1. 习题4.1

(2) $\int \frac{e^{3x}+1}{e^x+1}dx = \int (e^{2x}-e^x+1)dx = \frac{1}{2}e^{2x}-e^x+x+C$

(4) $\int \tan^2 x\,dx = \int (\frac{1}{\cos^2 x}-1)\,dx = \int \frac{1}{\cos^2 x}dx - \int 1\,dx = \tan x - x + C$

(6) $\int \frac{1+\cos^2 x}{1+\cos 2x}dx = \int \frac{\cos^2 x+1}{-\sin^2(x)+\cos^2 x+1}dx = \int \frac{\cos^2 x+1}{2\cos^2 x}dx = \frac{1}{2}\int (\frac{1}{\cos^2 x}+1)dx = \frac{1}{2}\tan x + \frac{x}{2} + C$

2. (1) $\int (2x-1)^{100}dx = \frac{(2x-1)^{101}}{202} + C$

(2) $\int \frac{1}{x^2}\sin\frac{1}{x}\,dx \overset{u=\frac{1}{x}}{=} \int -\sin(u)\,du = \cos(u) \overset{+C}{=} \cos\frac{1}{x} + C$

(3) 令 $u=\sin x+\cos x+1$ $\quad du=(\cos x-\sin x)dx$

原式 $=\int \frac{1}{u}du = \ln(|u|) \overset{+C}{=} \ln(|\sin x+\cos x+1|)+C$

(4) 原式 $=\int \arctan(x)\,d\arctan(x) = \frac{1}{2}\arctan^2(x)+C$

(5) 令 $v=x^2$ $\quad dv=2x\,dx$

原式 $=\frac{1}{2}\int\sqrt{1-v}\,dv = -\frac{1}{3}(\sqrt{1-v})^3+C = -\frac{1}{3}(\sqrt{1-x^2})^3+C$

(6) 令 $u=\sqrt{x}$ $\quad du=\frac{dx}{2\sqrt{x}}$

原式 $=\int\frac{2}{1+u^2}du = 2\arctan u+C = 2\arctan\sqrt{x}+C$

(7) 令 $u=\arctan(\frac{1}{x})$ $\quad du=-\frac{1}{x^2+1}dx$

原式 $=\int -u\,du = -\frac{1}{2}u^2+C = -\frac{1}{2}\arctan^2(\frac{1}{x})+C$

(8) 令 $u=x\ln x$ $\quad du=(1+\ln x)dx$

原式 $=\int\frac{1}{1+u}du = \ln(|1+u|)+C = \ln(|1+x\ln x|)+C$

3. (1) 令 $x=\ln(t^2+2)$ $\quad dx=\frac{2t}{t^2+2}$

$\int\sqrt{e^x-2}\,dx = \int\frac{2t^2}{t^2+2}dt = \int(2-\frac{4}{t^2+2})dt = 2t-2\sqrt{2}\arctan\frac{t}{\sqrt{2}}+C$

$= 2\sqrt{e^x-2}-2\sqrt{2}\arctan\sqrt{\frac{e^x-2}{2}}+C$

3. (2) 令 $x = a\tan t \quad dx = a\sec^2 t\,dt$

原式 $= \int a\cdot\sec t\cdot a\sec^2 t\,dt = a^2\int \sec^3 t\,dt$

$$\int \sec^3 t\,dt = \int \sec t\,d(\tan t) = \sec t\tan t - \int \tan t\,d(\sec t)$$
$$= \sec t\tan t - \int \tan^2 t\sec t\,dt$$
$$= \sec t\tan t - \int (\sec^2 t - 1)\sec t\,dt$$
$$= \sec t\tan t - \int \sec^3 t\,dt + \underline{\int \sec t\,dt} \underset{①}{=} \ln|\sec t + \tan t$$
$$= \sec t\tan t + \ln|\sec t + \tan t|) - \ln|\sec t + \tan t| + C$$

$$\Rightarrow \int \sec^3 t\,dt = \frac{1}{2}(\sec t\tan t + \ln|\sec t + \tan t|) - \ln|\sec t+\tan t| + C$$

