

Singular equivalences and Auslander-Reiten conjecture

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Abstract

Auslander-Reiten conjecture, which says that an Artin algebra does not have any non-projective generator with vanishing self-extensions in all positive degrees, is shown to be invariant under certain singular equivalences induced by adjoint pairs, which occur often in matrix algebras, recollements and change of rings. Accordingly, several reduction methods are established to study this conjecture.

1 Introduction

In the representation theory of algebras, the celebrated Nakayama conjecture (**NC** for short) states that an Artin algebra A is self-injective provided that all terms in a minimal injective resolution of A are projective. In [23], Müller restated the Nakayama conjecture and proved that the Nakayama conjecture holds for all algebras if and only if for each algebra A , a generator-cogenerator M of A with $\text{Ext}_A^i(M, M)$ vanishing for all $i > 0$ is necessarily projective. In [2], Auslander and Reiten proposed a conjecture which is an analogue of Müller' theorem.

Auslander-Reiten conjecture (ARC): *A generator (may not be a cogenerator) M of A with $\text{Ext}_A^i(M, M) = 0$ for all $i > 0$ must be projective, or equivalently, a finitely generated A -module X satisfying $\text{Ext}_A^i(X, X \oplus A) = 0$ for all $i > 0$ is necessarily projective.*

ARC is closely connected with NC and several famous conjectures in the representation theory of Artin algebras. For example, let A be an Artin algebra.

Finitistic dimension conjecture(FDC): *The finitistic dimension of A (the supremum of the projective dimensions of finitely generated A -modules with finite projective dimension) is finite.*

Strong Nakayama conjecture(SNC): *For each nonzero finitely generated A -module M , there is an integer $n \geq 0$ such that $\text{Ext}_A^n(M, A) \neq 0$ [11].*

Generalized Nakayama conjecture(GNC): *Every indecomposable injective A -module occurs as a direct summand of a term in a minimal injective resolution of ${}_A A$ [2].*

All these conjectures are widely open. The relationship among them is stated as follows. For more details, we refer the readers to [2], [37, Theorem 3.4.3].

- For an individual algebra, $\text{FDC} \Rightarrow \text{SNC} \Rightarrow \text{GNC} \Rightarrow \text{NC}$.
- GNC holds for all Artin algebras if and only if so does ARC for all Artin algebras.

Thus, FDC holds for all algebras implies that ARC holds for all algebras. However, the implication is unknown for individual algebras. For example, the finitistic dimension conjecture is known true for self-injective algebras, but ARC is still open for these algebras [18, Theorem 3.4].

ARC is known to be true for several special classes of algebras. For example, algebras of finite representation type, torsionless-finite algebras, symmetric biserial algebras, algebras

with radical square zero, local algebras with radical cube zero [2, 35, 36]. ARC also holds for an algebra satisfying the **(Fg)** condition [14]. This kinds of algebras include group algebras of finite groups [7], finite group schemes [15], commutative complete intersections [3], quantum complete intersections where $q_{i,j}$ are roots of unity [26], and so on.

This article is devoted to studying ARC in the context of singularity categories of algebras. The singularity category $\mathcal{D}_{\text{sg}}(A)$ of an algebra A is defined as the Verdier quotient

$$\mathcal{D}_{\text{sg}}(A) = \mathcal{D}^b(A\text{-mod}) / \mathcal{K}^b(A\text{-proj})$$

of the bounded derived category $\mathcal{D}^b(A\text{-mod})$ by the subcategory of bounded complexes of projective modules [6]. Two algebras are called *singularly equivalent* if their singularity categories are equivalent as triangulated categories.

It is well-known that there is a full embedding

$$\underline{\perp}A \hookrightarrow \mathcal{D}_{\text{sg}}(A)$$

sending a module to the corresponding stalk complex concentrated in degree zero, where

$$\underline{\perp}A := \{X \in A\text{-mod} \mid \text{Ext}_A^i(X, A) = 0 \text{ for all } i > 0\}$$

and $\underline{\perp}A$ is the additive quotient category of $\underline{\perp}A$ modulo projective modules. The embedding induces an isomorphism

$$\text{Ext}_A^i(X, Y) \cong \text{Hom}_{\mathcal{D}_{\text{sg}}(A)}(X, Y[i])$$

for each $X, Y \in \underline{\perp}A$ and for each $i > 0$.

ARC holds for A precisely means that $\underline{\perp}A$ has no nonzero objects X with $\text{Ext}_A^i(X, X) = 0$ for all positive i . Recall that an object T in a triangulated category \mathcal{T} is *presilting* if $\text{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for all $i > 0$. As a consequence, if $\mathcal{D}_{\text{sg}}(A)$ has no nonzero presilting objects, then ARC holds for A . Therefore, it is natural to conjecture: *The singularity category of any algebra contains no nonzero presilting objects.* We call this singular presilting conjecture (SPC). Observe that SPC implies ARC, and the converse is also true if A is a Gorenstein algebra — the above embedding is an equivalence in this case. Hence SPC might be used as a tool to study ARC. Obviously, SPC is invariant under singular equivalences. This implies that ARC is preserved under singular equivalences between Gorenstein algebras. One can ask a more general question.

Question: *Do singular equivalences preserve ARC?*

If two algebras are derived equivalent (they are certainly also singularly equivalent), then the answer to the above question is yes. This was proved by Wei in [34] (see [29, 13] for derived invariance of a generalized version of ARC). In this paper, we shall consider singular equivalences induced by adjoint pairs and show that many singular equivalences do preserve ARC.

Our first result is the following, which is listed in Theorem 3.6.

Theorem 1.1. *Let A, B be two algebras. Suppose that there are triangle functors*

$$\begin{array}{ccc} & L & \\ & \curvearrowright & \\ \mathcal{D}(B\text{-Mod}) & \xrightarrow{F} & \mathcal{D}(A\text{-Mod}) \\ & \curvearrowleft & \\ & G & \\ & \curvearrowright & \\ & H & \\ & \curvearrowleft & \\ & K & \end{array}$$

between the unbounded derived categories of A and B such that $(L, F), (F, G), (G, H)$ and (H, K) are all adjoint pairs. Assume that G induces a singular equivalence, and that H preserves bounded complexes of projective modules. Then the ARC holds for A if and only if it holds for B .

Applying Theorem 1.1 to ladders (see Section 3), we get the following theorem (see Theorem 3.8).

Theorem 1.2. *Let A , B and C be three algebras, and there is a ladder of height 3*

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j^*} & \\ & \xleftarrow{i_*} & & \xleftarrow{j_*} & \\ \mathcal{D}(B\text{-Mod}) & \xleftarrow{i^!} & \mathcal{D}(A\text{-Mod}) & \xleftarrow{j^!} & \mathcal{D}(C\text{-Mod}) \\ & \xleftarrow{i^?} & & \xleftarrow{j^?} & \end{array}.$$

Then ARC holds for A implies that it holds for B . Moreover, if the ladder can be completed to a ladder of height 4, then the following statements hold.

- (1). *If the ARC holds on A , then it holds on B and C ;*
- (2). *If C (resp. B) has finite global dimension, then ARC holds on A if and only if it holds on B (resp. C).*

Theorem 1.1 can also be applied to singular equivalences induced by tensor functors. The following result is listed in Theorem 4.1.

