# New Rank-One Matrix Decomposition Techniques and Applications to Signal Processing

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# Outline

- Trust-region subproblems in nonlinear programming
- Radar code selection problems
- The new matrix rank-one decomposition techniques
- Theoretical applications
- Optimal transmit beamforming in cognitive radio networks
- Summary

## Trust-Region Subproblem

- The trust-region subproblem[1]:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} - 2\boldsymbol{b}_0^T \boldsymbol{x} \\ \text{subject to} & \|\boldsymbol{x}\|^2 \leq \delta \end{array}$$

- Such type of programs are solved repeatedly in the trust region approach to unconstrained optimization.

- It is a non-convex Quadratically Constrained Quadratic Program (QCQP).

[1] A. Conn, N. Gould, and P. Toint, *Trust-Region Methods*, MPS-SIAM Series on Optimization, 2000.

# The CDT Trust-Region Subproblem

- The CDT (Celis, Dennis, Tapia, 1985) trust-region subproblem:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} - 2\boldsymbol{b}_0^T \boldsymbol{x} \\ \text{subject to} & \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2 \leq \delta_1 \\ & \|\boldsymbol{x}\|^2 \leq \delta_2 \end{array}$$

- To solve it in polynomial time, a sufficient condition is needed[2].

- QCQP is *homogeneous*, if all the quadratic functions (of objective and constraint) have no linear term; otherwise, *inhomogeneous*.

- For instance, the previous two QCQP problems are in a inhomogeneous form.

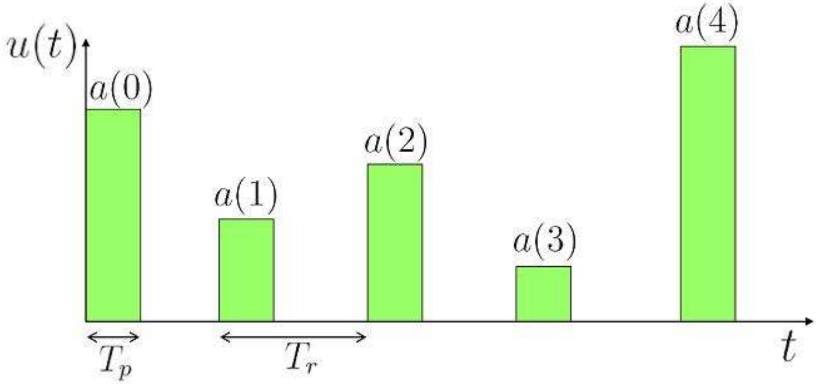
[2] A. Beck and Y. Eldar, "Strong duality in nonconvex quadratic optimization with two quadratic constraints," *SIAM Journal on Optimization*, 2006.

# Optimal Design of Radar Waveform

Consider a scenario of optimum radar detection in the presence of colored disturbance (caused by interference, clutter, and operating enviroment)[3].

[3] Y. Huang, A. De Maio, and S. Zhang, "Semidefinite programming, matrix decomposition, and radar code design," in *Convex Optimization in Signal Processing and Communications*, D. P. Palomar and Y. C. Eldar, Eds., Cambridge University Press, 2020, ch. 6.

- The class of linearly coded pulse trains are considered.




- A sequence of radar code in the transmitted waveforms is determined, with the goals:
  - maximal detection probability;
  - constraining CRB for the target Doppler estimation;
  - controlling the shape of the resulting coded waveform similar to a known radar code;
  - energy constraint.

# Problem Formulation

- A coherent burst of pulses transmitted at the radar:

$$s(t) = a_t u(t) \exp[i(2\pi f_0 t + \phi)]$$

  - $a_t$ is the transmit signal amplitude,
  - $u(t)$ is the signal's complex envelop having the form:

$$u(t) = \sum_{j=0}^{N-1} a(j) p(t - jT_r) \, ,$$

  - $[a(0), a(1), \ldots, a(N-1)]^T \in \mathbb{C}^N$ is the radar code (the optimization variable),
  - $p(t)$ is the signature of the transmitted pulse,
  - $T_r$ is the Pulse Repetition Time (PRT),
  - $f_0$ is the carrier frequency,
  - $\phi$ is a random phase.