①式证明: $\int \sec t\,dt = \int \frac{\cos t}{\cos^2 t}dt = \int \frac{1}{1-\sin^2 t}d(\sin t) \overset{u=\sin t}{=\!=} \int \frac{1}{1-u^2}du = \int \frac{1}{(1-u)(1+u)}du$

$$= \frac{1}{2}\int \left(\frac{1}{u+1} - \frac{1}{u-1}\right)du$$
$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C$$
$$= \ln\sqrt{\left|\frac{u+1}{u-1}\right|} + C$$

代回

$$= \ln\sqrt{\frac{1+2\sin t+\sin^2 t}{\cos^2 t}} + C = \ln|\sec t + \tan t| + C$$

(3) 设 $x = \frac{a}{\cos\theta}$，则 $\cos\theta = \frac{a}{x}$，$\sin\theta = \sqrt{1-\frac{a^2}{x^2}}$

$$\int \frac{1}{(x^2-a^2)^{\frac{3}{2}}}dx = \int \frac{1}{\frac{a^3\sin^3\theta}{\cos^3\theta}}\cdot\frac{a\sin\theta}{\cos^2\theta}d\theta = \frac{1}{a^2}\int \frac{1}{\sin^2\theta}d(\sin\theta)$$
$$= -\frac{1}{a^2}\frac{x}{\sqrt{x^2-a^2}} + C$$

(4) $\int \frac{x^2}{\sqrt{a^2-x^2}}dx = a^2\int \frac{(\frac{x}{a})^2}{\sqrt{1-(\frac{x}{a})^2}}d(\frac{x}{a}) \qquad$ 令 $v = \frac{x}{a}$

$$= a^2\int \frac{v^2}{\sqrt{1-v^2}}dv \qquad v = \sin u$$
$$= a^2\int \frac{\sin^2 u}{\cos u}\cdot\cos u\,du = a^2\int \sin^2 u\,du$$
$$= \frac{a^2}{2}\int (1-\cos 2u)\,du = \frac{a^2}{2}(u + \frac{1}{2}\sin 2u) + C$$
$$= \frac{a^2}{2}(\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}) + C$$
$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

(5) 设 $t = \sqrt{x+1}$，则 $x = t^2 - 1$

$$\int \frac{1}{1+\sqrt{x+1}}dx = \int \frac{1}{1+t}d(t^2-1) = 2\int (1-\frac{1}{1+t})dt = 2t - 2\ln(1+t) + C$$
$$= 2\sqrt{x+1} - 2\ln(1+\sqrt{x+1}) + C$$

(6) $\int \frac{x\ln x}{(1+x^2)^{\frac{3}{2}}}dx$ 令 $v=x^2$

$$原式 = \frac{1}{4}\int \frac{\ln v}{(1+v)^{\frac{3}{2}}}dv \quad 令\ u=\frac{1}{\sqrt{1+v}} \quad 则\ du=-\frac{dv}{2(1+v)^{\frac{3}{2}}}$$

$$= -\frac{1}{2}\int \ln\left(\frac{1}{u^2}-1\right)du$$

$$= \int \ln u\,du - \frac{1}{2}\int \ln(1-u)du - \frac{1}{2}\int \ln(1+u)du$$

$$= u\ln u - u - \frac{1}{2}\left[-(1-u)\ln(1-u)+(1-u) + (1+u)\ln(1+u)-(1+u)\right]$$

$$= u\ln u - \frac{1}{2}\ln\frac{1+u}{1-u} - \frac{u}{2}\ln(1-u^2)+C$$

$$= u\ln\frac{u}{\sqrt{1-u^2}} - \frac{1}{2}\ln\frac{1+u}{1-u}$$

$$= \frac{\ln x}{\sqrt{1+x^2}} - \frac{1}{2}\ln\frac{(\sqrt{x^2+1}+1)^2}{(\sqrt{x^2+1}-1)(\sqrt{x^2+1}+1)}+C$$

$$= \frac{\ln x}{\sqrt{1+x^2}} - \frac{1}{2}\ln(\sqrt{x^2+1}+1)^2 + \frac{1}{2}\ln x^2 + C$$

$$= \frac{\ln x}{\sqrt{1+x^2}} - \ln(\sqrt{x^2+1}+1) + \ln x + C$$

(7)

$$\int \frac{1-\ln x}{(x-\ln x)^2}dx = \int \frac{1}{\left(1-\frac{\ln x}{x}\right)^2}d\left(\frac{\ln x}{x}\right) = \frac{x}{x-\ln x}+C$$