Theorem 1.3. *Suppose that X^\bullet is a B - A -bimodules complex which is perfect over B and A , and assume $Y^\bullet := \mathbf{R}\mathbf{Hom}_B(X^\bullet, B)$ is a perfect complex of A -modules. If $X^\bullet \otimes_A^{\mathbf{L}} -$ induces singular equivalences between A and B , then ARC holds for B implies that it holds for A . If moreover $\mathbf{R}\mathbf{Hom}_A(Y^\bullet, A) \in \mathcal{K}^b(B\text{-proj})$, then ARC holds for A if and only if it holds for B .*

Our result can be applied to give several reduction methods on ARC. Let A be a lower triangular matrix, and e be an idempotent. Under certain conditions, A satisfies ARC if and only if so does eAe (see Corollary 3.9). As an immediate application, ARC is preserved under one-point (co-)extensions (see Example 5.3). Thus, for quiver algebras, ARC is invariant when we remove a sink or a source. Let AeA be a hereditary ideal of A such that eA has finite injective dimension as a right A -module. We also show that ARC holds for A if and only if it holds for A/AeA (see Example 5.4).

This paper is organized as follows. We collect necessary facts in Section 2, and investigate ARC under singular equivalences induced by adjoint pairs and recollements in Section 3 and 4. Some examples are given in the final section.

2 Preliminaries

In this section we fix our notation and recall some basic facts for later proofs.

As we mentioned at the beginning of the previous section, all algebras are finite dimensional algebras over a fixed field k . Let A be such an algebra. The opposite algebra of A is denoted by A^{op} and the enveloping algebra $A \otimes_k A^{\text{op}}$ is denoted by A^e . We identify A - A -bimodules with left A^e -modules. We denote by $A\text{-Mod}$ the category of left A -modules, by $A\text{-mod}$ its subcategory consisting of finitely generated left A -modules, by $A\text{-Proj}$ its subcategory consisting of projective left A -modules, and by $A\text{-proj}$ its subcategory consisting of finitely generated projective left A -modules.

Let \mathcal{X} be a full subcategory of $A\text{-Mod}$. If \mathcal{X} is not contained in $A\text{-mod}$, then we define

$${}^\perp \mathcal{X} := \{Y \in A\text{-Mod} \mid \text{Ext}_A^i(Y, X) = 0, \forall i > 0, X \in \mathcal{X}\},$$

otherwise, we write

$${}^\perp \mathcal{X} := \{Y \in A\text{-mod} \mid \text{Ext}_A^i(Y, X) = 0, \forall i > 0, X \in \mathcal{X}\}.$$

The corresponding additive quotient modulo projectives is denoted by ${}^\perp \underline{\mathcal{X}}$. An A -module M contained in ${}^\perp M$ is called *self-orthogonal*.

The following result is well-known.

Lemma 2.1. *Let A be an algebra. Then*

$${}^{\perp}(A\text{-Proj}) \cap A\text{-mod} = {}^{\perp}A.$$

Proof. It is clear that the left hand is contained in the right hand. For any $X \in {}^{\perp}A \subseteq A\text{-mod}$, one has

$$\text{Ext}_A^i(X, \coprod_{j \in I} A) \simeq \coprod_{j \in I} \text{Ext}_A^i(X, A)$$

for any index set I . Thus, $X \in {}^{\perp}(A\text{-Proj})$. \square

For an algebra A , the unbounded derived category of $A\text{-Mod}$ is denoted by $\mathcal{D}(A\text{-Mod})$. As usual, we write $\mathcal{D}^b(A\text{-Mod})$ (resp. $\mathcal{D}^b(A\text{-mod})$) for the full subcategory consisting of bounded complexes of left A -modules (resp. finitely generated left A -modules), and write $\mathcal{K}^b(A\text{-Proj})$ (resp. $\mathcal{K}^b(A\text{-proj})$) for the full subcategory consisting of bounded complexes of projective modules (resp. finitely generated projective modules). From [20], the “big” singularity category of A is defined to be the following Verdier quotient

$$\mathcal{D}_{\text{sg}}(A\text{-Mod}) = \mathcal{D}^b(A\text{-Mod}) / \mathcal{K}^b(A\text{-Proj}).$$

It is well-known that the above mentioned unbounded and bounded derived categories, and the singularity category are all triangulated categories. We refer to Happel’s book [17] for basic results on triangulated categories. For an object X in a triangulated category, we write $X[n]$ for the object obtained from X by applying the shift functor n times. The object X is called *presilting* if $\text{Hom}(X, X[n]) = 0$ for all $n > 0$.

Note that there is a full embedding

$$\underline{{}^{\perp}(A\text{-Proj})} \hookrightarrow \mathcal{D}_{\text{sg}}(A\text{-Mod})$$

which induces a natural isomorphism

$$\text{Ext}_A^i(X, Y) \cong \text{Hom}_{\mathcal{D}_{\text{sg}}(A\text{-Mod})}(X, Y[i])$$

for each $i > 0$. Thus, every self-orthogonal object in $\underline{{}^{\perp}(A\text{-Proj})}$ is a presilting object in $\mathcal{D}_{\text{sg}}(A\text{-Mod})$.

A complex X^{\bullet} of (finitely generated) A -modules is a sequence of (finitely generated) A -module homomorphisms

$$X^{\bullet} : \cdots \rightarrow X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} X^{i+2} \rightarrow \cdots$$

such that $d^{i+1}d^i = 0$ for all $i \in \mathbb{Z}$. The i -th homology of X^{\bullet} is denoted by $H^i(X^{\bullet})$. For a right A -module M and a left A -module N , denote by $M \otimes_A X^{\bullet}$ and $\text{Hom}_A(X^{\bullet}, N)$ the complexes

$$\cdots \rightarrow M \otimes_A X^i \xrightarrow{1 \otimes d^i} M \otimes_A X^{i+1} \rightarrow \cdots$$

and

$$\cdots \rightarrow \text{Hom}_A(X^{i+1}, N) \xrightarrow{\text{Hom}_A(d^i, N)} \text{Hom}_A(X^i, N) \rightarrow \cdots$$

respectively. Note that the i -th term of $\text{Hom}_A(X^{\bullet}, N)$ is $\text{Hom}_A(X^{-i}, N)$ for all $i \in \mathbb{Z}$.

The following will be useful in our later discussion.

Lemma 2.2 ([24, Theorem 4.1 and 5.1] and [1, Lemma 2.7]). *Let A and B be two algebras, and $F : \mathcal{D}(A\text{-Mod}) \rightarrow \mathcal{D}(B\text{-Mod})$ be a triangle functor with a right adjoint G . Consider the following conditions*

- (1) F preserves $\mathcal{K}^b(\text{proj})$;
- (2) G preserves coproducts;
- (3) G admits a right adjoint;
- (4) G preserves $\mathcal{D}^b(\text{mod})$;
- (5) G preserves $\mathcal{D}^b(\text{Mod})$.

Then we have $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5)$.

3 Singular equivalences induced by adjoint tuples

In this section, we consider ARC and singular equivalences induced by adjoint pairs. First, let us recall from [19] the definition of non-negative functors.