- Signal backscattered by a target, and received at the radar:

$$r(t) = \alpha_r e^{i2\pi(f_0+f_d)(t-\tau)} u(t-\tau) + n(t)$$

  - $\tau$ is the two-way time delay of the backscattered signal,
  - $\alpha_r$ is the complex echo amplitude (accounting for the transmit amplitude, phase, target reflectivity, and channels propagation effects),
  - $f_d$ is the target Doppler frequency,
  - $n(t)$ is additive disturbance due to clutter and thermal noise.

# Discrete Signal Model

- The received signal is
  - down-converted to baseband, and
  - filtered through a linear system with impulse response $h(t) = p^*(-t)$, and
  - sampled at $t_k = \tau + kT_r$, $k = 0, 1, \ldots, N-1$.

- The samples $v(t_k)$ form the vector $\boldsymbol{v} = [v(t_0), v(t_1), \ldots, v(t_{N-1})]^T$ satisfying

$$\boldsymbol{v} = \alpha \boldsymbol{c} \odot \boldsymbol{p} + \boldsymbol{w}$$

  - $\alpha = \alpha_r e^{-i2\pi f_0 \tau}$,
  - $\boldsymbol{c} = [a(0), a(1), \ldots, a(N-1)]^T$ is the radar code vector,
  - $\boldsymbol{p} = [1, e^{i2\pi f_d T_r}, \ldots, e^{i2\pi(N-1) f_d T_r}]^T$ is the temporal steering vector,
  - $\boldsymbol{w} = [w(t_0), w(t_1), \ldots, w(t_{N-1})]^T$ is the filtered disturbance samples, assumed to be a zero-mean circular Gaussian vector with known covariance

$$\mathsf{E}[\boldsymbol{w}\boldsymbol{w}^H] = \boldsymbol{M}.$$

# Detection Issues: GLRT Detector

- The problem of detecting a target is formulated in terms of the following binary hypotheses test:

$$\begin{cases} H_0: & \boldsymbol{v} = \boldsymbol{w} \\ H_1: & \boldsymbol{v} = \alpha \boldsymbol{c} \odot \boldsymbol{p} + \boldsymbol{w} \end{cases}$$

- The GLRT is given by

$$|\boldsymbol{v}^H \boldsymbol{M}^{-1} (\boldsymbol{c} \odot \boldsymbol{p})|^2 \underset{H_0}{\overset{H_1}{\gtrless}} G$$

where $G$ is the detection threshold set according to a desired value of $P_{fa}$.

- The detection probability $P_d$ has the analytical expression:

$$P_d = Q\left( \underbrace{\sqrt{2|\alpha|^2 (\boldsymbol{c} \odot \boldsymbol{p})^H \boldsymbol{M}^{-1} (\boldsymbol{c} \odot \boldsymbol{p})}}_{\text{SNR}}, \sqrt{-2 \ln P_{fa}} \right)$$

where $Q(\cdot, \cdot)$ denotes the Marcum $Q$ function of order 1.

# Optimal Radar Code Problem

- The radar code is optimally selected, so that
  - maximize the detection performance (the detection probability), while
  - providing a control both on the target Doppler estimation accuracy and on the similarity with a given radar code $\boldsymbol{c}_0$.

- The optimal radar code problem is formulated as:

$$\begin{array}{ll} \underset{\boldsymbol{c}}{\text{maximize}} & \boldsymbol{c}^H \boldsymbol{R} \boldsymbol{c} \\ \text{subject to} & \boldsymbol{c}^H \boldsymbol{R}_1 \boldsymbol{c} \geq \delta_a \\ & \|\boldsymbol{c} - \boldsymbol{c}_0\|^2 \leq \epsilon \\ & \boldsymbol{c}^H \boldsymbol{c} = 1 \end{array}$$

  - $\boldsymbol{R} = \boldsymbol{M}^{-1} \odot (\boldsymbol{p}\boldsymbol{p}^H)^*$,
  - $\boldsymbol{R}_1 = \boldsymbol{M}^{-1} \odot (\boldsymbol{p}\boldsymbol{p}^H)^* \odot (\boldsymbol{u}\boldsymbol{u}^H)^*$, with $\boldsymbol{u} = [0, i2\pi, \ldots, i2\pi(N-1)]^T$,
  - the feasibility of the problem depends on the parameters $\delta_a$, $\epsilon$, and the pre-fixed code $\boldsymbol{c}_0$ of unit norm.