(8) $\int \frac{1}{x^2\sqrt{x^2+a^2}}dx$ 令 $u=\frac{x}{a}$

$$原式 = \int \frac{a\,du}{a^2u^2\sqrt{a^2u^2+a^2}} = \frac{1}{a^2}\int\frac{du}{u^2\sqrt{u^2+1}} \quad 令\ u=\tan v \quad du=\frac{1}{\cos^2 v}dv$$

$$= \frac{1}{a^2}\int \frac{\cos v}{\sin^2 v}dv$$

$$= \frac{1}{a^2}\int \frac{1}{\sin^2 v}d(\sin v)$$

$$= -\frac{1}{a^2}\left(\frac{1}{\sin v}\right) = -\frac{1}{a^2}\frac{\sqrt{x^2+a^2}}{x}+C$$

4. (1) $\int |x|dx = \int x\cdot \mathrm{sgn}(x)dx = \mathrm{sgn}(x)\cdot\frac{x^2}{2}+C$

$$= \frac{|x|\cdot x}{2}+C$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

$\cancel{(2)\int \max(1,x^2)\,dx =}$

但不能这么写 要有C.

$$(2) \int \max(1,x^2) = \begin{cases} \frac{x^3}{3} & x\le -1 \\ \frac{2}{3}+x & -1<x\le 1 \\ \frac{4}{3}+\frac{x^3}{3} & x>1 \end{cases}$$

$$\max\{1,x^2\} = \begin{cases} x^2 & x\le -1 \\ 1 & x\in(-1,1) \\ x^2 & x\ge 1 \end{cases}$$

$$\Rightarrow \int \max\{1,x^2\}dx = \begin{cases} \int x^2 & x\le -1 \\ \int 1 & x\in(-1,1) \\ \int x^2 & x\ge 1 \end{cases}$$

$$= \begin{cases} \frac{1}{3}x^3+C_1 & x\le -1 \\ x+C_2 & x\in(-1,1) \\ \frac{1}{3}x^3+C_3 & x\ge 1 \end{cases}$$

取 $C_2=C$ 利用 $\begin{cases} -\frac{1}{3}+C_1=C_2-1 \\ C_2+1=\frac{1}{3}+C_3 \end{cases} \Rightarrow \begin{cases} C_1=C-\frac{2}{3} \\ C_3=C+\frac{2}{3} \end{cases}$ 代入 即可.

5.

(7) $\int x\arcsin x\,dx$

$= x\arcsin x - \int x\frac{1}{\sqrt{1-x^2}}\,dx$

$= x\arcsin x + \frac{\sqrt{1-x^2}}{2} + C$

(8) $\int x(\arctan x)^2\,dx$

$= \int(\arctan x)^2\,d\left(\frac{x^2}{2}\right)$

$= \frac{x^2}{2}(\arctan x)^2 - \frac{1}{2}\int\frac{x^2}{1+x^2}2\arctan x\,dx$

$= \frac{x^2}{2}(\arctan x)^2 - \int\arctan x\cdot\frac{1+x^2-1}{1+x^2}\,dx$

$= \frac{x^2}{2}(\arctan x)^2 - \int\arctan x + \int\arctan x\,d\arctan x$

$= \frac{1}{2}(\arctan x)^2x^2 - \left(x\arctan x - \int x\frac{1}{1+x^2}dx\right) + \frac{1}{2}(\arctan x)^2$

$= \frac{x^2+1}{2}(\arctan x)^2 - x\arctan x + \frac{1}{2}\ln(1+x^2) + C$

(9) $\int(\arcsin x)^2\,dx$

$= x(\arcsin x)^2 - 2\int\arcsin x\cdot\frac{x}{\sqrt{1-x^2}}\,dx$

$= x(\arcsin x)^2 + 2\int\frac{d(1-x^2)}{2\sqrt{1-x^2}}\arcsin x$

$= x(\arcsin x)^2 + 2\int\arcsin x\,d(\sqrt{1-x^2})$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(10) $\int\ln(x+\sqrt{x^2+1})\,dx$

