Introduction to Nuclear Physics

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Chapter 1 Introduction

1.1 Why nuclear physics

- Aim: understand the properties of nuclei and their applications
- One of the most important branches and milestones in modern physics in the last century
- Intrinsic connections to quantum mechanics, a laboratory for quantum physics
- Applications
  - The origin of nuclei in the universe, e.g. nuclear nucleosynthesis;
  - Nuclear energy, nuclear fission and fusion;
  - Nuclear transmutation of radioactive waste with neutrons;
  - Radiotherapy for cancer with and proton and heavy ion beams;

- Applications (Continue)
  - Medical Imaging, e.g. MRI (Nuclear magnetic resonance), X-rays (better detectors lower doses), PET (Positron-electron tomography), etc.;
  - Radioactive Dating, e.g. C-14/C-12 dating for dead lives, Kr-81 dating for ground water;
  - Element analysis, e.g. forenesic (as in hair), biology (elements in blood cells), archaeology (provenance via isotope ratios).
- Early history
  - 1895 Roentgen – X-ray
  - 1896 Becquerel – radio activity
  - 1897 Thomson – electron
  - 1898 Curies – Radium
  - 1909 Rutherford – nucleus

1.2 From cgs-Gaussian to natural unit (1)

- cgs unit: [mass, length and time] $\rightarrow$ [g, cm, s] $\rightarrow \mathrm{g}^a \mathrm{cm}^b \mathrm{s}^c$
- natural unit: [action, velocity, energy] $\rightarrow [\hbar, c, \mathrm{eV}] \rightarrow \hbar^\alpha c^\beta \mathrm{eV}^\gamma$
- So the quantity with the dimension $[D] = \mathrm{g}^a \mathrm{cm}^b \mathrm{s}^c = \hbar^\alpha c^\beta \mathrm{eV}^\gamma$, then we can derive following relations

$$\begin{aligned} \alpha &= b + c \\ \beta &= -2a + b \\ \gamma &= a - b - c \end{aligned} \tag{1}$$

1.2 From cgs-Gaussian to natural unit (2)

- natural unit → cgs unit

$$
\begin{aligned}
1\,c &= 3\times 10^{10}\,\mathrm{cm}\cdot\mathrm{s}^{-1} \\
1\,\hbar &= 1.05\times 10^{-27}\,\mathrm{g}\cdot\mathrm{cm}^{2}\cdot\mathrm{s}^{-1} \\
1\,\mathrm{eV} &= 1.6\times 10^{-12}\,\mathrm{g}\cdot\mathrm{cm}^{2}\cdot\mathrm{s}^{-2} \\
1\,k_{\mathrm{B}} &= 1.3806488\times 10^{-16}\mathrm{g}\cdot\mathrm{cm}^{2}\cdot\mathrm{s}^{-2}\cdot\mathrm{K}^{-1}
\end{aligned} \tag{2}
$$

- cgs unit → natural unit

$$
\begin{aligned}
1\,\mathrm{s} &= 1.52\times 10^{15}\,\hbar\cdot\mathrm{eV}^{-1} \\
1\,\mathrm{cm} &= 5.06\times 10^{4}\,\hbar\cdot\mathrm{eV}^{-1}\cdot c \\
1\,\mathrm{g} &= 5.6\times 10^{32}\,\mathrm{eV}\cdot c^{-2} \\
1\,\mathrm{K} &= 8.617\times 10^{-5}\,\mathrm{eV}\cdot k_{\mathrm{B}}^{-1}
\end{aligned} \tag{3}
$$

Unrationalized Gaussian units in electromagnetics. In this unit, the factor 4π appears in the Maxwell's equation and it is absent in the Coulomb force law.