# Commonalities

- In general, QCQP has the form:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & q_0(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{Q}_0 \boldsymbol{x} - 2\text{Re}\ \boldsymbol{b}_0^H \boldsymbol{x} \\ \text{subject to} & q_j(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{Q}_j \boldsymbol{x} - 2\text{Re}\ \boldsymbol{b}_j^H \boldsymbol{x} + c_j \leq 0,\ j = 1, \ldots, m. \end{array}$$

- The trust-region problems and the radar code selction problem are non-convex QCQP, with a few constraints, in either real or complex variables.

# Matrix Rank-One Decomposition: Symmetric PSD Cases

- **Theorem**[4] Let $\boldsymbol{A} \in \mathbb{S}^n$. Let $\boldsymbol{X} \in \mathbb{S}^n_+$ with rank $r$. Then there is a rank-one decomposition for $\boldsymbol{X}$, i.e., $\boldsymbol{X} = \sum_{j=1}^r \boldsymbol{x}_j \boldsymbol{x}_j^T$, such that

$$\boldsymbol{x}_j^T \boldsymbol{A} \boldsymbol{x}_j = \frac{\boldsymbol{A} \bullet \boldsymbol{X}}{r}, \ j = 1, \ldots, r.$$

- The theorem is true for $\boldsymbol{X}$ being a Hermitian PSD.

- It can be shown easily by example that it is only possible to get a complete rank-one decomposition for one matrix parameter (i.e., $\boldsymbol{A}$).

- For two matrix parameters, it is possible to get a partial decomposition:

[4] J. Sturm and S. Zhang, "On cones of nonnegative quadratic functions," *Mathematics of Operations Research*, vol. 28, no. 2, pp. 246-267, 2003.

- **Theorem**[5] Let $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{S}^n$, and $\boldsymbol{X} \in \mathbb{S}_+^n$ with rank $r$. If $r \geq 3$, then there is a rank-one decomposition for $\boldsymbol{X}$, i.e., $\boldsymbol{X} = \sum_{j=1}^r \boldsymbol{x}_j \boldsymbol{x}_j^T$, such that

$$\boldsymbol{x}_j^T \boldsymbol{A}_1 \boldsymbol{x}_j = \frac{\boldsymbol{A}_1 \bullet \boldsymbol{X}}{r}, \quad j = 1, \ldots, r$$

$$\boldsymbol{x}_j^T \boldsymbol{A}_2 \boldsymbol{x}_j = \frac{\boldsymbol{A}_2 \bullet \boldsymbol{X}}{r}, \quad j = 1, \ldots, r-2.$$

- How about the corresponding Hermitian PSD case?

[5] W. Ai and S. Zhang, "Strong duality for the CDT subproblem: A Necessary and sufficient condition," *SIAM Journal on Optimization*, vol. 19, no. 4, pp. 1735-1756, 2009.

# Matrix Rank-One Decomposition: Hermitian PSD Cases

- **Theorem**[6] Let $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{H}^n$, and $\boldsymbol{X} \in \mathbb{H}_+^n$ with rank $r$. Then there is a rank-one decomposition for $\boldsymbol{X}$, i.e., $\boldsymbol{X} = \sum_{j=1}^r \boldsymbol{x}_j \boldsymbol{x}_j^H$, such that

$$\boldsymbol{x}_j^H \boldsymbol{A}_1 \boldsymbol{x}_j = \frac{\boldsymbol{A}_1 \bullet \boldsymbol{X}}{r}, \quad j = 1, \ldots, r$$

$$\boldsymbol{x}_j^H \boldsymbol{A}_2 \boldsymbol{x}_j = \frac{\boldsymbol{A}_2 \bullet \boldsymbol{X}}{r}, \quad j = 1, \ldots, r.$$

- Can we do more?

[6] Y. Huang and S. Zhang, "Complex matrix decomposition and quadratic programming," *Mathematics of Operations Research*, vol. 32, no. 3, pp. 758-768, 2007.