Definition 3.1 ([19, Definition 4.1]). Let A and B be two algebras. A triangle functor

$$G : \mathcal{D}^b(B\text{-Mod}) \rightarrow \mathcal{D}^b(A\text{-Mod})$$

is called *non-negative* if G satisfies the following two conditions:

- (1) $G(X)$ is isomorphic to a complex with zero homology in all negative degrees for all $X \in B\text{-Mod}$;
- (2) $G(P)$ is isomorphic to a complex in $\mathcal{K}^b(A\text{-Proj})$ with zero terms in all negative degrees for all $P \in B\text{-Proj}$.

The following proposition taken from [19] on non-negative functors will be crucial for our discussion.

Proposition 3.2 ([19, Proposition 4.8 and 5.2]). *Let $G : \mathcal{D}^b(B\text{-Mod}) \rightarrow \mathcal{D}^b(A\text{-Mod})$ be a non-negative triangle functor admitting a right adjoint H which preserving $\mathcal{K}^b(\text{Proj})$. Then there is a commutative diagram*

$$\begin{array}{ccc} \underline{\perp}(B\text{-Proj}) & \xrightarrow{\bar{G}} & \underline{\perp}(A\text{-Proj}) \\ \downarrow \iota_B & & \downarrow \iota_A \\ \mathcal{D}_{\text{sg}}(B\text{-Mod}) & \xrightarrow{\tilde{G}} & \mathcal{D}_{\text{sg}}(A\text{-Mod}), \end{array} \quad (3.1)$$

where ι_B and ι_A are natural embeddings.

In certain cases, the above commutative diagram indicates the relationship between the validity of ARC for A and B , as is shown in the the following lemma.

Lemma 3.3. *Assume that we have the above commutative diagram (3.1). If \bar{G} preserves finitely generated modules, and \tilde{G} is fully faithful, then ARC holds for A implies that it holds for B .*

Proof. Let X be a self-orthogonal object in $\underline{\perp}B$. Then X is finitely generated and contained in $\underline{\perp}(B\text{-Proj})$ by Lemma 2.1. Since the embedding ι_B takes self-orthogonal objects to presilting objects, the object $\iota_B(X)$ and thus $\tilde{G}\iota_B(X)$ is presilting. By the commutative diagram (3.1), one has $\iota_A\bar{G}(X) = \tilde{G}\iota_B(X)$. It follows that $\bar{G}(X)$ is self-orthogonal. By our assumption $\bar{G}(X)$ is still finitely generated and thus belongs to $\underline{\perp}A$ by Lemma 2.1. By assumption ARC holds for A . This implies that $\bar{G}(X)$ is isomorphic to the zero object in $\underline{\perp}A$. Since \tilde{G} is fully faithful, by the commutative diagram (3.1), one can see that \tilde{G} is fully faithful. Hence X must be isomorphic to the zero object in $\underline{\perp}B$.

Altogether, we have shown that $\underline{\perp}B$ does not contain any nonzero self-orthogonal objects, that is, ARC holds for B . \square

Lemma 3.3 will serve as our main idea to study the relationship between singular equivalences and ARC. However, at this stage, we don't know how to get non-negative functors that inducing singular equivalences or fully faithful functor between the singularity categories.

If G is a derived equivalence, then $G[i]$ is non-negative for some integer i . In general, the following lemma shows that adjoint pairs of triangle functors between derived categories may give rise to non-negative functors.

Lemma 3.4. *Let*

$$\mathcal{D}(B\text{-Mod}) \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{D}(A\text{-Mod})$$

be an adjoint pair with both F and G preserving $\mathcal{K}^b(\text{proj})$. Then, up to shifts, G restricts to a non-negative functor from $\mathcal{D}^b(B\text{-Mod})$ to $\mathcal{D}^b(A\text{-Mod})$.

Proof. Since F preserves $\mathcal{K}^b(\text{proj})$, it follows from Lemma 2.2 that G preserves $\mathcal{D}^b(\text{Mod})$ and coproducts.

By assumption both $G(B)$ and $F(A)$ are bounded complexes of finitely generated projectives. Assume that $F(A)$ and $G(B)$ are of the following form:

$$G(B) : \quad \cdots \rightarrow 0 \rightarrow P^{-m} \rightarrow \cdots \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots \rightarrow P^n \rightarrow 0 \rightarrow \cdots,$$

$$F(A) : \quad \cdots \rightarrow 0 \rightarrow Q^{-r} \rightarrow \cdots \rightarrow Q^0 \rightarrow Q^1 \rightarrow \cdots \rightarrow Q^s \rightarrow 0 \rightarrow \cdots.$$

Set $t = \max\{m, s\}$, $\hat{F} = F[t]$ and $\hat{G} = G[-t]$. Then $\hat{G}(B) = G(B)[-t]$ is a complex in $\mathcal{K}^b(A\text{-proj})$ with zero terms in all negative degrees, and $\hat{F}(A) = F[A][t]$ is a complex in $\mathcal{K}^b(B\text{-proj})$ with zero terms in all positive degrees. Since \hat{G} preserves coproducts, the complex $\hat{G}(P)$ is isomorphic to a complex in $\mathcal{K}^b(A\text{-Proj})$ with zero terms in all negative degrees for all $P \in B\text{-Proj}$.

For any $X \in B\text{-Mod}$ and any integer i , we have isomorphisms

$$\begin{aligned} H^i(\hat{G}(X)) &\simeq \text{Hom}_{\mathcal{D}^b(A\text{-Mod})}(A, \hat{G}(X)[i]) \\ &\simeq \text{Hom}_{\mathcal{D}^b(B\text{-Mod})}(\hat{F}(A), X[i]) \\ &\simeq \text{Hom}_{\mathcal{K}^b(B\text{-Mod})}(\hat{F}(A), X[i]), \end{aligned}$$

where the second isomorphism follows from the adjointness of \hat{F} and \hat{G} . Therefore, $H^i(\hat{G}(X)) = 0$ for any $i < 0$, that is, $\hat{G}(X)$ has no homology in negative degrees. This proves that \hat{G} is non-negative. \square

Corollary 3.5. *Suppose that (F, G, H) in the following diagram is an adjoint triple of triangulated functors.*

$$\mathcal{D}(B\text{-Mod}) \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \\ \xleftarrow{H} \end{array} \mathcal{D}(A\text{-Mod})$$

Assume that G preserves $\mathcal{K}^b(\text{proj})$ and H preserves $\mathcal{K}^b(\text{Proj})$. Then we have the commutative diagrams (3.1). Moreover, if G induces a fully faithful functor between the singularity categories of B and A , then ARC holds for A implies that it holds for B .

Proof. Since G has a right adjoint H , by Lemma 2.2, we have that F preserves $\mathcal{K}^b(\text{proj})$. It follows from Lemma 3.4 that, up to shifts, G restricts to a non-negative functor from $\mathcal{D}^b(B\text{-Mod})$ to $\mathcal{D}^b(A\text{-Mod})$. By Proposition 3.2, one gets the commutative diagram (3.1). Moreover, F preserves $\mathcal{K}^b(\text{proj})$ implies that G preserves $\mathcal{D}^b(\text{mod})$. Taking $X \in B\text{-mod}$ and using [19, Section 4.2], we have that $\overline{G}(X)$ is the cokernel of some map appeared in the projective resolution of $G(X)$. Therefore, the fact that G preserves $\mathcal{D}^b(\text{mod})$ implies that \overline{G} preserves finitely generated modules. Now the statement follows from Lemma 3.3. \square

A sequence of functors (F_1, F_2, \dots, F_r) between two categories is called an *adjoint tuple* if (F_i, F_{i+1}) is an adjoint pair for all $i = 1, \dots, r-1$.