- Maxwell equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 4\pi\rho, \\
\nabla \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},
\end{aligned} \tag{4}$$

- The inverse-square force laws

$$\begin{aligned} \mathbf{F} &= \frac{q_1 q_2}{r^3}\mathbf{r}, \\ \mathbf{F} &= \frac{1}{c^2}\int\int\frac{I_1 d\mathbf{l}_1 \times (I_2 d\mathbf{l}_2 \times \mathbf{r})}{r^3}. \end{aligned} \quad (5)$$

- Lorentz-Heaviside and rationalized Gaussian units

$$\begin{aligned} \mathbf{E}_{\mathrm{LH}} &= \frac{1}{\sqrt{4\pi}}\mathbf{E}_{\mathrm{unrat-Gauss}}, \\ q_{\mathrm{LH}} &= \sqrt{4\pi}q_{\mathrm{unrat-Gauss}}. \end{aligned} \quad (6)$$

Useful conversion factors

$$\begin{aligned} \hbar c &= 197\,\mathrm{MeV}\cdot\mathrm{fm} \\ e^2 &\approx \frac{1}{137}\hbar c \end{aligned} \tag{7}$$

In contrast this relation becomes $e^2/(4\pi) \approx \hbar c/137$ in rationalized Gaussian or Lorentz-Heaviside units.

In the cgs-Gaussian units, the charge is in the electrostatic unit (esu) which can be determined from the Coulomb law

$$\begin{aligned} F &= \frac{q^2}{r^2} \rightarrow \mathrm{esu}^2 = \mathrm{g} \cdot \mathrm{cm} \cdot \mathrm{s}^{-2} \times \mathrm{cm}^2 = \mathrm{g} \cdot \mathrm{cm}^3 \cdot \mathrm{s}^{-2} \\ &\rightarrow \mathrm{esu} = \mathrm{g}^{1/2} \cdot \mathrm{cm}^{3/2} \cdot \mathrm{s}^{-1} \end{aligned} \quad (8)$$

We know that the Coulomb force law in the SI system has the following form

$$F = \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2} \quad (9)$$

Here we have

$$\begin{aligned}\epsilon_0 &= 8.8542\times 10^{-12}\ \mathrm{C^2N^{-1}m^{-2}} \\ \frac{1}{4\pi\epsilon_0} &= 8.99\times 10^{9}\ \mathrm{C^{-2}N^{1}m^{2}} \qquad (10)\end{aligned}$$

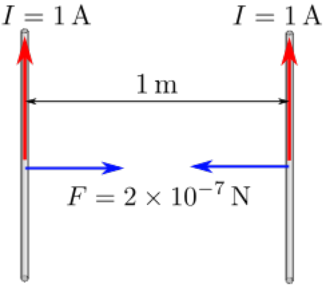
and the charge unit is Coulomb (C). We can determine the conversion rule for C to esu. In the SI system, we have $q = 1\ \mathrm{C}$ and $r = 1$ m, then the force is $F = 8.99\times 10^9$ N. In the unrationalized Gaussian units, we have $q = 1\ \mathrm{esu}$ and $r = 1$ cm, then the force is $F = 1\ \mathrm{dyn} = 1\ \mathrm{g\cdot cm\cdot s^{-2}} = 10^{-5}\ \mathrm{N}$. Comparing two units, we obtain

$$\begin{aligned}1\ \mathrm{C} &= 3\times 10^{9}\ \mathrm{esu} \\ 1\ \mathrm{e} &= 1.602\times 10^{-19}\ \mathrm{C} = 4.8\times 10^{-10}\ \mathrm{esu} \qquad (11)\end{aligned}$$

One ampere is defined as that the force between two wires as shown in figure is 2×10^{-7}N/m. The vacuum permeability is:

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N} \cdot \mathrm{A}^{-2} \quad (12)$$

Figure : The unit of Ampere. Force per unit length: $F/L = \mu_0 I_1 I_2/(2\pi d)$, where d is the distance between two wires.



In the SI system, the unit of the electric field is $\mathrm{Volt}/\mathrm{m} = \mathrm{N}/\mathrm{C}$, while in the unrationalized Gaussian units, the electric and magnetic fields have the same unit: Gauss (G). So we have

$$
\begin{aligned}
1\,\mathrm{G} &= \frac{\mathrm{dyn}}{\mathrm{esu}} = \mathrm{g}^{1/2}\cdot\mathrm{cm}^{-1/2}\cdot\mathrm{s}^{-1} \\
&= 6.92\times 10^{-2}\,(\hbar c)^{-3/2}\cdot\mathrm{eV}^2 \\
1\,\mathrm{Volt} &= 1\,\mathrm{N}\cdot\mathrm{m}/\mathrm{C} = \frac{10^7\,\mathrm{dyn}\cdot\mathrm{cm}}{3\times 10^9\,\mathrm{esu}} \\
&= \frac{1}{3}\times 10^{-2}\mathrm{g}^{1/2}\cdot\mathrm{cm}^{1/2}\cdot\mathrm{s}^{-1} \qquad (13)
\end{aligned}
$$

Then we obtain $1\,\mathrm{eV} = 1.6\times 10^{-12}\,\mathrm{g}\cdot\mathrm{cm}^2\cdot\mathrm{s}^{-2}$.