- **Theorem**[7] Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3 \in \mathbb{H}^n$, and $\boldsymbol{X} \in \mathbb{H}_+^n$ with rank $r$. If $r \geq 3$, then there is a rank-one decomposition for $\boldsymbol{X}$, i.e., $\boldsymbol{X} = \sum_{j=1}^{r} \boldsymbol{x}_j \boldsymbol{x}_j^H$, such that

$$\boldsymbol{x}_j^H \boldsymbol{A}_1 \boldsymbol{x}_j = \frac{\boldsymbol{A}_1 \bullet \boldsymbol{X}}{r}, \quad j = 1, \ldots, r$$

$$\boldsymbol{x}_j^H \boldsymbol{A}_2 \boldsymbol{x}_j = \frac{\boldsymbol{A}_2 \bullet \boldsymbol{X}}{r}, \quad j = 1, \ldots, r$$

$$\boldsymbol{x}_j^H \boldsymbol{A}_3 \boldsymbol{x}_j = \frac{\boldsymbol{A}_3 \bullet \boldsymbol{X}}{r}, \quad j = 1, \ldots, r-2.$$

[7] W. Ai, Y. Huang, and S. Zhang, "New results on Hermitian matrix rank-one decomposition," *Mathematical Programming: Series A*, vol. 128, no. 1-2, pp. 253-283, June 2011.

# Computational Complexities and Matlab Programs

- The computational complexity of each decomposition theorem is of $O(n^3)$.
- The respective proofs of the theorems are constructive, so that it is convenient to write Matlab programs to perform the specific rank-one decomposition.
- The software release (with a short user guide), based on Matlab, is online at

**http://www.math.hkbu.edu.hk/~huang/dcmp/dcmp.html**

# Solving QCQP by Matrix Decomposition

- QCQP has the general form:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & q_0(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{Q}_0 \boldsymbol{x} - 2\mathrm{Re}\ \boldsymbol{b}_0^H \boldsymbol{x} \\ \text{subject to} & q_j(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{Q}_j \boldsymbol{x} - 2\mathrm{Re}\ \boldsymbol{b}_j^H \boldsymbol{x} + c_j \leq 0,\ j = 1, \ldots, m. \end{array}$$

- Let $\boldsymbol{M}(q_0) = \begin{bmatrix} 0 & -\boldsymbol{b}_0^H \\ -\boldsymbol{b}_0 & \boldsymbol{Q}_0 \end{bmatrix}$, and $\boldsymbol{M}(q_j) = \begin{bmatrix} c_j & -\boldsymbol{b}_j^H \\ -\boldsymbol{b}_j & \boldsymbol{Q}_j \end{bmatrix}$, $j = 1, \ldots, m$.

- QCQP is recast into the homogeneous form (with one more variable and one more constraint):

$$\begin{array}{ll} \underset{\boldsymbol{x},\, t}{\text{minimize}} & \boldsymbol{M}(q_0) \bullet \begin{bmatrix} t \\ \boldsymbol{x} \end{bmatrix} \begin{bmatrix} t \\ \boldsymbol{x} \end{bmatrix}^H = \boldsymbol{x}^H \boldsymbol{Q}_0 \boldsymbol{x} - 2\mathrm{Re}\ \boldsymbol{b}_0^H \boldsymbol{x} t^* \\ \text{subject to} & \boldsymbol{M}(q_j) \bullet \begin{bmatrix} t \\ \boldsymbol{x} \end{bmatrix} \begin{bmatrix} t \\ \boldsymbol{x} \end{bmatrix}^H = \boldsymbol{x}^H \boldsymbol{Q}_j \boldsymbol{x} - 2\mathrm{Re}\ \boldsymbol{b}_j^H \boldsymbol{x} t^* + c_j |t|^2 \leq 0,\ \forall j \\ & |t|^2 = 1. \end{array}$$

# SDP Relaxation

- The matrix form of the homogenous QCQP can be further written equivalently as

$$\begin{array}{ll} \underset{\boldsymbol{X}}{\text{minimize}} & \boldsymbol{M}(q_0) \bullet \boldsymbol{X} \\ \text{subject to} & \boldsymbol{M}(q_j) \bullet \boldsymbol{X} \leq 0,\ j = 1, \ldots, m \\ & \boldsymbol{I}_{00} \bullet \boldsymbol{X} = 1 \\ & \boldsymbol{X} \succeq \mathbf{0},\ \text{rank}(\boldsymbol{X}) = 1 \end{array}$$

where $\boldsymbol{I}_{00} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{S}^{n+1}$.

- If the rank-one constraint is removed, it becomes an SDP, which is called the SDP relaxation problem.