令 $u = \ln(x+\sqrt{x^2+1})$

$\int\ln(x+\sqrt{x^2+1})\,dx = x\ln(x+\sqrt{x^2+1}) - \int x\,d\ln(x+\sqrt{x^2+1})$

$= x\ln(x+\sqrt{x^2+1}) - \int\frac{x}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x\right)d$

$= x\ln(x+\sqrt{x^2+1}) - \int\frac{x}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)dx$

$= x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1} + C$

7 (1) $\int \frac{1}{1+e^x}dx$

$= \int (1-\frac{e^x}{1+e^x})dx$

$= x - \ln(1+e^x) + C.$

(2) $\int \frac{x^2-1}{x^4+x^2+1}dx$

$= \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+1}dx$

$= \int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-1}$

$= \int \frac{dy}{y^2-1} = \frac{1}{2}\int(\frac{1}{y-1}-\frac{1}{y+1})dy$

$= \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| + C$

$= \frac{1}{2}\ln|x+\frac{1}{x}-1| - \frac{1}{2}\ln|x+\frac{1}{x}+1| + C$

(3) $\int \frac{1}{x^4+x^6}dx$

$= \int (\frac{1}{x^4} - \frac{1}{x^2} + \frac{1}{1+x^2})dx$

$= -\frac{1}{3x^3} + \frac{1}{x} + \arctan x + C$

(4) $\int x\sqrt{x-2}\,dx$

令 $x = t^2+2.$

$= \int (t^2+2)t\cdot 2t\,dt$

$= \frac{2}{5}t^5 + \frac{4}{3}t^3 + C$

$= \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)\sqrt{x-2} + C.$

(5) $\int \frac{\sqrt{x-1}\arctan\sqrt{x-1}}{x}dx$

设 $u=\sqrt{x-1} \Leftrightarrow x=u^2+1.\quad dx=2u\,du$

原式 $= \int \frac{u\arctan u}{u^2+1}2u\,du$

$= 2\int \frac{u^2\arctan u}{u^2+1}du$

$= 2\int \arctan u\,du - 2\int \frac{\arctan u}{u^2+1}du$

$= 2u\arctan u - 2\int \frac{u}{u^2+1}du - 2\int \arctan u\,d(\arctan u)$

$= 2\sqrt{x-1}\arctan\sqrt{x-1} - (\arctan\sqrt{x-1})^2 - \ln x + C.$

(6) $\int \frac{xe^x}{\sqrt{e^x-2}}dx$

令 $u=\sqrt{e^x-2} \Rightarrow u^2=e^x-2 \Rightarrow x=\ln(u^2+2)$

$= \int 2\ln(u^2+2)\,du$

$= 2u\ln(u^2+2) - 2\int \frac{2u^2}{u^2+2}du$

$= 2u\ln(u^2+2) - 4u + 4\sqrt{2}\arctan\frac{u}{\sqrt{2}} + C$

7. $\int xe^x \sin x\,dx$

$= \int x\sin x\,de^x$

$= xe^x\sin x - \int e^x(\sin x + x\cos x)dx$

$= xe^x\sin x - \int(\sin x + x\cos x)de^x$

$= e^x(x\sin x - \sin x - x\cos x)$

$+ \int e^x(2\cos x - x\sin x)dx$

$= e^x(x\sin x - \sin x - x\cos x)$

$+ 2\int e^x\cos x\,dx - \int xe^x\sin x\,dx$

$= \frac{e^x}{2}[x(\sin x - \cos x) + \cos x] + C$

8. $\int \frac{1}{(1+\tan x)\sin^2 x}dx$

$d(\frac{1}{\tan x}) = -(\tan x)^{-2}\cdot\frac{1}{\cos^2 x}$

$= -\frac{1}{\sin^2 x}$

$\Rightarrow$ 原式 $= -\int \frac{1+\tan x-\tan x}{1+\tan x}d(\frac{1}{\tan x})$

$= -\frac{1}{\tan x} + \int\frac{\tan x}{1+\tan x}d\frac{1}{\tan x}$