Also we obtain

$$\begin{aligned} e^2 &= 2.304\times 10^{-19}\,\mathrm{esu}^2 = 2.304\times 10^{-19}\,\mathrm{g}\cdot\mathrm{cm}^3\cdot\mathrm{s}^{-2} \\ \hbar c &= 3.15\times 10^{-17}\,\mathrm{g}\cdot\mathrm{cm}^3\cdot\mathrm{s}^{-2} \\ e^2 &\approx \frac{1}{137}\hbar c \end{aligned} \tag{14}$$

We can express $\hbar c$ by

$$\begin{aligned} \hbar c &= 3.15\times 10^{-17}\,(\mathrm{g}\cdot\mathrm{cm}^2\cdot\mathrm{s}^{-2})\cdot\mathrm{cm} \\ &= 3.15\times 10^{-4}(\mathrm{g}\cdot\mathrm{cm}^2\cdot\mathrm{s}^{-2})\cdot\mathrm{fm} \\ &= 197\,\mathrm{MeV}\cdot\mathrm{fm} \end{aligned} \tag{15}$$

In natural unit, we take

$$\hbar = c = k_B = 1 \tag{16}$$

where k_B is the Boltzmann constant.

For electric and magnetic fields in cgs units, we have

$$\begin{aligned} 1\,\mathrm{Gauss} &= \mathrm{g}^{1/2}\cdot\mathrm{cm}^{-1/2}\cdot\mathrm{s}^{-1} \\ &= 6.92\times 10^{-2}\,(\hbar c)^{-3/2}\cdot\mathrm{eV}^2 \end{aligned} \tag{17}$$

We can convert the proton and neutron masses in cgs unit to natural unit.

$$\begin{aligned} m_p &= 1.672621\times 10^{-24}\,\mathrm{g} = 938.27\,\mathrm{MeV}\cdot c^{-2} \\ m_n &= 1.674927\times 10^{-24}\,\mathrm{g} = 939.57\,\mathrm{MeV}\cdot c^{-2} \end{aligned} \tag{18}$$

With another natural unit k_B. The shear viscosity is defined by $F = \eta A \frac{dv}{dx}$ and entropy density is defined as thermal energy sVT.

$$\begin{aligned} [s] &= [\mathrm{cm}^{-3} \cdot k_B] \\ [\eta] &= [\mathrm{g} \cdot \mathrm{cm}^{-1} \cdot \mathrm{s}^{-1}] \end{aligned} \quad (19)$$

So the ratio η/s has the dimension

$$[\eta/s] = [\mathrm{g} \cdot \mathrm{cm}^{2} \cdot \mathrm{s}^{-1} \cdot k_B^{-1}] = \hbar \cdot k_B^{-1} \quad (20)$$

Chapter 2 Properties of Nuclei

2.1 Discover atomic nucleus (1)

Figure : Rutherford alpha scattering experiments in 1909.