- The dual problem is:

$$\begin{array}{ll} \underset{\boldsymbol{Z}, \{y_j\}}{\text{maximize}} & y_{m+1} \\ \text{subject to} & \boldsymbol{Z} = \boldsymbol{M}(q_0) + \sum_{j=1}^{m} y_j \boldsymbol{M}(q_j) - y_{m+1} \boldsymbol{I}_{00} \succeq \mathbf{0} \\ & y_j \geq 0,\ j = 1, \ldots, m,\ y_{m+1} \in \mathbb{R}. \end{array}$$

# Complementary Slackness

- Under suitable conditions, the primal and dual problems have complementary optimal solutions, $\boldsymbol{X}^\star$ and $\boldsymbol{Z}^\star$:

$$\boldsymbol{X}^\star \boldsymbol{Z}^\star = \mathbf{0}.$$

- If we can decompose $\boldsymbol{X}^\star = \sum_{j=1}^{r} \boldsymbol{x}_j^\star \boldsymbol{x}_j^{\star H}$, so that some $\boldsymbol{x}_j^\star \boldsymbol{x}_j^{\star H}$ satisfying all the constraints of the primal SDP problem, then the rank-one matrix will be optimal.

- Now, our matrix rank-one decomposition theorems can help provide a rank-one optimal solution.

# Consequences of the Matrix Decomposition Theorems

- Generally, the following cases of QCQP are polynomially solvable:
  - real QCQP:
    * $m = 1$ ($m = 2$ for the homogeneous instance);
    * $m = 2$ ($m = 3$ for the homogeneous instance) and rank$\boldsymbol{X}^\star \geq 3$;
    * $m = 2$ and one inequality constraint is inactive at $\boldsymbol{X}^\star$.
  - complex QCQP:
    * $m = 2$ ($m = 3$ for the homogeneous instance);
    * $m = 3$ ($m = 4$ for the homogeneous instance) and rank$\boldsymbol{X}^\star \geq 3$;
    * $m = 3$ and one inequality constraint is inactive at $\boldsymbol{X}^\star$.

- Particularly, the optimal radar code selection problem is a complex inhomogeneous QCQP with $m = 3$, however, it is solvable, thanks to the problem structure that two of the constraint functions share the same Hessian.

- The solvability is irrelevant to the convexity of the functions.

# Further Theoretical Applications

- Field of values
  - The field of values of a $n \times n$ matrix $\boldsymbol{A}$ is given by

$$\mathcal{F}(\boldsymbol{A}) = \{\boldsymbol{x}^H \boldsymbol{A} \boldsymbol{x} \mid \boldsymbol{x}^H \boldsymbol{x} = 1\} \subseteq \mathbb{C}.$$

  - It is known to be convex[8].

- Joint numerical range
  - In general, the joint numerical range of matrices is defined by

$$\mathcal{F}(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_m) = \left\{ \begin{bmatrix} \boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x} \\ \vdots \\ \boldsymbol{x}^H \boldsymbol{A}_m \boldsymbol{x} \end{bmatrix} \middle| \ \boldsymbol{x}^H \boldsymbol{x} = 1, \ \boldsymbol{x} \in \mathbb{C}^n \right\} \in \mathbb{C}^m.$$

[8] R. A. Horn and C. R. Johnson. *Topics in Matrix analysis*. Cambridge University Press, 1991, ch. 1.

- The convexity of joint numerical range has a long history.

- **Theorem**[9] If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are Hermitian, then $\mathcal{F}(\boldsymbol{A}_1, \boldsymbol{A}_2)$ is a convex set.

- Brickman generalizes the above Hausdorff theorem:

- **Theorem**[10] If $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and $\boldsymbol{A}_3$ are Hermitian, then the set

$$\left\{ \begin{bmatrix} \boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x} \\ \boldsymbol{x}^H \boldsymbol{A}_2 \boldsymbol{x} \\ \boldsymbol{x}^H \boldsymbol{A}_3 \boldsymbol{x} \end{bmatrix} \middle| \ \boldsymbol{x} \in \mathbb{C}^n \right\} \in \mathbb{R}^3.$$

  is a convex cone.

[9] F. Hausdorff, "Der Wertvorrat einer Bilinearform," *Mathematische Zeitschrift*, vol. 3, pp. 314-316, 1919.

[10] L. Brickman, "On the field of values of a matrix," *Proceedings of the American Mathematical Society*, vol. 12, pp. 61-66, 1961.