Theorem 3.6. *Let A and B be two algebras. Suppose that the sequence $(F_1, F_2, F_3, F_4, F_5)$ of triangle functors in the following diagram is an adjoint tuple.*

$$\begin{array}{ccc}
 & F_1 & \\
 & \curvearrowright & \\
 & F_2 & \\
 \mathcal{D}(B\text{-Mod}) & \xrightarrow{F_3} & \mathcal{D}(A\text{-Mod}) \\
 & \curvearrowleft & \\
 & F_4 & \\
 & F_5 &
 \end{array}$$

Assume that F_3 induces a singular equivalence between B and A . If F_4 preserves $\mathcal{K}^b(\text{Proj})$, then ARC holds for A if and only if it holds for B .

Proof. By Lemma 2.2, the functors F_1, F_2 and F_3 preserve $\mathcal{K}^b(\text{proj})$. Now consider the adjoint triple (F_2, F_3, F_4) . Since F_4 preserves $\mathcal{K}^b(\text{Proj})$ and F_3 induces a singular equivalence, by Corollary 3.5, ARC holds for A implies that it holds for B .

Conversely, consider the adjoint triple (F_1, F_2, F_3) . By Lemma 2.2, the functors F_2 and F_3 preserve both $\mathcal{K}^b(\text{proj})$ and $\mathcal{D}^b(\text{mod})$. Thus, F_2 and F_3 induce an adjoint pair between the singularity categories of A and B . Since F_3 induces a singular equivalence, so does F_2 . Moreover F_3 also preserves $\mathcal{K}^b(\text{Proj})$ since it preserves both $\mathcal{K}^b(\text{proj})$ and coproducts. Hence, applying Corollary 3.5 to the adjoint triple (F_1, F_2, F_3) , we conclude that ARC holds for B implies that it holds for A . \square

Corollary 3.7. *Let A and B be two algebras. Suppose that the sequence (F_1, \dots, F_6) of triangle functors in the following diagram is an adjoint tuple.*

$$\begin{array}{ccc}
 & F_1 & \\
 & \curvearrowright & \\
 & F_2 & \\
 \mathcal{D}(B\text{-Mod}) & \xrightarrow{F_3} & \mathcal{D}(A\text{-Mod}) \\
 & \curvearrowleft & \\
 & F_4 & \\
 & F_5 & \\
 & F_6 &
 \end{array}$$

Assume that F_3 induces a singular equivalence between B and A . Then ARC holds for A if and only if it holds for B .

Proof. According to Theorem 3.6, it suffices to prove that F_4 preserves $\mathcal{K}^b(\text{Proj})$. However, this follows easily from the fact that F_4 preserves $\mathcal{K}^b(\text{proj})$ and coproducts by Lemma 2.2. \square

An immediate consequence of Corollary 3.7 is that derived equivalences preserve ARC, which was proved by Wei in [34]. Actually, a derived equivalence F and its quasi-inverse F^{-1} give rise to an adjoint tuple $(F, F^{-1}, F, F^{-1}, \dots)$ of arbitrary length, and F induces a singular equivalence.

Adjoint tuples typically occur in recollements and ladders.

Let \mathcal{T}_1 , \mathcal{T} and \mathcal{T}_2 be triangulated categories. A *recollement* [4] of \mathcal{T} relative to \mathcal{T}_1 and \mathcal{T}_2 is a diagram

$$\begin{array}{ccccc}
 & i^* & & j_! & \\
 & \curvearrowright & & \curvearrowleft & \\
 & i_* = i_! & \xrightarrow{\quad} & j^! = j^* & \\
 & \curvearrowleft & & \curvearrowright & \\
 & i^! & & j_* &
 \end{array}
 \quad \mathcal{T}_1 \quad \mathcal{T} \quad \mathcal{T}_2$$

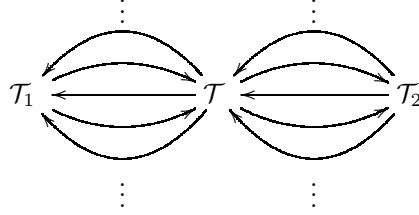
of triangulated categories and triangle functors such that

- (1) $(i^*, i_*, i^!)$, $(j_!, j^!, j_*)$ are adjoint triples of triangle functors;
- (2) i_* , $j_!$ and j_* are full embeddings;
- (3) $j^! i_* = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);
- (4) for each $X \in \mathcal{T}$, there are triangles

$$\begin{array}{c} j_! j^! X \rightarrow X \rightarrow i_* i^* X \rightarrow \\ i_! i^! X \rightarrow X \rightarrow j_* j^* X \rightarrow \end{array},$$

where the maps are given by adjunctions.

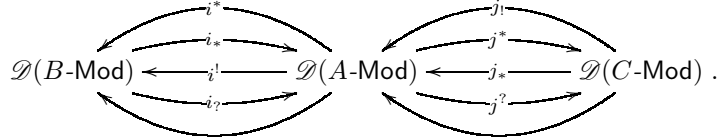
A *ladder* [1] is a finite or infinite diagram of triangulated categories and triangle functors



such that any three consecutive rows form a recollement. The *height* of a ladder is the number of recollements contained in it (counted with multiplicities).

Now, we are ready to apply our discussion to ladders.

Theorem 3.8. *Suppose that A , B and C are algebras, and there is a ladder of height 3*

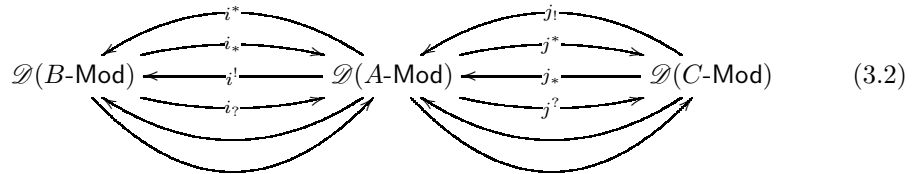


Then ARC holds for A implies that it holds for B . Moreover, if the ladder can be completed to a ladder of height 4, then the following statements hold.

- (1). If the ARC holds for A , then it holds for B and C ;
- (2). i_* (resp. j_*) induces a singular equivalence if and only if C (resp. B) has finite global dimension, and in this case, the ARC holds for A if and only if it holds for B (resp. C).

Proof. It follows from Lemma 2.2 that i^* , i_* and $i^!$ preserve $\mathcal{K}^b(\text{proj})$, and $i^!$ preserves coproducts. Therefore, $i^!$ preserves $\mathcal{K}^b(\text{Proj})$. Moreover, i_* is a fully faithful functor which preserve $\mathcal{K}^b(\text{proj})$ and $\mathcal{D}^b(\text{mod})$, and then the induced functor $\tilde{i}_* : \mathcal{D}_{\text{sg}}(B) \rightarrow \mathcal{D}_{\text{sg}}(A)$ is also fully faithful, see [28, Lemma 1.2]. Applying Corollary 3.5 to the adjoint triple $(i^*, i_*, i^!)$, we get that the ARC holds for B provided it holds for A .