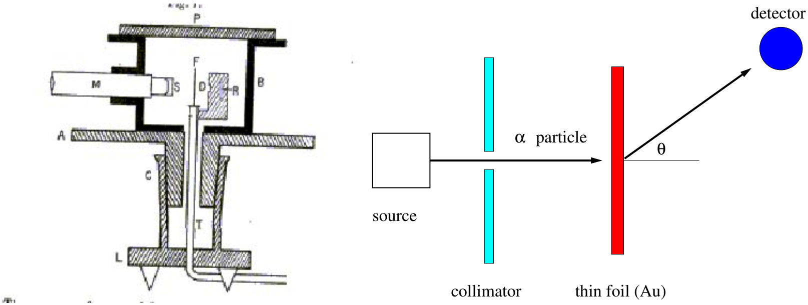
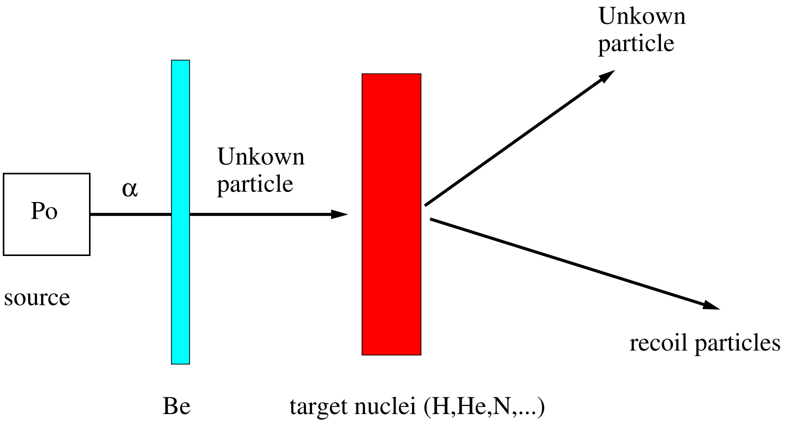


Figure : Chadwick found neutrons via reaction ${}^{4}_{2}He + {}^{9}_{4}Be \rightarrow {}^{12}_{6}C + n$ in 1932. The unknown particles carry no charges and has almost the same mass as proton.



2.1 Discover atomic nucleus (3)

Atomic mass unit is defined as 1/12 of the mass of ${}^{12}_{6}C$, i.e. N_A^{-1} g where $N_A = 6.022142 \times 10^{23}$ is the Avogadro constant. Atomic unit is

$$1u = 1.66 \times 10^{-27}\ \mathrm{kg} = 931.5\ \mathrm{MeV}/c^2, \tag{21}$$

where

$$\begin{aligned} 1\ \mathrm{eV} &= 1.6022 \times 10^{-19}\ \mathrm{J} = 1.78 \times 10^{-36}\ \mathrm{kg} \cdot c^2 \\ 1\ \mathrm{GeV} &= 1.6022 \times 10^{-10}\ \mathrm{J} = 1.78 \times 10^{-27}\ \mathrm{kg} \cdot c^2 \end{aligned} \tag{22}$$

The proton and neutron mass are

$$\begin{aligned} m_p &= 1.007276u = 938.27\ \mathrm{MeV}/c^2 \\ m_n &= 1.008665u = 939.57\ \mathrm{MeV}/c^2 \end{aligned} \tag{23}$$

2.1 Discover atomic nucleus (4)

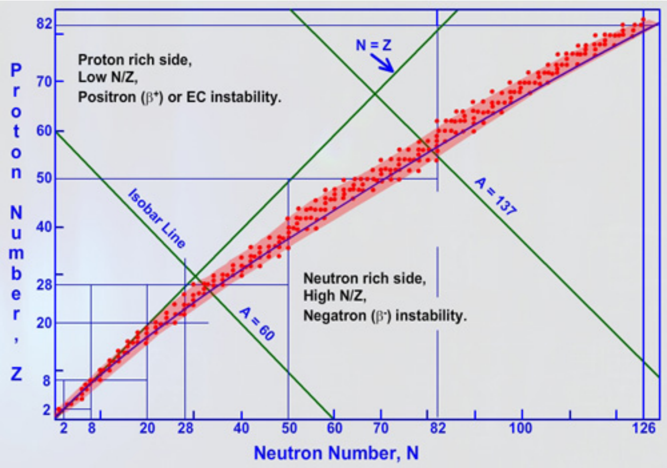
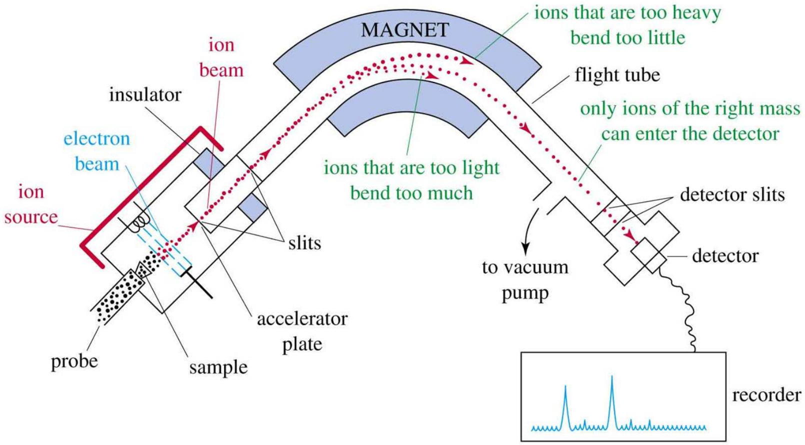