# An Extension of Brickman's Theorem

- **Theorem**[11] Suppose that $\boldsymbol{A}_j \in \mathbb{H}^n$ with $n \geq 3$. If

$$(\boldsymbol{A}_1 \bullet \boldsymbol{X}, \boldsymbol{A}_2 \bullet \boldsymbol{X}, \boldsymbol{A}_3 \bullet \boldsymbol{X}, \boldsymbol{A}_4 \bullet \boldsymbol{X}) \neq (0, 0, 0, 0)$$

for any nonzero $\boldsymbol{X} \in \mathbb{H}_+^n$, then

$$\left\{ \left[ \begin{array}{c} \boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x} \\ \boldsymbol{x}^H \boldsymbol{A}_2 \boldsymbol{x} \\ \boldsymbol{x}^H \boldsymbol{A}_3 \boldsymbol{x} \\ \boldsymbol{x}^H \boldsymbol{A}_4 \boldsymbol{x} \end{array} \right] \middle| \; \boldsymbol{x} \in \mathbb{C}^n \right\} \in \mathbb{R}^4$$

is a pointed closed convex cone.

[11] W. Ai, Y. Huang, and S. Zhang, "New results on Hermitian matrix rank-one decomposition," *Mathematical Programming: Series A*, vol. 128, no. 1-2, pp. 253-283, June 2011.

# The $S$-Procedure

- It is often useful to consider the following implication:

$$G_1(\boldsymbol{x}) \geq 0, \ \ldots, G_m(\boldsymbol{x}) \geq 0 \Rightarrow F(\boldsymbol{x}) \geq 0.$$

- A sufficient condition is:

$$\exists \tau_1 \geq 0, \ldots, \tau_m \geq 0, \text{ such that } F(\boldsymbol{x}) - \sum_{j=1}^{m} \tau_j G_j(\boldsymbol{x}) \geq 0, \ \forall \boldsymbol{x}.$$

- If the condition is also necessary, then this procedure is called lossless.

- **S-lemma** (real-valued case)[12] Suppose that $m = 1$, and $F$, $G_1$ are real quadratic forms (i.e., $F(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{F} \boldsymbol{x}$, $G_1(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{G}_1 \boldsymbol{x}$, and $\boldsymbol{F}$, $\boldsymbol{G}_1$ are symmetric). Moreover, there is $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $\boldsymbol{x}_0^T \boldsymbol{G}_1 \boldsymbol{x}_0 > 0$. Then the $S$-procedure is lossless.

[12] V. A. Yakubovich, "S-procedure in Nonlinear Control Theory," *Vestnik Leninggrad Univ.*, vol. 4, no. 1, pp. 73-93, 1977. (In Russian 1971.)

- **S-lemma** (complex-valued case)[13] Suppose $m = 2$, and $F$, $G_1$, $G_2$ are Hermitian quadratic form (i.e., $F(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{F} \boldsymbol{x}$, $G_1(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{G}_1 \boldsymbol{x}$, $G_2(\boldsymbol{x}) = \boldsymbol{x}^H \boldsymbol{G}_2 \boldsymbol{x}$, and $\boldsymbol{F}$, $\boldsymbol{G}_1$, $\boldsymbol{G}_2$ are Hermitian). Moreover, there is $\boldsymbol{x}_0 \in \mathbb{C}^n$ such that $\boldsymbol{x}_0^H \boldsymbol{G}_j \boldsymbol{x}_0 > 0$, $j = 1, 2$. Then the $S$-procedure is lossless.

[13] A. L. Fradkov and V. A. Yakubovich, "The S-procedure and duality relations in nonconvex problems of quadratic programming," *Vestnik Leninggrad Univ.*, vol. 6, pp. 101-109, 1979. (In Russian 1973.)

# Extensions on A Result of Yuan

- **Theorem**[14] Let $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{S}^n$. If

$$\max\{\boldsymbol{x}^T \boldsymbol{A}_1 \boldsymbol{x}, \boldsymbol{x}^T \boldsymbol{A}_2 \boldsymbol{x}\} \geq 0, \forall \boldsymbol{x} \in \mathbb{R}^n, \quad (1)$$

  then there are $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_1 + \mu_2 = 1$ such that

$$\mu_1 \boldsymbol{A}_1 + \mu_2 \boldsymbol{A}_2 \succeq \boldsymbol{0}.$$

- By our decomposition theorems, we can re-prove it. Indeed, we show that (1) amounts to

$$\max\{\boldsymbol{A}_1 \bullet \boldsymbol{X}, \boldsymbol{A}_2 \bullet \boldsymbol{X}\} \geq 0, \forall \boldsymbol{X} \in \mathbb{S}^n_+.$$

- For the Hermitian case, we can do more.