Assume the ladder can be completed to a ladder of height 4. Without loss of generality, we may assume it extend one step downwards, and the upward case can be proved similarly. Consider the following ladder of height 4



Then, it follows from Lemma 2.2 that j_* and $j^?$ preserve $\mathcal{K}^b(\text{proj})$. Moreover, $j^?$ preserves coproducts and then it preserves $\mathcal{K}^b(\text{Proj})$. On the other hand, j_* induces a fully faithful functor between the corresponding singularity categories. Now, applying Corollary 3.5 to the adjoint triple $(j^*, j_*, j^?)$, we get that the ARC holds for C if it holds for A .

By [21, Proposition 2.5], the functors i_* , $i^!$, $i_?$, j^* , j_* and $j^?$ induce a recollement between the corresponding singularity categories. Therefore, i_* induces a singular equivalence if and only if $\mathcal{D}_{\text{sg}}(C) = 0$, and this occurs precisely when C has finite global

dimension. In this case, $i^!$ also induces a singular equivalence. Applying Corollary 3.7 to the left part of the ladder 3.2, we obtain that the ARC holds for A if and only if it holds for B . Similarly, j^* induces a singular equivalence if and only if B has finite global dimension, and in this case, the ARC holds for A if and only if it holds for C . \square

Theorem 3.8 can be applied to triangle matrix algebras.

Corollary 3.9. *Let $A = \begin{pmatrix} B & 0 \\ {}_C M_B & C \end{pmatrix}$ be a triangular matrix algebra, where B, C are algebras and M a finitely generated C - B -bimodules. Then following statements hold.*

- (1). *If $\text{gl.dim } B < \infty$ and $\text{proj.dim}_C M < \infty$, then the ARC holds for A if and only if it holds for C ;*
- (2). *If $\text{gl.dim } C < \infty$ and $\text{proj.dim}(M)_B < \infty$, then the ARC holds for A if and only if it holds for B .*

Proof. Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It follows from [1, Example 3.4] that there is a ladder of height 2

$$\begin{array}{ccccc} & & & & Ae_1 \otimes_B^{\mathbf{L}} - \\ & & & & \nearrow \\ \mathcal{D}(C\text{-Mod}) & \xrightarrow{Ae_2 \otimes_C^{\mathbf{L}} -} & \mathcal{D}(A\text{-Mod}) & \xrightarrow{\quad} & \mathcal{D}(B\text{-Mod}). \\ & \xleftarrow{e_2 A \otimes_A^{\mathbf{L}} -} & \nwarrow & & \\ & & & & \end{array} \quad (3.3)$$

If $\text{gl.dim } B < \infty$ and $\text{proj.dim}_C M < \infty$, then $\text{proj.dim } M_B < \infty$ and by [1, Example 3.4], the ladder (3.3) can be extended to a ladder of height 4. Applying Theorem 3.8 to this ladder, we have that the ARC holds for A if and only if it holds for C . The second case can be proved similarly. \square

Recall that an A -module M is *Gorenstein projective* if there are short exact sequences

$$0 \longrightarrow X_i \longrightarrow P_i \longrightarrow X_{i+1} \longrightarrow 0, i \in \mathbb{Z}$$

in $A\text{-mod}$ with $X_i \in {}^\perp A$ for all i such that $M = X_0$. Particularly, all Gorenstein projective modules are contained in ${}^\perp A$. Denote by $A\text{-Gproj}$ the full subcategory of $A\text{-mod}$ consisting of all Gorenstein projective modules. The stable category of $A\text{-Gproj}$ is a triangulated category and is contained in ${}^\perp A$. Thus, one gets full embeddings

$$\underline{A\text{-Gproj}} \hookrightarrow {}^\perp A \hookrightarrow \mathcal{D}_{\text{sg}}(A).$$

If ARC holds for A , that is, ${}^\perp A$ does not contain any nonzero self-orthogonal objects, then there is no nonzero self-orthogonal objects in $\underline{A\text{-Gproj}}$. It is then natural to conjecture:

Gorenstein projective Conjecture (GPC for short): A finitely generated Gorenstein projective module M over A is projective if $\text{Ext}_A^i(M, M) = 0$, for any $i \geq 1$.

This conjecture was stated by Luo and Huang in [22]. Let $G : \mathcal{D}^b(B\text{-mod}) \rightarrow \mathcal{D}^b(A\text{-mod})$ be a non-negative triangle functor (a small module version of definition 3.1) admitting a right adjoint H which preserving $\mathcal{K}^b(\text{proj})$. Due to [19, Proposition 5.3], we have a commutative diagram

$$\begin{array}{ccc} \underline{B\text{-Gproj}} & \xrightarrow{\bar{G}} & \underline{A\text{-Gproj}} \\ \downarrow \iota_B & & \downarrow \iota_A \\ \mathcal{D}_{\text{sg}}(B) & \xrightarrow{\bar{G}} & \mathcal{D}_{\text{sg}}(A), \end{array}$$

analogous to Proposition 3.2. Therefore, if we replace ARC by GPC, all results in this section (e.g., Theorem 3.6 and Theorem 3.8) still hold and the proofs are almost identical to the case of ARC.

4 Singular equivalences induced by tensor functors

Let A and B be two algebras, and X^\bullet be a complex of B - A -bimodules. Then the tensor product functor $F = X^\bullet \otimes_A^L - : \mathcal{D}(A\text{-Mod}) \rightarrow \mathcal{D}(B\text{-Mod})$ has a right adjoint $\mathbf{RHom}_B(X^\bullet, -)$. It is natural to ask when F induces an equivalence between the singularity categories, and under which conditions this singular equivalence preserves ARC.

It does happen often that F induces a singular equivalence. Typical examples are singular equivalences of Morita type (with level) when X^\bullet is a module ([38, Theorem 3.1], [33, Definition 2.1]). See [8, Proposition 4.8] for recent progress in this direction. In the general case, it was proved recently by Dalezios [12, Theorem 3.6] that F induces a singular equivalence if $Y^\bullet := \mathbf{RHom}_B(X^\bullet, B)$ is a perfect complex of A -modules, $\mathbf{RHom}_B(X^\bullet, X^\bullet) \cong A$ in $\mathcal{D}_{\text{sg}}(A^e)$, and $X^\bullet \otimes_A^L \mathbf{RHom}_B(X^\bullet, B) \cong B$ in $\mathcal{D}_{\text{sg}}(B^e)$.

The main result of this section is the following theorem.