Figure : Chart of Nuclide.

The nucleus mass can be measured by mass spectrometer where a magnetic field B is used to bend charged particles. If v is fixed or selected by $v = \frac{E}{B}$, $qvB = m\frac{v^2}{R}$ $\rightarrow$ momentum selector, $mv = qRB$.



2.2 The size and density distribution of nucleus (1)

A nucleus is a collection of protons and neutrons which can be regarded as a bulk of nuclear matter. If the nucleus is treated as a sphere, while the volume of a nucleus is proportional to the number of nucleons A, then the radius of the nucleus is in the form,

$$\begin{aligned} R &= r_0 A^{1/3}, \; r_0 \approx 1.2 \,\mathrm{fm} \\ \rho_0 &= A/V \sim 0.14/\mathrm{fm}^3 \end{aligned} \tag{24}$$

We can estimate the density of a nucleus. The number and mass densities are

$$\begin{aligned} n &= A/V \approx A/(4/3\pi r_0^3 A) \approx 10^{38} \,\mathrm{cm}^{-3} \\ \rho &= n m_N \approx 1.66 \times 10^{14} \,\mathrm{g/cm}^3 \end{aligned} \tag{25}$$

2.2 The size and density distribution of nucleus (2)

One can use the Woods-Saxon distribution (or the Fermi distribution) to discribe the nuclear charge density,

$$\rho(r) = \frac{\rho_0}{1+\exp[(r-R)/a]} \tag{26}$$

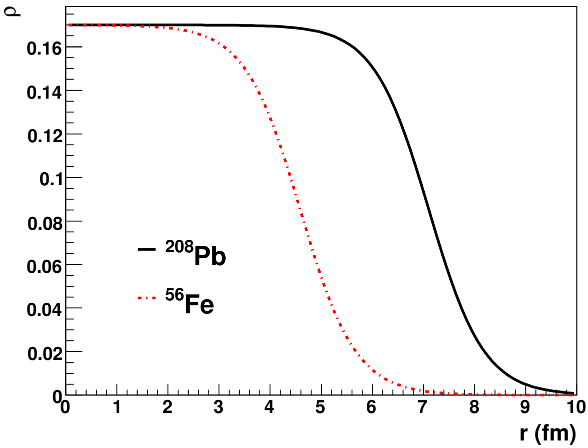
where $a \approx 0.54$ fm. The width of the surface t can be defined by the criterion that the density drops from 90% to 10% of ρ_0,

$$t \approx 4.4a \approx 2.4 \text{ fm} \tag{27}$$

One can introduce the angular dependence of the radius $R(\theta)$ to describe the non-spherical shapes of the nuclei,

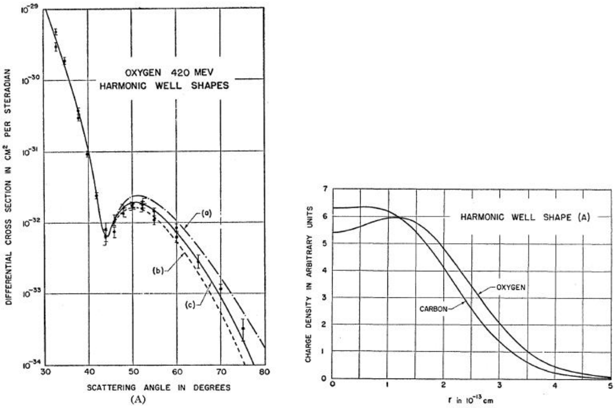
$$R(\theta) = R_0[1 + \beta_2 Y_{20}(\theta) + \beta_4 Y_{40}(\theta) + \cdots] \tag{28}$$

Figure : The nuclear density distribution.



