Homework-2

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Problem.2

$$\begin{cases} x &= \gamma(x' + vt') \\ t &= \gamma(t' + \frac{v}{c^2}x') \end{cases} \tag{1}$$

Solution 1:

solve linear equation directly

Solution 2:

take V as -v and exchange σ' and σ

answer:

$$\begin{cases} x' &= \gamma(x - vt) \\ t' &= \gamma(t - \frac{v}{c^2}x) \end{cases} \tag{2}$$

Problem.3

Prove:

$$A^{\alpha} B_{\alpha} = A'^{\alpha} B'_{\alpha} \tag{3}$$

Solution:

use the transformation matrix :

$$T_{trans} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and we get:

$$A'^{\alpha} = T_{trans} A^{\alpha} \qquad B'_{\alpha} = T'_{trans} B_{\alpha}$$

So:

$$A^\alpha B_\alpha = T_{trans} A^\alpha T'_{trans} B'_\alpha$$

(Alert: $T_{trans} T'_{trans} = I$)

$$\begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

(matrix multiplication)

Problem.4

Prove:

$$\begin{cases} r &= r' + (\gamma - 1)\dfrac{r' \cdot V}{V^2} + \gamma V t' \\ t &= \gamma\left(t' + \dfrac{r' \cdot V}{c^2}\right) \end{cases} \tag{4}$$

Solution:

separate $r_{\|}$ *and* $r_{\perp}$ as it hints

$$r_{\|} = \frac{r \cdot V}{V^2} V$$

$$r_{\perp} = r - \frac{r \cdot V}{V^2} V$$

Then transform $r_{\|}$ with Lorentz rules and we get :

$$\begin{cases} r_{\|} &= \gamma(r'_{\|} + vt') \\ t &= \gamma(t' + \frac{v}{c^2}|r_{\|}|) \end{cases} \tag{5}$$

and left $r_{\perp}$ as it is:

$$r_{\perp} = r'_{\perp} \tag{6}$$

add up (5) and (6) and rewrite $r'_{\|}$ with $\dfrac{r' \cdot V}{V^2} V$ to get :

$$\begin{cases} r &= r' + (\gamma - 1)\dfrac{r' \cdot V}{V^2} + \gamma V t' \\ t &= \gamma\left(t' + \dfrac{r' \cdot V}{c^2}\right) \end{cases} \tag{7}$$

Problem.5

Derive a tensor's Lorentz transformation

Solution:

separate into two sections

#A

symmetry part : $A_{ij} = A_{ji}$

solution-1:

take A_{ij} as $M_i N_j$

and for vector M_i and N_j we have:

$$\begin{cases} M_i & = & T'_{trans} M'_i \\ N_j & = & T'_{trans} N'_j \end{cases}$$

can infer :

$$M_i N_j = T'_{trans} M'_i T'_{trans} N'_j$$

get element clearly presented and get the result.

solution-2:
use matrix transformation in frame transform:
for:

$$M_j = T'_{trans} M'_j$$

there is:

$$A_{ij} = T_{trans} A'_{ij} T'_{trans}$$

in element form:

$$\begin{pmatrix} A^{00} & A^{00} & A^{01} & A^{02} \\ A^{03} & A^{10} & A^{20} & A^{30} \\ A^{20} & A^{21} & A^{22} & A^{23} \\ A^{30} & A^{31} & A^{32} & A^{33} \end{pmatrix} =$$

$$\begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} A'^{00} & A'^{00} & A'^{01} & A'^{02} \\ A'^{03} & A'^{10} & A'^{20} & A'^{30} \\ A'^{20} & A'^{21} & A'^{22} & A'^{23} \\ A'^{30} & A'^{31} & A'^{32} & A'^{33} \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

so we get the answer:

$$
\begin{array}{rcl}
A^{00} & = & \gamma^2(A'^{00} + 2\beta A'^{01} + \beta^2 A'^{11}) \\
A^{01} & = & \gamma((1+\beta^2)A'^{01} + \beta(A'^{00} + A'^{11})) \\
A^{02} & = & \gamma(A'^{02} + \beta A'^{12}) \\
A^{03} & = & \gamma(A'^{03} + \beta A'^{13}) \\
 & & \\
A^{11} & = & \gamma^2(A'^{11} + 2\beta A'^{01} + \beta^2 A'^{00}) \\
A^{12} & = & \gamma(A'^{12} + \beta A'^{02}) \\
A^{13} & = & \gamma(A'^{13} + \beta A'^{03}) \\
 & & \\
A^{22} & = & A'^{22} \\
A^{23} & = & A'^{23} \\
 & & \\
A^{33} & = & A'^{33}
\end{array}
\tag{8}
$$

with $A_{ij} = A_{ji}$

#B
anti-symmetry part : $A_{ij} = -A_{ji}$
use $A_{ii} = 0$

to simplify the answer of #A

so we get the answer:

$$\begin{aligned}
A^{01} &= \gamma((1-\beta^2)A'^{01} + \beta(A'^{00} + A'^{11})) \\
A^{02} &= \gamma(A'^{02} + \beta A'^{12}) \\
A^{03} &= \gamma(A'^{03} + \beta A'^{13}) \\
\\
A^{12} &= \gamma(A'^{12} + \beta A'^{02}) \\
A^{13} &= \gamma(A'^{13} + \beta A'^{03}) \\
\\
A^{23} &= A'^{23}
\end{aligned} \tag{9}$$

$$A^{00} = A^{11} = A^{22} = A^{33} = 0$$

Problem.6

derive the Lorentz transform of velocity vector u^α

Solution: use the transform matrix. T_{trans}

$$\begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} u'^0 \\ u'^1 \\ u'^2 \\ u'^3 \end{pmatrix}$$

tips:

$$A^\alpha = (\phi, \vec{A}) \qquad x^\alpha = (ct, \vec{x}) = (\frac{cdt}{d\tau}, \frac{d\vec{x}}{d\tau})$$

$$v^\alpha = (c\gamma, \vec{c}\gamma) \qquad p^\alpha = (\frac{\epsilon}{c}, \vec{p}) \qquad j^\alpha = (\rho c, \vec{j})$$

alert :

$$v^\alpha = (c\gamma, \vec{c}\gamma) \text{ in } \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} u'^0 \\ u'^1 \\ u'^2 \\ u'^3 \end{pmatrix}$$

there are 3 γ (in u^α and T_{trans} and u'^α)