Beyond Heuristics: Applying Alternating Direction Method of Multipliers in Nonconvex Territory

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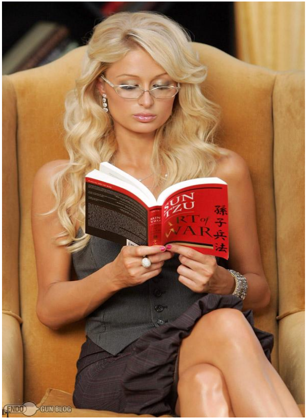
2012 International Workshop on Signal Processing, Optimization, and Control
USTC, Hefei, Anhui, China

Outline

Section I. Introduction and Application

An Ancient Strategy



- “远交近攻，各个击破”，
  “分而治之”
  ——《孙子兵法》(《SUN TZU, ART OF WAR》)
  孙子(535 – 470 BC)
- “Divide et impera”
  Julius Caesar (100 – 44 BC)

Mathematical Point of View: Split and Alternate

Splitting Techniques

Case 1: Nondifferentiable Term

$$\min f(x) + g(Bx)$$

$$\Downarrow$$

$$\min f(x) + g(y) \qquad \text{s.t.} \;\; Bx - y = 0.$$

Case 2: Highly Nonconvex

$$\min f(g(x))$$

$$\Downarrow$$

$$\min f(y) \qquad \text{s.t.} \;\; g(x) - y = 0.$$

Case 3: Inconsistent Objective and Constraint

$$\min f(x) \qquad \text{s.t.} \;\; c(x) = 0$$

$$\Downarrow$$

$$\min f(x) \qquad \text{s.t.} \;\; c(y) = 0, \; x = y.$$

Splitting Instances

Instance 1: Compressive Sensing

$$\min \|Wx\|_1 + \frac{\mu}{2}\|Ax - b\|_2^2$$

$$\Downarrow$$

$$\min \|y\|_1 + \frac{\mu}{2}\|Ax - b\|_2^2 \quad \text{s.t.} \quad Wx - y = 0.$$

Instance 2: Nonlinear ℓ_1 Minimization

$$\min \ \|f(x)\|_1.$$

$$\Downarrow$$

$$\min \ \|y\|_1 \quad \text{s.t.} \ f(x) = y.$$

Instance 3: Dual Problem of Compressive Sensing (Yang-Zhang 2009)

$$\min \; -b^{\mathsf{T}}y + \frac{1}{2\mu}\|y\|_2^2 \qquad \text{s.t. } \; \|W^{-\mathsf{T}}A^{\mathsf{T}}y\|_\infty \le 1.$$

$$\Downarrow$$

$$\min \; -b^{\mathsf{T}}y + \frac{1}{2\mu}\|y\|_2^2 \qquad \text{s.t. } \; \|z\|_\infty \le 1, \quad z = W^{-\mathsf{T}}A^{\mathsf{T}}y.$$

Equality Constrained Problems

$$\min f(x) \qquad \text{s.t.} \;\; c(x) = 0.$$

Augmented Lagrangian Function (Henstenes 1969, Powell 1969, Rockafellar 1973)

$$\mathcal{L}_\beta(x, \lambda) = f(x) - \lambda^\top c(x) + \frac{\beta}{2}\|c(x)\|_2^2.$$

Augmented Lagrangian Method

$$\text{ALM} : \begin{cases} x^{k+1} \leftarrow \arg\min \mathcal{L}_\beta(x, \lambda^k); \\ \lambda^{k+1} \leftarrow \lambda^k - \tau \beta\, c(x^{k+1}); \\ \text{update } \beta \text{ if necessary }. \end{cases}$$

Augmented Lagrangian Method (Cont'd)

Problems with Equality Constraints

$$\min_{x\in\Omega} f(x) \qquad \text{s.t.} \;\; c(x) = 0.$$

Augmented Lagrangian Method – Extension

$$\text{ALM}: \begin{cases} x^{k+1} \leftarrow \arg\min_{x\in\Omega} \mathcal{L}_\beta(x, \lambda^k); \\ \lambda^{k+1} \leftarrow \lambda^k - \tau\beta\, c(x^{k+1}); \\ \text{update } \beta \text{ if necessary }. \end{cases}$$

Block Structure

$$\{x \mid x \in \Omega\} = \bigcap_{i=1}^{p} \{x \mid x_i \in \Omega_i\}.$$

(Augmented Lagrangian) Alternating Direction Method (of Multiplier)

(Glowinski-Marocco 1975, Gabay-Mercier 1976, ...)

$$\text{ADMM}: \begin{cases} x_1^{k+1} \leftarrow \arg\min_{x_1 \in \Omega_1} \mathcal{L}_\beta(x_1, x_2^k, \dots, x_p^k, \lambda^k); \\ x_2^{k+1} \leftarrow \arg\min_{x_2 \in \Omega_2} \mathcal{L}_\beta(x_1^{k+1}, x_2, x_3^k, \dots, x_p^k, \lambda^k); \\ \dots\dots \\ x_p^{k+1} \leftarrow \arg\min_{x_p \in \Omega_p} \mathcal{L}_\beta(x_1^{k+1}, \dots, x_{p-1}^{k+1}, x_p, \lambda^k); \\ \lambda^{k+1} \leftarrow \lambda^k - \tau\beta\, c(x_1^{k+1}, \dots, x_p^{k+1}). \end{cases}$$

Phase Retrieval (Wen-Yang-L.)