2.2 The size and density distribution of nucleus (4)

Figure : Elastic electron scattering of the charge distribution of ^{16}O.



2.3 Spin and magnetic moment (1)

Now let us consider the Maxwell equations (4) for magnetostatics,

$$\begin{aligned} \nabla \times \mathbf{B} &= 4\pi \mathbf{j}, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{29}$$

Using $\mathbf{B} = \nabla \times \mathbf{A}$, we obtain

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = 4\pi \mathbf{j}. \tag{30}$$

We impose Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ and the above becomes

$$\nabla^2 \mathbf{A} = -4\pi \mathbf{j}, \tag{31}$$

The solution is

$$\mathbf{A}(\mathbf{r}) = \int d^3 r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{32}$$

2.3 Spin and magnetic moment (2)

We can make expansion of the integrand for $r = |\mathbf{r}| \gg r' = |\mathbf{r}'|$,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r} - r_i' \, \partial_i \frac{1}{|\mathbf{r}-\mathbf{r}'|}\bigg|_{r'=0} + \cdots = \frac{1}{r} + r_i' \frac{r_i}{r^3} + \cdots \qquad (33)$$

Then Eq. (32) becomes

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{1}{r}\int d^3r' \mathbf{j}(\mathbf{r}') + \frac{r_i}{r^3}\int d^3r' r_i' \mathbf{j}(\mathbf{r}') + \cdots \\ &= \frac{r_i}{r^3}\int d^3r' r_i' \mathbf{j}(\mathbf{r}') + \cdots \end{aligned} \qquad (34)$$

where the first term is vanishing $\int d^3r' \mathbf{j}(\mathbf{r}') = 0$.

This can be shown by using $\nabla \cdot \mathbf{j} = 0$ and

$$\begin{aligned} 0 &= \int d^3 r' \nabla' \cdot (r'_i \mathbf{j}) = \int d^3 r' (j_i + r'_i \nabla' \cdot \mathbf{j}) \\ &= \int d^3 r' j_i \end{aligned} \tag{35}$$

We can also have the following identity for $\mathbf{j}$,

$$\begin{aligned} 0 &= \int d^3 r' \nabla' \cdot (r'_i r'_k \mathbf{j}) = \int d^3 r' (r'_k j_i + r'_i j_k + r'_i r'_k \nabla' \cdot \mathbf{j}) \\ &= \int d^3 r' (r'_k j_i + r'_i j_k) \end{aligned} \tag{36}$$

Then we can re-write Eq. (34) as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= -\mathbf{e}_k \frac{r_i}{2r^3}\int d^3r'[r'_k j_i(\mathbf{r}') - r'_i j_k(\mathbf{r}')] \\ &= -\mathbf{e}_k \frac{1}{2r^3}\epsilon_{kil} r_i \int d^3r'[\mathbf{r}' \times \mathbf{j}(\mathbf{r}')]_l \\ &= -\frac{1}{r^3}\mathbf{r} \times \boldsymbol{\mu} \qquad (37)\end{aligned}$$

where the magnetic moment is defined by

$$\boldsymbol{\mu} = \frac{1}{2}\int d^3r' \mathbf{r}' \times \mathbf{j}(\mathbf{r}') \qquad (38)$$

If we consider a system of charged particles, the current density is given by

$$\mathbf{j}(\mathbf{r}) = \sum_i q_i \mathbf{v}_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{39}$$

where $\mathbf{r}_i$ and $\mathbf{v}_i$ are position and velocity of the particle i. Then the magnetic moment in Eq. (38) can be re-written as

$$\begin{aligned} \boldsymbol{\mu} &= \frac{1}{2m} \sum_i q_i \int d^3 r' (\mathbf{r}' \times \mathbf{p}_i) \delta(\mathbf{r} - \mathbf{r}_i) \\ &= \frac{1}{2m} \sum_i q_i \mathbf{L}_i \end{aligned} \tag{40}$$

where $\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$ is the oribital angluar momentum for the particle i.