$= -\frac{1}{\tan x} + \int\frac{1}{1+\frac{1}{\tan x}}d\frac{1}{\tan x}$

$= -\frac{1}{\tan x} + \ln\left|1+\frac{1}{\tan x}\right| + C$

9. $\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\,dx$

$1-x=t^2$

$$= \int \frac{t\,(-2t)}{1-\sqrt{1-t^2}}\,dt$$

$$= \int \frac{-2t^2}{1-\sqrt{1-t^2}}\,dt$$

$$= -2\int \left(1+\sqrt{1-t^2}\right)dt$$

$t=\sin\theta$

$$= -2\int (1+\cos\theta)\cos\theta\,d\theta$$

$$= -2\sin\theta - \cancel{\theta} - \frac{1}{2}\sin 2\theta + C$$

$$= -2t - \arcsin t - \cancel{\theta}\ t\sqrt{1-t^2} + C$$

$$= -2\sqrt{1-x} - \arcsin\sqrt{1-x} - x\sqrt{1-x} + C.$$

10. $\int \frac{1}{x^2}\sqrt{\frac{x-1}{x+1}}\,dx$

10. $t=\sqrt{\frac{x-1}{x+1}}$ $\quad t^2(x+1)=(x-1)$ $\quad x=\frac{1+t^2}{1-t^2}=-1+\frac{2}{1-t^2}$

$$原式=\int t\cdot\left(\frac{1-t^2}{1+t^2}\right)^2 \frac{2\cdot 2t}{(1-t^2)^2}\,dt$$

$$=\int \frac{4t^2}{(1+t^2)^2}\,dt$$

$$=4\int\left(-\frac{1}{(1+t^2)^2}+\frac{1}{1+t^2}\right)dt.$$

由例4.1.14 $\quad \int\frac{dt}{(1+t^2)^2}=\frac{t}{2(t^2+1)}+\frac{1}{2}\int\frac{1}{t^2+1}\,dt$

$$=\frac{t}{2(t^2+1)}+\frac{1}{2}\arctan t$$

$I=2\arctan t-\frac{2t}{(t^2+1)}$ 将t代回即得结果

4.2 节

1.

(1)

$$\int \frac{1}{(x+2)(x-1)}dx = \frac{1}{3}\int\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$$

(2) $$\int \frac{x^4}{x^2+1}dx = \int (x^2-1) + \frac{1}{x^2+1}dx = \frac{1}{3}x^3 - x + \arctan x + C$$

(3) $$\int \frac{x^2+1}{x^3-x}dx = \int\left(1+\frac{x+1}{x(x-1)(x+1)}\right)dx = \int\left(1+\frac{1}{x-1}-\frac{1}{x}\right)dx$$

$$= x + \ln\left|\frac{x-1}{x}\right| + C$$

(4) $$\frac{1}{(x^2+1)(x+1)x} = \frac{1}{x(x^2+1)} - \frac{1}{(x+1)(x^2+1)} = \left(\frac{1}{x}-\frac{x}{x^2+1}\right) - \frac{1}{2}\left(\frac{1}{x+1}-\frac{x-1}{x^2+1}\right)$$

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(1)

$$\int \frac{1}{(x+2)(x-1)}dx = \frac{1}{3}\int\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx = \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$$

(2) $$\int \frac{x^4}{x^2+1}dx = \int (x^2-1)+\frac{1}{x^2+1}dx = \frac{1}{3}x^3 - x + \arctan x + C$$

T2(1) 令 $t=\tan\frac{x}{2}$，则 $\sin x=\frac{2t}{1+t^2}$，$\cos x = \frac{1-t^2}{1+t^2}$，$\frac{dx}{dt} = \frac{2}{1+t^2}dt$

$$\int \frac{1+\sin x}{\sin x(1+\cos x)}dx = \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\cdot\frac{2}{1+t^2}}\cdot\frac{2}{1+t^2}dt$$

$$= \frac{1}{2}\int \frac{(t+1)^2}{t}dt.$$

$$= \frac{1}{4}t^2 + t + \frac{1}{2}\ln|t| + C$$

$$= \frac{1}{4}\tan^2\frac{x}{2} + \tan\frac{x}{2} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C$$