[14] Y. X. Yuan, "On a subproblem of trust region algorithms for constrained optimization," *Mathematical Programming*, vol. 47, pp. 53-63, 1990.

- **Theorem**[15] Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3 \in \mathbb{H}^n$. If

$$\max\{\boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x}, \boldsymbol{x}^H \boldsymbol{A}_2 \boldsymbol{x}, \boldsymbol{x}^H \boldsymbol{A}_3 \boldsymbol{x}\} \geq 0, \; \forall \boldsymbol{x} \in \mathbb{C}^n,$$

then there are $\mu_1, \mu_2, \mu_3 \geq 0$, $\mu_1 + \mu_2 + \mu_3 = 1$ such that

$$\mu_1 \boldsymbol{A}_1 + \mu_2 \boldsymbol{A}_2 + \mu_3 \boldsymbol{A}_3 \succeq \mathbf{0}.$$

- **Theorem**[15] Suppose that $\boldsymbol{A}_j \in \mathbb{H}^n$, $j = 1, 2, 3, 4$, with $n \geq 3$, and suppose that there are $\lambda_j \in \mathbb{R}$ such that $\lambda_1 \boldsymbol{A}_1 + \lambda_2 \boldsymbol{A}_2 + \lambda_3 \boldsymbol{A}_3 + \lambda_4 \boldsymbol{A}_4 \succ \mathbf{0}$. If

$$\max\{\boldsymbol{x}^H \boldsymbol{A}_1 \boldsymbol{x}, \boldsymbol{x}^H \boldsymbol{A}_2 \boldsymbol{x}, \boldsymbol{x}^H \boldsymbol{A}_3 \boldsymbol{x} \, \boldsymbol{x}^H \boldsymbol{A}_4 \boldsymbol{x}\} \geq 0, \; \forall \boldsymbol{x} \in \mathbb{C}^n,$$

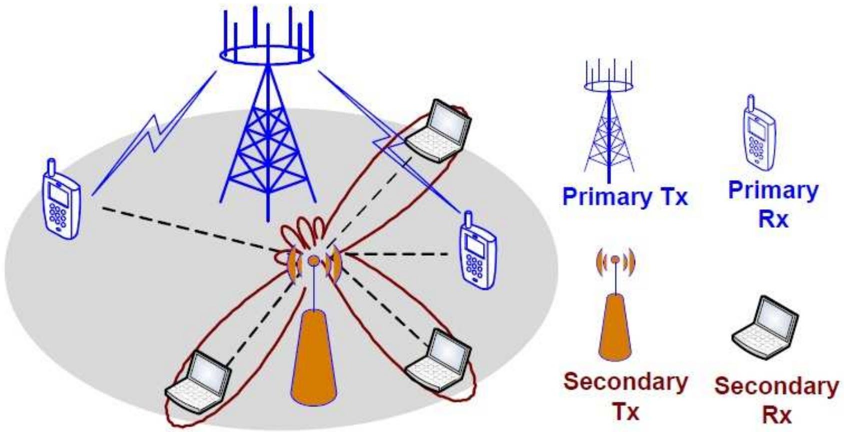
then there are $\mu_j \geq 0$, $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1$ such that

$$\mu_1 \boldsymbol{A}_1 + \mu_2 \boldsymbol{A}_2 + \mu_3 \boldsymbol{A}_3 + \lambda_4 \boldsymbol{A}_4 \succeq \mathbf{0}.$$

[15] W. Ai, Y. Huang, and S. Zhang, "New results on Hermitian matrix rank-one decomposition," *Mathematical Programming: Series A*, vol. 128, no. 1-2, pp. 253-283, June 2011.

# One More Application: Multicast Beamforming in CR Networks

Consider a scenario of single-group multicast transmission between secondary users in a *spectrum sharing* cognitive radio network[16].