In classical picture, the magnetic moment of a charged particle is given by $\mu = \pi r^2 \frac{qv}{2\pi r} = \frac{q}{2m} l$.
In quantum mechanics, the magnetic moment of a particle in an external magnetic field can be derived as follows,

$$\begin{aligned}
\frac{\mathbf{p}^2}{2m} &\rightarrow \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} \rightarrow \frac{(-i\vec{\nabla} - q\mathbf{A})^2}{2m} \\
&\rightarrow \frac{iq(\vec{\nabla} \cdot \mathbf{A} + \mathbf{A} \cdot \vec{\nabla})}{2m} = \frac{iq}{m} \mathbf{A} \cdot \vec{\nabla} \\
&= -\frac{iq}{2m} (\mathbf{r} \times \mathbf{B}) \cdot \vec{\nabla} = \frac{iq}{2m} \mathbf{B} \cdot (\mathbf{r} \times \vec{\nabla}) \\
&= -\frac{q}{2m} \mathbf{B} \cdot \mathbf{L} \qquad (41)
\end{aligned}$$

where $\mathbf{A} = -\frac{1}{2} \mathbf{r} \times \mathbf{B}$.

One can verify

$$\begin{aligned}(\nabla \times \mathbf{A})_k &= -\frac{1}{2}\epsilon_{ijk}\partial_i(\mathbf{r}\times\mathbf{B})_j = -\frac{1}{2}\epsilon_{ijk}\epsilon_{lsj}\partial_i(r_l B_s) \\ &= \frac{1}{2}(\delta_{il}\delta_{ks}-\delta_{is}\delta_{kl})B_s\partial_i r_l \\ &= \frac{1}{2}B_k\partial_i r_i - \frac{1}{2}B_i\partial_i r_k = B_k \\ \vec{\nabla}\cdot\mathbf{A} &= \partial_i A_i = -\frac{1}{2}\epsilon_{ijk}B_k\partial_i r_j = 0 \end{aligned} \tag{42}$$

One can define the orbital magnetic moment from Eq. (41),

$$\boldsymbol{\mu}_L = \frac{q}{2m}\mathbf{L} \tag{43}$$

so that the magnetic energy due to orbital angular momentum is

$$H_L = -\boldsymbol{\mu}_L\cdot\mathbf{B} \tag{44}$$

2.3 Spin and magnetic moment (8)

In quantum mechanics, orbital and spin angular momentum eigenstates for a particle

$$|L, m_z\rangle\,,\ |S, S_z\rangle \tag{45}$$

The angular momentum operators act on the eigenstate

$$\begin{aligned}
\hat{\mathbf{L}}^2 |L, m_z\rangle &= L(L+1)|L, m_z\rangle \\
\hat{L}_z |L, m_z\rangle &= m_z |L, m_z\rangle\,,\ m_z = -L, -L+1, \cdots, L-1, L \\
\hat{\mathbf{S}}^2 |S, S_z\rangle &= S(S+1)|S, S_z\rangle \\
\hat{S}_z |S, S_z\rangle &= S_z |S, S_z\rangle\,,\ S_z = -S, -S+1, \cdots, S-1, S
\end{aligned} \tag{46}$$

The spin magnetic interaction can only be derived from the Dirac equation,

$$H_L = -\boldsymbol{\mu}_S \cdot \mathbf{B} \tag{47}$$

where

$$\boldsymbol{\mu}_S = g\frac{q}{2m}\mathbf{S} \tag{48}$$

with the factor of g and the spin $\mathbf{S}$ of the particle. We see that a non-zero spin always gives a non-zero magnetic moment. For electrons with $q = -e$, where e is the charge modula of the electron, we have $g_e = 2$ following the Dirac equation.

We know that nucleons have magnetic moments,

$$\begin{aligned} \boldsymbol{\mu}_p &= g_p \mu_N \mathbf{S}_p, \quad g_p = 5.586 \\ \boldsymbol{\mu}_n &= g_n \mu_N \mathbf{S}_n, \quad g_n = -3.82 \end{aligned} \tag{49}$$

where $\mu_N \equiv \frac{e}{2m_p}$ is the nuclear magneton for nucleons with the proton mass m_p, $S_p = S_n = 1/2$. Normally use the maximum value in unit μ_N to denote $\boldsymbol{\mu}$, e.g. $\boldsymbol{\mu}_p \rightarrow 5.586/2 = 2.793$ and $\boldsymbol{\mu}_n \rightarrow -3.82/2 = -1.91$. Note that $\boldsymbol{\mu}_n \sim \boldsymbol{\mu}_e$ in sign indicating an inhomogeneous charge distribution.

