph Γ with an admissible automorphism σ as follows. The vertex set of Γ is given by $\Gamma_0 = \{(i, l_i) \mid 1 \leq i \leq n, 0 \leq l_i < c_i\}$. There is an edge between (i, l_i) and (j, l_j) if and only if $i \neq j, c_{ij} \neq 0$ and $l_i \equiv l_j \pmod{\text{gcd}(c_i, c_j)}$, in which case there are exactly $\text{gcd}(c_{ij}, c_{ji})$ such edges. The automorphism σ sends (i, l_i) to $(i, l_i + 1)$, where we identify $(i, 0)$ with (i, c_i) .

Table 2
Non-simply-laced Euclidean graphs Γ (with \tilde{A}_{12} omitted).

Γ	Minimal symmetrizer D	Γ'	(Γ', σ) in [11]
\tilde{B}_n : ($n \geq 2$)	$\text{diag}(2, 1, \dots, 1, 2)$	If $p = 2, \tilde{B}_n$ If $p \neq 2, \tilde{D}_{n+2}$	[11, 14.1.5 (c)]
\tilde{C}_n : ($n \geq 2$)	$\text{diag}(1, 2, \dots, 2, 1)$	If $p = 2, \tilde{C}_n$ If $p \neq 2, \tilde{A}_{2n-1}$	[11, 14.1.5 (a)]
\tilde{A}_{11} :	$\text{diag}(4, 1)$	If $p = 2, \tilde{A}_{11}$ If $p \neq 2, \tilde{D}_4$	[11, 14.1.5 (e)]
\widetilde{BC}_n : ($n \geq 2$)	$\text{diag}(4, 2, \dots, 2, 1)$	If $p = 2, \widetilde{BC}_n$ If $p \neq 2, \tilde{D}_{2n+2}$	[11, 14.1.5 (e)]
\widetilde{BD}_n : ($n \geq 3$)	$\text{diag}(2, 1, \dots, 1)$	If $p = 2, \widetilde{BD}_n$ If $p \neq 2, \tilde{D}_{n+1}$	[11, 14.1.5 (b)]
\widetilde{CD}_n : ($n \geq 3$)	$\text{diag}(1, 2, \dots, 2)$	If $p = 2, \widetilde{CD}_n$ If $p \neq 2, \tilde{D}_{2n}$	[11, 14.1.5 (d)]
\tilde{F}_{41} :	$\text{diag}(2, 2, 2, 1, 1)$	If $p = 2, \tilde{F}_{41}$ If $p \neq 2, \tilde{E}_7$	[11, 14.1.5 (i)]
\tilde{F}_{42} :	$\text{diag}(1, 1, 1, 2, 2)$	If $p = 2, \tilde{F}_{42}$ If $p \neq 2, \tilde{E}_6$	[11, 14.1.5 (g)]
\tilde{G}_{21} :	$\text{diag}(3, 3, 1)$	If $p = 3, \tilde{G}_{21}$ If $p \neq 3, \tilde{E}_6$	[11, 14.1.5 (h)]
\tilde{G}_{22} :	$\text{diag}(1, 1, 3)$	If $p = 3, \tilde{G}_{22}$ If $p \neq 3, \tilde{D}_4$	[11, 14.1.5 (f)]

The following well-known correspondence between Cartan matrices and graphs with admissible automorphisms is due to [11, Section 14.1]. We use slightly different terminologies.

Proposition 6.8. *The above assignment yields an injective map between the corresponding sets of isomorphism classes*

$$\left\{ (C, D) \mid \begin{array}{l} C \text{ a Cartan matrix,} \\ D \text{ its symmetrizer} \end{array} \right\} \hookrightarrow \left\{ (\Gamma, \sigma) \mid \begin{array}{l} \Gamma \text{ a finite graph, } \sigma \text{ an admissible} \\ \text{automorphism of } \Gamma \end{array} \right\}.$$

Here, we say that (C, D) and (C_1, D_1) are isomorphic provided that there is an isomorphism between the corresponding associated valued graphs of C and C_1 , which is compatible with D and D_1 .

The above map is in general not surjective; compare [6, the second paragraph of Introduction].

Proof. The partial inverse map is given as follows. We assume that there are exactly n σ -orbits in Γ_0 . We will index the rows and columns of the matrices by the orbit set Γ_0/σ . For a σ -orbit $[i]$, we set $c_{[i]}$ to be its cardinality. This defines the diagonal matrix D . For $[i] \neq [j]$, we define $c_{[i],[j]} = -\frac{N}{c_{[i]}}$, where N is the total number of edges between i' and j' with all possible $i' \in [i]$ and $j' \in [j]$. This defines the Cartan matrix C . \square

We now add the orientations into consideration. Then we have the following immediate consequence.

Corollary 6.9. *There is an injective map between the corresponding sets of isomorphism classes*

$$\{(C, D, \Omega) \mid a \text{ Cartan triple}\} \hookrightarrow \left\{ (\Delta, \sigma) \mid \begin{array}{l} \Delta \text{ a finite acyclic quiver,} \\ \sigma \text{ an admissible automorphism of } \Delta \end{array} \right\}.$$

Let (C, D, Ω) be a Cartan triple and $\mathcal{C} = \mathcal{C}(C, D, \Omega)$ be its free EI category. Let k be a field with characteristic p . Then the category algebra $k\mathcal{C}$ is hereditary if and only if p is zero or coprime to each entry of D ; see [8, Theorem 1.2].

We have the following immediate consequence of Theorem 4.3. Roughly speaking, the algebra isomorphism in the hereditary case yields an algebraic enrichment of the correspondence in Corollary 6.9. For a different enrichment, we refer to the isomorphism in [6, Lemma 21], which is over finite fields.

Proposition 6.10. *Let $\mathcal{C} = \mathcal{C}(C, D, \Omega)$ be as above, and let p , the characteristic of k , be zero or coprime to each entry of D . Assume that k has enough roots of unity. Then there is an isomorphism of algebras*

$$k\mathcal{C} \xrightarrow{\sim} k\Delta,$$

where Δ is the acyclic quiver corresponding to (C, D, Ω) in Corollary 6.9.

Proof. Recall the construction of (C', D', Ω') in Subsection 4.2. It follows that D' is the identity matrix and C' is symmetric. In particular, the algebra $H(C', D', \Omega')$ is isomorphic to the path algebra $k\Delta$ for some acyclic quiver Δ ; compare Remark 3.2. Indeed, the quiver Δ is exactly the one in Corollary 6.9, which corresponds to (C, D, Ω) . Then we are done by Theorem 4.3. \square

Remark 6.11. It is not clear how the admissible automorphism σ of Δ interacts with the above algebra isomorphism. In particular, it would be nice to link this isomorphism to the representation theory of quivers with automorphisms; see [6].

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