Theorem 4.1. *Suppose that X^\bullet is a B - A -bimodules complex which is perfect over B and A , and assume $Y^\bullet := \mathbf{RHom}_B(X^\bullet, B)$ is a perfect complex of A -modules. If $X^\bullet \otimes_A^L - : \mathcal{D}(A\text{-Mod}) \rightarrow \mathcal{D}(B\text{-Mod})$ induces a singular equivalence, then ARC holds for B implies that it holds for A . If moreover $\mathbf{RHom}_A(Y^\bullet, A) \in \mathcal{K}^b(B\text{-proj})$, then ARC holds for A if and only if it holds for B .*

Proof. Note that there is an adjoint tuple

$$\begin{array}{ccc} & L & \\ \mathcal{D}(B\text{-Mod}) & \begin{array}{c} \xleftarrow{X^\bullet \otimes_A^L -} \\ \xrightarrow{Y^\bullet \otimes_B^L -} \end{array} & \mathcal{D}(A\text{-Mod}) \\ & \mathbf{RHom}_A(Y^\bullet, -) & \end{array}$$

with $X^\bullet \otimes_A^L -$ and $Y^\bullet \otimes_B^L -$ preserving $\mathcal{K}^b(\text{proj})$. The existence of L follows from the assumption that X^\bullet is a perfect complex. Clearly, $Y^\bullet \otimes_B^L -$ preserves coproducts and thus $Y^\bullet \otimes_B^L -$ preserves $\mathcal{K}^b(\text{Proj})$. Now notice that $X^\bullet \otimes_A^L -$ induce a singular equivalence between B and A , by Corollary 3.5, ARC holds for B implies that it holds for A .

Since $Y^\bullet \otimes_B^L -$ preserves $\mathcal{K}^b(\text{proj})$, it follows from Lemma 2.2 that $\mathbf{RHom}_A(Y^\bullet, -)$ preserves coproducts, and it admits a right adjoint K . Hence, the condition $\mathbf{RHom}_A(Y^\bullet, A) \in \mathcal{K}^b(B\text{-proj})$ implies that $\mathbf{RHom}_A(Y^\bullet, -)$ preserves $\mathcal{K}^b(\text{Proj})$. On the other hand, $(X^\bullet \otimes_A^L -, Y^\bullet \otimes_B^L -)$ gives an adjoint pair between the singularity categories. Since $X^\bullet \otimes_A^L -$ induces a singular equivalence, so does $Y^\bullet \otimes_B^L -$. Therefore, there is an adjoint tuple

$$\begin{array}{ccc} & L & \\ \mathcal{D}(B\text{-Mod}) & \begin{array}{c} \xleftarrow{X^\bullet \otimes_A^L -} \\ \xrightarrow{Y^\bullet \otimes_B^L -} \end{array} & \mathcal{D}(A\text{-Mod}) \\ & \begin{array}{c} \mathbf{RHom}_A(Y^\bullet, -) \\ K \end{array} & \end{array}$$

with $\mathbf{RHom}_A(Y^\bullet, -)$ preserving $\mathcal{K}^b(\text{Proj})$ and $Y^\bullet \otimes_B^L -$ inducing a singular equivalence between B and A . Then we have done by Theorem 3.6. \square

Now we focus on singular equivalences from change of rings, which was studied in [27, 12]. Let $f : A \rightarrow B$ be a morphism of algebras with $\text{proj. dim}_A B < \infty$ and $\text{proj. dim } B_A < \infty$. Recall from [27, Lemma 3.2] that there is an adjoint pair

$$\mathcal{D}(B\text{-Mod}) \begin{array}{c} \xleftarrow{{}_B B \otimes_A^L -} \\ \xrightarrow{{}_A B \otimes_B^L -} \end{array} \mathcal{D}(A\text{-Mod}). \quad (4.1)$$

If $\text{Cone}(f) \in \mathcal{K}^b(A^e\text{-proj})$ (that is, $B \cong A$ in $\mathcal{D}_{\text{sg}}(A^e)$) and ${}_B B \otimes_A^{\mathbf{L}} B_B \xrightarrow{\sim} {}_B B_B$ in $\mathcal{D}_{\text{sg}}(B^e)$, then ${}_B B \otimes_A^{\mathbf{L}} -$ and ${}_A B \otimes_B^{\mathbf{L}} -$ induce mutual equivalences between the singularity categories; see [12, Corollary 3.7].

In particular, $f : A \rightarrow B$ is a *homological epimorphism* if the induced functor

$$f_* = {}_A B \otimes_B^{\mathbf{L}} - : \mathcal{D}(B\text{-Mod}) \rightarrow \mathcal{D}(A\text{-Mod})$$

is a full embedding, or equivalently, there is an isomorphism ${}_B B \otimes_A^{\mathbf{L}} B_B \xrightarrow{\sim} {}_B B_B$ in $\mathcal{D}(B^e\text{-Mod})$; see [16]. In this case, ${}_A B \otimes_B^{\mathbf{L}} -$ induces a fully faithful functor between the corresponding singularity categories.

Corollary 4.2. *Let $f : A \rightarrow B$ be a morphism of algebras with $\text{proj. dim}_A B < \infty$ and $\text{proj. dim}_B A < \infty$. Then the following hold.*

- (1). *If $\text{Cone}(f) \in \mathcal{K}^b(A^e\text{-proj})$, then ARC holds for B implies that it holds for A ;*
- (2). *If moreover f is a homological epimorphism and $\mathbf{RHom}_A(B, A) \in \mathcal{K}^b(B\text{-proj})$, then the ARC holds for A if and only if it holds for B .*

Proof. (1) Since $\text{proj. dim}_B A < \infty$, it follows from [1, Lemma 2.8] that the functor ${}_B B \otimes_A^{\mathbf{L}} -$ in diagram (4.1) has a left adjoint. Moreover, $\text{proj. dim}_A B < \infty$ implies that ${}_A B \otimes_B^{\mathbf{L}} -$ preserves $\mathcal{K}^b(\text{Proj})$, and $\text{Cone}(f) \in \mathcal{K}^b(A^e\text{-proj})$ yields that ${}_B B \otimes_A^{\mathbf{L}} -$ induces a fully faithful functor between the corresponding singularity categories, see [27, Proposition 3.7]. Now the statement (1) follows immediately from Corollary 3.5.

(2) Since $\text{Cone}(f) \in \mathcal{K}^b(A^e\text{-proj})$ and f is a homological epimorphism, we infer that ${}_B B \otimes_A^{\mathbf{L}} -$ and ${}_A B \otimes_B^{\mathbf{L}} -$ induce mutual equivalences between the singularity categories. Now we finish our proof by Theorem 4.1. \square

A special case of algebra homomorphism is the canonical map from an algebra A to its quotient A/I for some ideal I . Applying Corollary 4.2, we get the following results:

- (1). If I has finite projective dimension as an A - A -bimodule, then ARC holds for A/I implies that it holds for A . Indeed, $\text{proj. dim}_{A^e} I < \infty$ yields that $\text{proj. dim}_A(A/I) < \infty$ and $\text{proj. dim}(A/I)_A < \infty$. Moreover, it is clear that the cone of $A \rightarrow A/I$ is $I[1]$, which belongs to $\mathcal{K}^b(A^e\text{-proj})$ by assumption. So, the statement follows from Corollary 4.2 (1).
- (2). A special case of (1) occurs when $I \cong M \otimes_k N$, where M and N are left and right A -modules, respectively, both have finite projective dimension. Another example is that I is a 1-dimensional ideal which has finite projective dimension as left and as right module. We refer to [27, Corollary 3.10 and 3.11] for more explanations.
- (3). In general, it is difficult to check whether the condition $\mathbf{RHom}_A(B, A) \in \mathcal{K}^b(B\text{-proj})$ in Corollary 4.2 (2) is satisfied. However, it is the case when A is a Gorenstein algebra. In fact, assume all conditions in Corollary 4.2 are satisfied, except for $\mathbf{RHom}_A(B, A) \in \mathcal{K}^b(B\text{-proj})$. Then it follows from the proof of Theorem 4.1 that there are adjoint functors

$$\begin{array}{ccc} & \begin{array}{c} \xleftarrow{i^* = B \otimes_A^{\mathbf{L}} -} \\ \xrightarrow{i^! = B \otimes_B^{\mathbf{L}} -} \end{array} & \\ \mathcal{D}(B\text{-Mod}) & \xrightarrow{i_* = B \otimes_B^{\mathbf{L}} -} & \mathcal{D}(A\text{-Mod}) \\ & \begin{array}{c} \xleftarrow{i^!} \\ \xrightarrow{i^*} \end{array} & \end{array}$$

where the functor i_* is fully faithful. Therefore, we get $i^* i_* \cong 1_B$ and $i^! i_* \cong 1_B$, and thus $B \cong i^! i_* B \in i^!(\mathcal{K}^b(A\text{-proj})) = i^!(\mathcal{K}^b(A\text{-inj}))$, which is contained in $\mathcal{K}^b(B\text{-inj})$ by [30, Lemma 1]. As a result, we obtain $\text{inj. dim}_B B < \infty$, and dually, we also have $\text{proj. dim}_B \text{Hom}_k(B, k) < \infty$, that is, B is also a Gorenstein algebra. Therefore, $\mathbf{RHom}_A(B, A) = i^! A \in i^!(\mathcal{K}^b(A\text{-inj})) \subseteq \mathcal{K}^b(B\text{-inj}) = \mathcal{K}^b(B\text{-proj})$.

5 Examples

In this section, we illustrate our results by some examples.

Example 5.1. (Tiled orders) Let A be a finite dimensional algebra, $I_{i,j}$ be an ideal of A . Let us consider tiled triangular rings, i.e., rings of the form

$$\Delta = \begin{pmatrix} A & I_{1,2} & \cdots & I_{1,n} \\ A & A & & \vdots \\ \vdots & & \ddots & I_{n-1,n} \\ A & \cdots & \cdots & A \end{pmatrix}.$$

Proposition 5.2. Assume that $\text{proj. dim}({}_A I_{1,i}) < \infty$, $\text{gl. dim}(A/I_{i-1,i}) < \infty$ for $i = 2, \dots, n$, and A satisfies ARC. Then Δ satisfies ARC.

Proof. As shown in [9, Proposition 4.14], the two algebras

$$\Delta = \begin{pmatrix} A & I_{1,2} & I_{1,3} & \cdots & I_{1,n} \\ A & A & I_{2,3} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & I_{n-1,n} \\ A & A & A & A & A \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} & 0 \\ A/I_{1,2} & A/I_{1,3} & & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} & 0 \\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} & A \end{pmatrix}$$

are derived equivalent. Let us denote $\Phi_1 :=$

$$\begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \\ A/I_{1,2} & A/I_{1,3} & \cdots & I_{3,n}/I_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ A/I_{1,2} & A/I_{1,3} & \cdots & A/I_{1,n} \end{pmatrix}.$$

Since $\text{proj. dim}({}_A I_{1,i}) < \infty$ for $i = 2, \dots, n$, $(A/I_{1,2}, A/I_{1,3}, \dots, A/I_{1,n})$ has finite projective dimension as a left A -module.

Now, we want to show that Φ_1 has finite global dimension. Let e be an idempotent of Φ_1 which has 1 in the $(1,1)$ -th position and zeros elsewhere. It is immediate that

$$\Phi_1 e \Phi_1 = \begin{pmatrix} A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \\ A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \\ \vdots & \vdots & & \vdots \\ A/I_{1,2} & I_{2,3}/I_{1,3} & \cdots & I_{2,n}/I_{1,n} \end{pmatrix}$$

which is projective as a right Φ_1 -module. Note that Δ is a ring. Then, by the multiplication of Δ , we have $I_{1,2}I_{2,i} \subseteq I_{1,i}$ for $i = 3, \dots, n$. So, $I_{2,i}/I_{1,i}$ is a left $A/I_{1,2}$ -module for $i = 3, \dots, n$. It follows from $\text{gl. dim}(A/I_{1,2}) < \infty$ that $I_{2,i}/I_{1,i}$ has finite projective dimension as a left $A/I_{1,2}$ -module for $i = 3, \dots, n$. Hence, $\Phi_1 e \Phi_1$ has finite projective dimension as a left Φ_1 -module.

It follows from $\text{Tor}_i^{\Phi_1}(\Phi_1/\Phi_1 e \Phi_1, \Phi_1/\Phi_1 e \Phi_1) = 0$ for $i > 0$ that the canonical ring homomorphism $\lambda : \Phi_1 \rightarrow \Phi_1/\Phi_1 e \Phi_1$ is a homological epimorphism. Thus, we have an adjoint triple

$$\begin{array}{ccc} & \xleftarrow{-\otimes_{\Phi_1}^L \Phi_1/\Phi_1 e \Phi_1} & \\ \mathcal{D}(\text{Mod-}\Phi_1/\Phi_1 e \Phi_1) & \xrightarrow{\lambda_*} & \mathcal{D}(\text{Mod-}\Phi_1) \\ & \xleftarrow{\text{RHom}_{\Phi_1}(\Phi_1/\Phi_1 e \Phi_1, -)} & \end{array}$$

where λ_* is an embedding, and $-\otimes_{\Phi_1}^L \Phi_1/\Phi_1 e\Phi_1$ and $\mathbf{RHom}_{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1, -)$ are the derived functors of $-\otimes_{\Phi_1} \Phi_1/\Phi_1 e\Phi_1$ and $\mathrm{Hom}_{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1, -)$, respectively. Note that $\Phi_1 e\Phi_1$ is projective as a right Φ_1 -module, and has finite projective dimension as a left Φ_1 -module. Then, the adjoint triple $(-\otimes_{\Phi_1}^L \Phi_1/\Phi_1 e\Phi_1, \lambda_*, \mathbf{RHom}_{\Phi_1}(\Phi_1/\Phi_1 e\Phi_1, -))$ restricts to $\mathcal{D}^b(\mathrm{mod})$ and $\mathcal{K}^b(\mathrm{proj})$, respectively.

