Homework-1

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April 2, 2016

$\boxed{\text{Problem.2}}$ Prove:

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \tag{1}$$

Properties of $\delta(x)$:

Obviously, we see that, from the defination, Dirac's delta function must be even in x, $\delta(-x) = \delta(x)$ If $a > 0$,

$$\delta(ax) = \frac{1}{a}\delta(x), a > 0. \tag{2}$$

Equation (2) can be proved by making the substitution $x = y/a$:

$$\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty} f(y/a)\delta(y)dy = \frac{1}{a}f(0). \tag{3}$$

Shift of origin:

$$\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx = f(x_0), \tag{4}$$

which can be proved by making the substitution $y = x - x_0$ and nothing that when $y = 0, x = x_0$.

If the argument of $\delta(x)$ is a function g(x) with simple zeros at points a_i on the real axis(and therefore $g'(a_i) \neq 0$),

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}. \tag{5}$$

To prove it, we write

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \sum_i \int_{a_i-\varepsilon}^{a_i+\varepsilon} f(x)\delta((x - a_i)g'(a_i))dx, \tag{6}$$

where we have decomposed the original integral into a sum of integrals over small in-tervals containing the zeros of g(x). In these intervals, we replaced g(x) by the leading term in its Taylor series. Applying Eq.(2) and Eq.(4) to each term of the sum, we confirm Eq.(5).

$\boxed{\text{Problem.3}}$

Prove:

The Galilean transformation of the coordinates is:

$$\begin{cases} x' &= x + v \cdot t \\ t' &= t \end{cases} \tag{7}$$

The Galilean transformation of the differential operator is:

$$\begin{cases} \dfrac{\partial}{\partial t'} &= \dfrac{\partial}{\partial t}\dfrac{\partial t}{\partial t'} + \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial t'} \\[2ex] \dfrac{\partial}{\partial x'} &= \dfrac{\partial}{\partial t}\dfrac{\partial t}{\partial x'} + \dfrac{\partial}{\partial x}\dfrac{\partial x}{\partial x'} \end{cases} \tag{8}$$

So apply Eq.(7) to Eq.(8), we have:

$$\begin{cases} \frac{\partial^2\phi}{\partial t'^2} & = & \frac{\partial}{\partial t'}[\frac{\partial\phi}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial t'}] \\ \frac{\partial^2\phi}{\partial x'^2} & = & \frac{\partial}{\partial x'}[\frac{\partial\phi}{\partial t}\frac{\partial t}{\partial x'} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x'}] \end{cases} \tag{9}$$

then,

$$\begin{cases} \frac{\partial^2\phi}{\partial t'^2} & = & \frac{\partial^2\phi}{\partial t^2} + 2\frac{\partial^2\phi}{\partial t\partial x}v + \frac{\partial^2\phi}{\partial x^2}v^2 \\ \frac{\partial^2\phi}{\partial x'^2} & = & \frac{\partial^2\phi}{\partial x^2} \end{cases} \tag{10}$$

Finally, we got

$$\frac{\partial^2\phi}{\partial t'^2} - c^2\frac{\partial^2\phi}{\partial x'^2} = \frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial^2\phi}{\partial t\partial x}v + \frac{\partial^2\phi}{\partial x^2}v^2 \tag{11}$$

so, in most cases, the equation is not invariant under Galilean transformation.

$\boxed{\text{Problem.4}}$

Please refer to Prof. Xin Tao's Lecture note, Page 103, it gives details about the derivation of 3-D Electromagnetic Wave Equation.

Problem.5

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla \cdot (\mathbf{BB} - \frac{1}{2}B^2\mathbf{I}) - (\nabla \cdot \mathbf{B})\mathbf{B} \qquad (12)$$

Prove:

$LHS = (\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla\mathbf{B} \cdot \mathbf{B}$(By middle-outer rule)

$RHS = \nabla \cdot (\mathbf{BB}) - \nabla \cdot (\frac{1}{2}B^2\mathbf{I}) - (\nabla \cdot \mathbf{B})\mathbf{B}$

$= \mathbf{B} \cdot \nabla\mathbf{B} + \underset{\sim\sim\sim}{(\nabla \cdot \mathbf{B})\mathbf{B}} - \frac{1}{2}\nabla B^2 - \underset{\sim\sim\sim}{(\nabla \cdot \mathbf{B})\mathbf{B}}$

$= \mathbf{B} \cdot \nabla\mathbf{B} - \frac{1}{2}\nabla(\mathbf{B} \cdot \mathbf{B})$

$= \mathbf{B} \cdot \nabla\mathbf{B} - \frac{1}{2}(\nabla\mathbf{B} \cdot \mathbf{B} + \nabla\mathbf{B} \cdot \mathbf{B})$

$= \mathbf{B} \cdot \nabla\mathbf{B} - \nabla\mathbf{B} \cdot \mathbf{B}$

Reference

Mathematics Methods for Physicists, Arfken

Lecture Notes on Electrodynamics, Xin Tao