- X-ray crystallography, transmission electron microscopy
- **Original model:**

$$\min_{\hat{\psi} \in \mathbb{C}^n} \sum_{i=1}^{k} \frac{1}{2} \left\| |\mathcal{F} Q_i \hat{\psi}| - b_i \right\|_2^2 .$$

- **Reformulation:**

$$\min_{\hat{\psi} \in \mathbb{C}^n, z \in \mathbb{C}^{m \times k}} \sum_{i=1}^{k} \frac{1}{2} \| |z_i| - b_i \|_2^2 \quad \text{s.t.} \ z_i = \mathcal{F} Q_i \hat{\psi}, \quad i = 1, ..., k.$$

- **Augmented Lagrange function:**

$$\mathcal{L}_\beta(z_i, \psi, y_i) = \sum_{i=1}^{k} \left(\frac{1}{2} \| |z_i| - b_i \|_2^2 + y_i^* (\mathcal{F} Q_i \psi - z_i) + \frac{\beta}{2} \| \mathcal{F} Q_i \psi - z_i \|_2^2 \right).$$

Portfolio Optimization (Wen-Peng-L.-Bai-Sun)

- Asset Allocation under the Basel Accord Risk Measures (Value-at-Risk) – integer programming
- **Original model:**

$$\min_{u \in \mathcal{U}_{r_0}} (-\tilde{R}u)_{(p)},$$

where $\mathcal{U}_{r_0} = \{u \in \mathbb{R}^d \mid \mu^\top u \geq r_0, \mathbf{1}^\top u = 1, u \geq 0\}$; $(\cdot)_{(p)}$ refers to the p-th smallest component of a vector.

- **Reformulation:**

$$\min_{u \in \mathcal{U}_{r_0},\, x \in \mathbb{R}^n} x_{(p)} \qquad \text{s.t.} \quad x + \tilde{R}u = 0.$$

- **Augmented Lagrange function:**

$$\mathcal{L}_\beta(x, u, \lambda) := x_{(p)} - \lambda^\top (x + \tilde{R}u) + \frac{\beta}{2}\|x + \tilde{R}u\|^2.$$

Applications III

Matrix Factorization (Zhang et al.)

- Nonnegative matrix factorization, structure enforcing matrix factorization
- **Original model:**

$$\min_{W\in\mathbb{R}^{m\times k},\ H\in\mathbb{R}^{n\times k}} \|A - WH^{\mathsf{T}}\|_{\mathsf{F}}^2 \quad \text{s.t.}\ \ W \in \mathbb{T}_1,\ H \in \mathbb{T}_2,$$

  where $\mathbb{T}_1$, $\mathbb{T}_2$ can be $\{X \mid X^{\mathsf{T}}X = I\}$, or $\{X \mid X \geq 0\}$, or any other matrix sets allowing 'easy projection'

- **Reformulation:**

$$\min_{W,\ H,\ S_1\in\mathbb{T}_1,\ S_2\in\mathbb{T}_2} \|A - WH^{\mathsf{T}}\|_{\mathsf{F}}^2 \quad \text{s.t.}\ \ W = S_1,\ H = S_2.$$

- **Augmented Lagrange function:**

$$\begin{aligned}\mathcal{L}_{(\beta_1,\,\beta_2)}(W, H, S_1, S_2, \Lambda) \;&=\; \|A - WH^{\mathsf{T}}\|_{\mathsf{F}}^2 - \Lambda_1 \bullet (W - S_1) \\ -\Lambda_2 \bullet (H - S_2) \;&+\; \frac{\beta_1}{2}\cdot\|W - S_1\|_{\mathsf{F}}^2 + \frac{\beta_2}{2}\cdot\|H - S_2\|_{\mathsf{F}}^2.\end{aligned}$$

Section II. Theoretical Results

Brief Introduction

Intuition

- "Splitting" brings easy subproblem
- "Splitting" induces equality constraint – Augmented Lagrange
- "Alternating" solves the split targets in turn
- From line search to ADM
  - Line search based optimization - one dimensional subspace
  - Subspace method - multi-dimensional subspace
  - ADMM - high-order subspaces