By [25, Example 5.3.4], the Verdier localization of $\mathcal{D}(\mathrm{Mod}\text{-}\Phi_1)$ via the essential image of $\mathcal{D}(\mathrm{Mod}\text{-}\Phi_1/\Phi_1 e\Phi_1)$ under λ_* is triangle equivalent to $\mathrm{Tria}_{\mathcal{D}(\mathrm{Mod}\text{-}\Phi_1)}(\Phi_1 e\Phi_1)$ which is the smallest full triangulated subcategory of $\mathcal{D}(\mathrm{Mod}\text{-}\Phi_1)$ containing $\Phi_1 e\Phi_1$ and closed under small coproducts, and $\mathrm{Tria}_{\mathcal{D}(\mathrm{Mod}\text{-}\Phi_1)}(\Phi_1 e\Phi_1)$ is triangle equivalent to the category $\mathcal{D}((\mathcal{C}_{\mathrm{dg}}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1))$ in which $(\mathcal{C}_{\mathrm{dg}}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1)$ is a dg algebra. Since $\Phi_1 e\Phi_1$ is a finitely generated projective right Φ_1 -module, we have a triangle equivalence between $\mathcal{D}((\mathcal{C}_{\mathrm{dg}}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1))$ and $\mathcal{D}(\mathrm{Mod}\text{-}\mathrm{H}^0((\mathcal{C}_{\mathrm{dg}}\Phi_1)(\Phi_1 e\Phi_1, \Phi_1 e\Phi_1)))$. And the latter one is triangle equivalent to $\mathcal{D}(\mathrm{Mod}\text{-}e\Phi_1 e)$. Note that $e\Phi_1 e = A/I_{1,2}$ and $\mathrm{gl.dim}(A/I_{1,2}) < \infty$. Then, Φ_1^{op} and $\Phi_1^{\mathrm{op}}/\Phi_1 e\Phi_1^{\mathrm{op}}$ are singularly equivalent, where Φ_1^{op} and $\Phi_1^{\mathrm{op}}/\Phi_1 e\Phi_1^{\mathrm{op}}$ are the opposite algebras of Φ_1 and $\Phi_1/\Phi_1 e\Phi_1$, respectively. Then, Φ_1 has finite global dimension if and only if so does $\Phi_1/\Phi_1 e\Phi_1$.

It is clear that

$$\Phi_1/\Phi_1 e\Phi_1 = \begin{pmatrix} A/I_{2,3} & I_{3,4}/I_{2,4} & \cdots & I_{3,n}/I_{2,n} \\ A/I_{2,3} & A/I_{2,4} & \cdots & I_{4,n}/I_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A/I_{2,3} & A/I_{2,4} & \cdots & A/I_{2,n} \end{pmatrix}.$$

We write Φ_2 for $\Phi_1/\Phi_1 e\Phi_1$. Recursively, Φ_1 has finite global dimension if and only if so does $A/I_{n-1,n}$. Thus, we have $\mathrm{gl.dim}(\Phi_1) < \infty$. Hence, by Corollary 3.9, Δ satisfies ARC if and only if it holds for A since derived equivalences preserve ARC. \square

Example 5.3. (One-point extensions) Let A be an algebra, and let M be a left A -module. The one-point extension algebra $A[M]$ is defined to be the triangular matrix algebra

$$\begin{bmatrix} k & 0 \\ M & A \end{bmatrix}$$

If M has finite projective dimension, then ARC holds for $A[M]$ if and only if it holds for A . This follows immediately from Corollary 3.9.

Example 5.4. (Quotient algebras) Let A be an algebra, and let e be a primitive idempotent in A such that the multiplication map $Ae \otimes_k eA \rightarrow AeA$ is an isomorphism and $eAe \simeq k$. The ideal AeA is called a heredity ideal in the literature. If furthermore the injective dimension of the right A -module eA is finite, then ARC holds for A if and only if it holds for A/AeA .

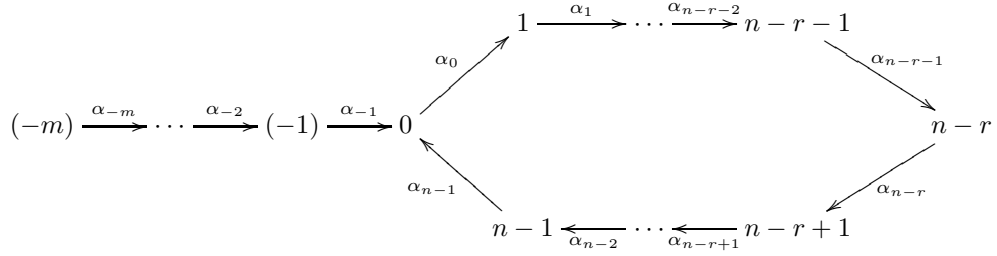
Indeed, it follows from [10] that $\mathcal{D}(A\text{-Mod})$ admits a recollement

$$\begin{array}{ccccc} & \xleftarrow{i^*} & & \xleftarrow{j!} & \\ \mathcal{D}(A/AeA\text{-Mod}) & \xrightarrow{i_*} & \mathcal{D}(A\text{-Mod}) & \xrightarrow{j^*} & \mathcal{D}(eAe\text{-Mod}) \\ & \xleftarrow{i^!} & & \xleftarrow{j_*} & \end{array} \quad (5.1)$$

where $i^* = A/AeA \otimes_A^L -$, $i_* = A/AeA \otimes_{A/AeA}^L -$, $i^! = \mathbf{RHom}_A(A/AeA, -)$, $j! = Ae \otimes_{eAe}^L -$, $j^* = eA \otimes_A^L -$ and $j_* = \mathbf{RHom}_{eAe}(eA, -)$. Clearly, $\mathrm{gl.dim} eAe < \infty$ implies that $j^*A = eA \in \mathcal{K}^b(eAe\text{-proj})$, and $\mathrm{inj.dim} eA_A < \infty$ yields that $j_*(eAe) = \mathbf{RHom}_{eAe}(eA, eAe) = \mathbf{RHom}_k(eA, k) \in \mathcal{K}^b(A\text{-proj})$. Hence, by [1, Lemma 2.5 and Proposition 3.2], the recollement (5.1) can be extended two steps downwards. Similarly, by [1, Lemma 2.8 and Proposition 3.2], $\mathrm{proj.dim} Ae_{eAe} < \infty$ implies that (5.1) can be extended one step upwards. Therefore, (5.1) is completed to a ladder of height 4, and by Theorem 3.8, ARC holds for A if and only if it holds for A/AeA .

Example 5.5. (Derived discrete algebras) From [32], an algebra A is said to be *derived discrete* provided for every positive element $\mathbf{d} \in K_0(A)^{(\mathbb{Z})}$ there are only finitely many isomorphism classes of indecomposable objects X in $\mathcal{D}^b(A)$ of cohomology dimension vector $(\underline{\dim} H^p(X))_{p \in \mathbb{Z}} = \mathbf{d}$. Now we apply our main theorem to show that ARC holds for all derived discrete algebras.

From [5, 32], a basic connected derived discrete algebra A is derived equivalent to either a piecewise hereditary algebra of Dynkin type, or a bound quiver algebra $\Lambda(r, n, m)$ given by



with the relations $\alpha_{n-1}\alpha_0, \alpha_{n-2}\alpha_{n-1}, \dots, \alpha_{n-r}\alpha_{n-r+1}$, where $1 \leq r \leq n$ and $m \geq 0$. Clearly, if $\text{gl.dim } A < \infty$ then ARC always hold true, and if $\text{gl.dim } A = \infty$ then A is derived equivalent to $\Lambda(n, n, m)$, which admits a series of infinite ladders, see [31, Lemma 17]. Note that the right terms of these ladders are k , and the left can be reduced to $\Lambda(n, n, 0)$ consecutively. Hence, applying Theorem 3.8 shows that ARC holds for A if and only if it holds for $\Lambda(n, n, 0)$, and the latter is known as 2-truncated cycle algebra, which is representation finite and then satisfies ARC by [2].

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