15-16 期末

6. 设 $f(x)$ 在 $\mathbb{R}$ 上连续，且满足 $f(x+a)=-f(x)$. 求证：

$$\int_0^{2a} x f(x)\,dx = -a\int_0^a f(x)\,dx.$$

Pr: $\int_0^{2a} x f(x)\,dx$

$$= \int_0^a x f(x)\,dx + \int_a^{2a} x f(x)\,dx$$

$$= \int_0^a x f(x)\,dx + \int_0^a (x+a) f(x+a)\,dx$$

$$= \int_0^a x f(x)\,dx + \int_0^a (x+a)\,{-f(x)}\,dx$$

$$= -a\int_0^a f(x)\,dx$$

#

7. 设 $f(x)$ 在 $[0,1]$ 连续，且 $0 \le f(x) \le 1$. 求证：

$$2\int_0^1 x f(x)\,dx \ge \left(\int_0^1 f(x)\,dx\right)^2$$

并求使上式取等的连续函数.

Pr: $F(t) = 2\int_0^t x f(x)\,dx - \left(\int_0^t f(x)\,dx\right)^2$ ☆

$$F'(t) = 2t f(t) - 2\left(\int_0^t f(x)\,dx\right)\cdot f(t)$$

$$= 2f(t)\left[t - \int_0^t f(x)\,dx\right] \ge 0.$$

故 $F(1) \ge F(0) = 0$.

取等 $\Leftrightarrow F'(t) = 0 \Leftrightarrow \begin{cases} f \equiv 0 \\ \text{或 } \int_0^t f(x)\,dx = t \Leftrightarrow f \equiv 1 \end{cases}$

故取等 $\Leftrightarrow f \equiv 0$ or $f \equiv 1$. #

12-13 期末

7. 设 $f(x)$ 在 $[0,1]$ 非负单调增的连续函数，$0<\alpha<\beta<1$

求证：

$$\int_0^1 f(x)\,dx \geq \frac{1-\alpha}{\beta-\alpha}\int_\alpha^\beta f(x)\,dx$$

且 $\frac{1-\alpha}{\beta-\alpha}$ 不能换成更大的数.

Pr: 原不等式

$$\Leftrightarrow (\beta-\alpha)\int_0^1 f(x)\,dx \geq (1-\alpha)\int_\alpha^\beta f(x)\,dx$$

$$\Leftrightarrow (\beta-\alpha)\left[\int_0^\alpha + \int_\alpha^\beta + \int_\beta^1\right] \geq (1-\alpha)\int_\alpha^\beta$$

$$\Leftrightarrow (\beta-\alpha)\left[\int_0^\alpha + \int_\beta^1\right] \geq (1-\beta)\int_\alpha^\beta$$

$$\Leftarrow (\beta-\alpha)\int_\beta^1 f(x)\,dx \geq (1-\beta)\int_\alpha^\beta f(x)\,dx$$

$$\Leftrightarrow \underbrace{\frac{1}{1-\beta}\int_\beta^1 f(x)\,dx}_{= f(\xi_1)} \geq \underbrace{\frac{1}{\beta-\alpha}\int_\alpha^\beta f(x)\,dx}_{= f(\xi_2)}$$

$\xi_1 \in (\beta, 1)$ $\quad$ $\xi_2 \in (\alpha, \beta)$

$\Rightarrow$ $\underline{\xi_1 > \xi_2}$.

取 $f(x) = \begin{cases} 0 & , \quad x \in [0, \alpha] \\ 1 & , \quad x \in (\alpha, 1] \end{cases}$

此时 左 $= 1-\alpha =$ 右

#

11-12 期末

8. 设 $f(x)$ 在区间 $[0,1]$ 上有连续导函数且 $f(0)=0$, $f(1)=1$. 求证:

$$\int_0^1 |\underline{f(x)+f'(x)}|\,dx \geq 1$$

Pr:

$$\text{左} = \int_0^1 |e^x f(x) + e^x f'(x)| \cdot \underset{\sim}{e^{-x}}\,dx$$

$$\geq \frac{1}{e}\int_0^1 |e^x f(x) + e^x f'(x)|\,dx$$

$$\geq \frac{1}{e}\int_0^1 d(e^x f(x)) = 1$$

#

08-09.（B卷）

6. 设 $f(x)$ 是定义在 $(-\infty,+\infty)$ 上满足函数方程

$$\int_0^x tf(t)dt=(x-1)\int_0^x f(t)dt$$ 的连续函数

求证：$f(x)\equiv 0$

Pr: $F(x)\triangleq\int_0^x tf(t)dt-(x-1)\int_0^x f(t)dt$

$$F'(x)=xf(x)-(x-1)f(x)-\int_0^x f(t)dt$$

$$=f(x)-\int_0^x f(t)dt$$

$\Rightarrow$ $f(x)=\int_0^x f(t)dt$ ①

$\Rightarrow$ f 可导.