Convergence Based on Strict Conditions

- Two blocks, joint convexity, separability (Gabay-Mercier 1976)
- Multiple blocks, variant versions (He, Yuan et al., Goldfarb and Ma, etc.)
  - complexity
  - acceleration
  - customization
- Two blocks, linear convergence rate (Yin-Deng 2012)

Towards a General Scheme

– nonconvex and nonseparable cases

- Local convergence and rate (Yang-L.-Zhang)
- Global convergence (L.-Yang-Zhang)
  - under some assumptions (ongoing)
  - special case: multiple blocks,
    separable + strongly convex + second order differentiable

Nonlinear Splitting and Iteration Scheme

- Original Nonlinear System: $F : \mathbb{R}^n \to \mathbb{R}^n$
- Splitting: $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$
  i.e. $G(x, x) := L(x) - R(x) \equiv F(x)$. $\partial_1 G \triangleq \partial_x G$, and $\partial_2 G \triangleq -\partial_x G$.
- Consider $G(x, x, \lambda)$ to be a splitting of $F := \nabla_x \mathcal{L}_\beta(x, \lambda)$
- A generalized ADMM scheme:

$$\text{GADMM} : \begin{cases} x^{k+1} \leftarrow G(x, x^k, \lambda^k) = 0; \\ \lambda^{k+1} \leftarrow \lambda^k - \tau\beta\, c(x^{k+1}). \end{cases}$$

Local Convergence Result

Error System

$$e^{k+1} = M(\tau)e^k + o(\|e^k\|)$$

where

$$M(\tau) = \begin{bmatrix} [\partial_1 G^*]^{-1}\partial_2 G^* & [\partial_1 G^*]^{-1}(\nabla c^*)^\top \\ -\tau\nabla c^*[\partial_1 G^*]^{-1}\partial_2 G^* & I - \tau\nabla c^*[\partial_1 G^*]^{-1}(\nabla c^*)^\top \end{bmatrix}$$

Local convergence:

- $e^k \triangleq ((x^k - x^*)^\top, (\lambda^k - \lambda^*)^\top)^\top$
- Implicit Function Theorem + Taylor Expansion
- Assumptions: $\nabla_{xx}\mathcal{L}_\beta(x^*, \lambda^*) \succ 0$ and $\nabla c(x^*)$ full row rank
- Results:
  - local convergence: $\exists\, \eta > 0, \quad \rho(M(\tau)) < 1, \quad \forall\, \tau \in (0, \eta)$;
  - R-linear rate: $\rho(M(\tau))$.

Global Convergence

Relative Error System

$$e^{k+1} = M(\tau)^k e^k$$

where

$$M(\tau)^k = \begin{bmatrix} [\bar{\partial}_1 G_L^k]^{-1} \bar{\partial}_2 G_U^k & [\bar{\partial}_1 G_L^k]^{-1} A^\top \\ -\tau A [\bar{\partial}_1 G_L^k]^{-1} \bar{\partial}_2 G_U^k & I - \tau A [\bar{\partial}_1 G_L^k]^{-1} A^\top \end{bmatrix}$$

Global convergence:

- $e^k \triangleq ((x^k - x^{k-1})^\top, (\lambda^k - \lambda^{k-1})^\top)^\top$
- Mean Value Theorem + Average Hessian $(\bar{\partial}_1 G_L^k, \bar{\partial}_2 G_U^k)$
- Difficulty: **non-stationary iteration**

  $\mathcal{L}_\beta$ strongly convex and $\nabla \mathcal{L}_\beta$ is Lipschitz continuous $\Rightarrow$

  $\rho(M(\tau)^k) \leq 1 - \epsilon$ $(\forall k)$ $\nLeftrightarrow$ global convergence

Global Convergence (Cont'd)

ℓ_2 Restriction (ongoing)

- $\|M(\tau)^k\|_2 \leq 1 - \epsilon$ $(\forall k)$
- Assumptions:
  - linear constraints • block-wise convexity
  - second order differentiability • block diagonal dominance
- Result: global convergence

Special Case

- Assumptions:
  - separability: $\bar{\partial}_2 G_U^k$ constant, $\bar{\partial}_1 G_L^k$ non-stationary in block diagonal
  - strongly convexity • linear constraints • second order differentiability
- Result:
  - $\exists \bar{\beta} > 0$ and $\exists \eta > 0$;
  - global convergence, $\forall \beta \in (0, \bar{\beta})$, $\forall \tau \in (0, \eta)$.

Section III. Conclusion and Future works

Conclusion

- Powerful tool for hard optimization problem with structure;
- Lack of convergence results for nonconvex problems;
- Excellent performance in practice.

Future Works

- There is still room for further improvement of the algorithm;
- Convergence results for lots of known successful cases are still unclear;
- Gap from the stationary point to the global optimizer.

Thank you for your attention!

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