- The secondary transmitter, equipped with an antenna array, sends common signals to its users, with the goals:
  - sufficient service quality to the secondary users
  - no excessive interference to the primary receivers
  - minimal transmission power

[16] Y. Huang, Q. Li, W.-K. Ma, and S. Zhang, "Robust multicast beamforming for spectrum sharing-based cognitive radios," *IEEE Transactions on Signal Processing*, vol. 60, no. 1, pp. 527-533, 2012.

# Signal Models

- Signal transmitted by the secondary transmitter

$$\boldsymbol{y}(t) = \boldsymbol{w}s(t)$$

where $s(t) \in \mathbb{C}$ is the information signal, and $\boldsymbol{w} \in \mathbb{C}^N$ is the beamvector.

- Signal received by $m$th secondary user:

$$\boldsymbol{x}_m(t) = \boldsymbol{H}_m^H \boldsymbol{y}(t) + \boldsymbol{n}_m(t)$$

where $\boldsymbol{H}_m$ is the channel matrix and $\boldsymbol{n}_m(t)$ is Gaussian noise vector having zero mean and covariance $\sigma_m^2 \boldsymbol{I}$.

## QoS Constraints and Interference Temperature Constraints

- SNR of the $m$th secondary user

$$\text{SNR}_m = \frac{\|\boldsymbol{H}_m^H \boldsymbol{w}\|^2}{\sigma_m^2}.$$

- The amount of interference generated to $k$th primal user

$$\|\boldsymbol{G}_k^H \boldsymbol{w}\|^2$$

  where $\boldsymbol{G}_k$ is the channel from the secondary transmitter to $k$th primary user.

- QoS constraints: $\text{SNR}_m \geq \tau_m$ for $m = 1, \ldots, M$.

- Interference temperature (IT) constraints:

$$\|\boldsymbol{G}_k^H \boldsymbol{w}\|^2 \leq \eta_k \quad \text{for} \quad k = 1, \ldots, K.$$

# Formulation of Robust Optimal Beamforming Problem

- **Non-Robust formulation**: Minimization of the secondary transmit power subject to QoS constraints and IT constraints:

$$\begin{array}{lll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^H \boldsymbol{w} & \\ \text{subject to} & \|\boldsymbol{H}_m^H \boldsymbol{w}\|^2 \geq \sigma_m^2 \tau_m, & m = 1, \ldots, M \\ & \|\boldsymbol{G}_k^H \boldsymbol{w}\|^2 \leq \eta_k, & k = 1, \ldots, K. \end{array}$$

- **Robust formulation**:

$$\begin{array}{lll} \underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^H \boldsymbol{w} & \\ \text{subject to} & \underset{\|\boldsymbol{\Delta}_m\| \leq \epsilon_m}{\text{minimize}} \; \|(\boldsymbol{H}_m + \boldsymbol{\Delta}_m)^H \boldsymbol{w}\|^2 \geq \sigma_m^2 \tau_m, & m = 1, \ldots, M \\ & \underset{\|\boldsymbol{\Delta}'_k\| \leq \epsilon'_k}{\text{maximize}} \; \|(\boldsymbol{G}_k + \boldsymbol{\Delta}'_k)^H \boldsymbol{w}\|^2 \leq \eta_k, & k = 1, \ldots, K. \end{array}$$

- The equivalent formulation of the robust problem can be derived:

$$
\begin{array}{lll}
\underset{\boldsymbol{w}}{\text{minimize}} & \boldsymbol{w}^H \boldsymbol{w} & \\
\text{subject to} & \|\boldsymbol{H}_m^H \boldsymbol{w}\| \geq \sigma_m \sqrt{\tau_m} + \epsilon_m \|\boldsymbol{w}\|, & m = 1, \ldots, M \\
& \|\boldsymbol{G}_k^H \boldsymbol{w}\| \leq \sqrt{\eta_k} - \epsilon'_k \|\boldsymbol{w}\|, & k = 1, \ldots, K.
\end{array}
$$

- By the matrix rank-one decomposition theorems, we identify several polynomially solvable scenarios (corresponding to different

# Summary

- We have presented our specific matrix rank-one decomposition techniques, which have manageable computational complexity. The software release based on Matlab has been ready.

- Efficiently solving some nonconvex QCQP problems has showcased one significant application.

- In connection to engineering applications, we have demonstrated two optimal design problems, one from radar and the other from wireless communications.

- The theoretical applications we have displayed include the CDT trust-region problems, $S$-lemma, the convexity of joint numerical range, and the extension on Yuan's theorem.