$$f'(x)=f(x)$$

令 $g(x)=e^{-x}f(x)$

$$g'(x)=e^{-x}[f'(x)-f(x)]=0.$$

$\Rightarrow g(x) = g(0)$

$e^{-x} f(x) = f(0)$

$\Rightarrow f(x) = f(0) \cdot e^{x}$ 代入 ①

得 $f(0) e^{x} = \int_0^x e^t f(0)\, dt = (e^x - 1) f(0)$

$\Rightarrow f(0) = 0 \Rightarrow f \equiv 0.$

#.

if $F(x) = \underline{\int_0^x \underline{f(t)}\, dt}$.

$\Rightarrow F'(x) = f(x)$.

7. 设 $f(x)$ 在区间 $[0,1]$ 上非负连续可微，$f(0) = f(1) = 0$. 且 $|f'(x)| \leq 1$. 求证：

$$\int_0^1 f(x)\,dx \leq \frac{1}{4}$$

Pr: $f(x) = f(0) + f'(\xi_1) x = f'(\xi_1) x$

$f(x) = f(1) + f'(\xi_2)(x-1) = f'(\xi_2)(x-1)$

$$\int_0^1 |f(x)|dx = \int_0^{\frac{1}{2}} |f(x)|dx + \int_{\frac{1}{2}}^1 |f(x)|dx$$
$$\le \int_0^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^1 |1-x|\,dx$$
$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

#

$\int_0^1 |f(x)|dx = \frac{1}{4}$ 取等条件？

08-09 (A卷)

5. 求证：当 $x>0$，$\alpha>1$ 时 有 $(1+x)^\alpha > 1+\alpha x$

Pr: $F(x) = (1+x)^\alpha - \alpha x - 1$
$$= e^{\alpha \ln(1+x)} - \alpha x - 1$$

$$F'(x) = e^{\alpha \ln(1+x)} \cdot \alpha \frac{1}{1+x} - \alpha$$
$$= \alpha\left[(1+x)^{\alpha-1} - 1\right] > 0$$

$F(x) > F(0) = 0.\quad \forall\ x>0.$

7. 设 $a_n = 2\sqrt{n} - \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}}$，求证：

(1) $1 \leq a_n \leq 2 - \frac{1}{\sqrt{n}}$

(2) 极限 $a = \lim\limits_{n\to+\infty} a_n$ 存在

(3) 数列 $\{\sqrt{n}(a - a_n)\}$ 有界.

Pr:

(1) $a_n = 2\sqrt{n} - \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}}$

$$2(\sqrt{n}-1) = 2(\sqrt{n}-\sqrt{n-1}) + 2(\sqrt{n-1}-\sqrt{n-2}) + \cdots + 2(\sqrt{2}-1)$$

$$= \underline{\frac{2}{\sqrt{n}+\sqrt{n-1}} + \frac{2}{\sqrt{n-1}+\sqrt{n-2}} + \cdots + \frac{2}{\sqrt{2}+\sqrt{1}}} \quad ①$$

$$① \leq \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{1}}$$

$$① \geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n-1}} + \cdots + \frac{1}{\sqrt{2}}$$

故 $a_n \in [1, 2 - \frac{1}{\sqrt{n}}]$

(2) $a_{n+1}-a_n = 2\sqrt{n+1}-2\sqrt{n}-\dfrac{1}{\sqrt{n+1}}$

$$= \frac{2}{\sqrt{n+1}+\sqrt{n}} - \frac{1}{\sqrt{n+1}} \geq 0.$$

故 $\{a_n\}_{n=1}^{\infty}$ 单增有上界.

$\Rightarrow$ a_n 有极限.

(3) $\sqrt{n}(a-a_n) = \underline{\sqrt{n}(a-a_{n+p})} + \overbrace{\underline{\sqrt{n}(a_{n+p}-a_n)}}^{\in[0,1]}$

$$\sqrt{n}(a_{n+p}-a_n) = \sqrt{n}\left[2\sqrt{n+p}-\sqrt{n}-\sum_{n+1}^{n+p}\frac{1}{\sqrt{k}}\right]$$

$$= \sqrt{n}\left[\frac{1}{\sqrt{n+p-1}}+\cdots+\frac{1}{\sqrt{n}}-\sum_{n+1}^{n+p}\frac{1}{\sqrt{k}}\right]$$

$$= \sqrt{n}\left[\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+p}}\right]$$

$$= 1-\sqrt{\frac{n}{n+p}}$$

$\underline{\sqrt{n}(a-a_{n+p})}$. 取 p 充分大.

故 $\sqrt{n}(a-a_n)$ 有界.

07-08

7. 设函数 $f(x)$ 在 $[0,+\infty)$ 上连续且满足

$$|f(x)| \le \underbrace{e^{kx} + k\int_0^x |f(t)|\,dt}_{} = y(x)$$

求证：$|f(x)| \le (kx+1)e^{kx}$

Pr: 令 $y(x) = e^{kx} + k\int_0^x |f(t)|\,dt$

$$y'(x) = ke^{kx} + k|f(x)| \le ke^{kx} + ky(x)$$

$$y'(x) - ky(x) \le ke^{kx}$$

$$e^{-kx}[y'(x) - ky(x)] \le k$$

$$(e^{-kx}y(x))' \le k$$

$$\Rightarrow e^{-kx}y(x) \le kx+1$$

$$y(x) \le e^{kx}(kx+1)$$

$$|f(x)| \le y(x) \le e^{kx}(kx+1)$$

#

05-06 期末

4. 设 $f(x)$ 是区间 $[-1,1]$ 上的连续函数，且对一切在 $[-1,1]$ 上可积的奇函数 $g(x)$，有

$$\int_{-1}^{1} f(x)g(x)\,dx = 0.$$

求证：$f(x)$ 是 $[-1,1]$ 上的偶函数.

Pr: $\int_{-1}^{1} f(x)g(x)\,dx$

$$= \int_{-1}^{0} f(x)g(x)\,dx + \int_{0}^{1} f(x)g(x)\,dx$$

$$= -\int_{1}^{0} f(-x)g(-x)\,dx + \int_{0}^{1} f(x)g(x)\,dx$$

$$= -\int_{0}^{1} f(-x)g(x)\,dx + \int_{0}^{1} f(x)g(x)\,dx$$

$$= \underline{\int_{0}^{1} [f(x)-f(-x)]\,g(x)\,dx = 0.}$$

$$g(x) = \begin{cases} 1, & x \in (a,b) \\ -1, & x \in (-b,-a) \end{cases}$$

$$\Rightarrow \int_{a}^{b} [f(x)-f(-x)]\,dx = 0.$$

令 $h(x) = f(x) - f(-x)$ 连续

$$\underbrace{\int_a^b h(x)\,dx = 0}$$

$\Rightarrow h \equiv 0$.

若 $h(x_0) > 0$, 存在 (c, d), s.t. $f((c,d)) > 0$

$\Rightarrow \int_c^d h(x)\,dx = 0$ 矛盾!

$f(x) = f(-x)$ 故 f 是 $[-1, 1]$ 偶函数.

#.

15-16. 第三次测试

5. 设 $f(x)$ 在 $[a,b]$ 上有连续导函数. 证明:

$$\max_{a\le x\le b}|f(x)| \le \frac{1}{b-a}\left|\int_a^b f(x)\,dx\right| + \int_a^b |f'(x)|\,dx$$

Pr: $f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

$$f(x)-f(\xi) = \int_\xi^x f'(t)\,dt$$

$$\begin{aligned} |f(x)| &= \left|f(\xi) + \int_\xi^x f'(t)\,dt\right| \\ &\le |f(\xi)| + \int_\xi^x |f'(t)|\,dt \\ &\le |f(\xi)| + \int_a^b |f'(t)|\,dt \end{aligned}$$

两边取 max

$$\max_{a\le x\le b}|f(x)| \le |f(\xi)| + \int_a^b |f'(x)|\,dx$$

#

13-14 第三次测试

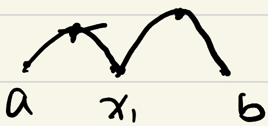
7. 设 $f(x)$ 在 $[a,b]$ 连续，且

$$\int_a^b f(x)dx = 0,\quad \int_a^b x f(x)dx = 0$$

证明：至少存在两点 x_1，$x_2 \in (a,b)$ 使 $f(x_1)=f(x_2)=0$

Pr: $F(x) = \int_a^x f(t)dt$.

$F(a) = F(b) = 0$.



$F(x_1)=0$.

$$0 = \int_a^b x f(x) dx$$

$$= \int_a^b x\, dF(x) = -\int_a^b F(x) dx.$$

故存在 $x_1 \in (a,b)$ $F(x_1) = 0$.

故 $\exists\ \xi_1 \in (a, x_1)$ $F'(\xi_1) = f(\xi_1) = 0$,

$\exists\ \xi_2 \in (x_1, b)$ $F'(\xi_2) = f(\xi_2) = 0$ #.