2.3 Spin and magnetic moment (11)

Using

$$\begin{aligned} 1\,\mathrm{Gauss} &= 6.92\times 10^{-2}\,(\hbar c)^{-3/2}\cdot \mathrm{eV}^2 \\ 1\,\mathrm{s} &= 1.52\times 10^{15}\,\hbar\cdot \mathrm{eV}^{-1} \end{aligned} \tag{50}$$

Let's estimate the magnitude of the nuclear magneton,

$$\begin{aligned} \mu_{\mathrm{N}} &= \frac{1/\sqrt{137}}{2\times 938.3}\mathrm{MeV}^{-1} = 4.55\times 10^{-11}\mathrm{eV}^{-1} \\ &\approx 4.55\times 10^{-11}\times 1.05184\times 10^{14}\,(\mathrm{s}^{-1}\cdot \mathrm{gauss}^{-1}) \\ &\approx 4.79\times 10^{3}\,\mathrm{s}^{-1}\cdot \mathrm{Gauss}^{-1} = 4.79\times 10^{7}\,\mathrm{Hz}\cdot \mathrm{T}^{-1}. \end{aligned} \tag{51}$$

2.3 Spin and magnetic moment (12)

Similarly a nucleus has a magnetic moment,

$$\begin{aligned} \boldsymbol{\mu}_A &= g_A \mu_N \mathbf{I}_A \\ \mathbf{I}_A &= \sum_{i=1,\cdots,A} (\boldsymbol{l}_i + \boldsymbol{s}_i) \end{aligned} \tag{52}$$

with g_A: nuclear g-factor and $\mathbf{I}_A$ the spin or total AM of the nucleus.

- For even-even nuclei, $I_A = 0$. Every two protons or neutrons pair, so that the total angluar momentum is vanishing.
- For even-odd nuclei, I_A is a half-integer and is determined by the unpaired nucleon.
- For odd-odd nuclei, I is an integer and is determined by unpaired nucleons. For nuclei with $A > 10$, nuclear spins come from JJ couplings of constituent nucleons, i.e. $\mathbf{I}_A = \sum_{i=1} \boldsymbol{j}_i$ where $\boldsymbol{j}_i = \boldsymbol{l}_i + \boldsymbol{s}_i$. For nuclei with $A < 10$, there are LS couplings, $\mathbf{J} = \mathbf{L} + \mathbf{S}$ where $\mathbf{L} = \sum_i \boldsymbol{l}_i$ and $\mathbf{S} = \sum_i \boldsymbol{s}_i$.

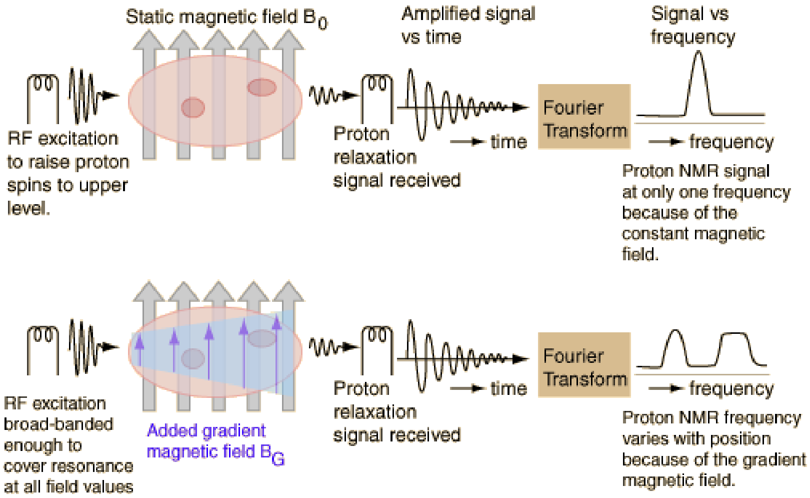
The nucleus magnetic moment can be measured by exerting an external magnetic field, the associated energy and energy difference of neighboring level are

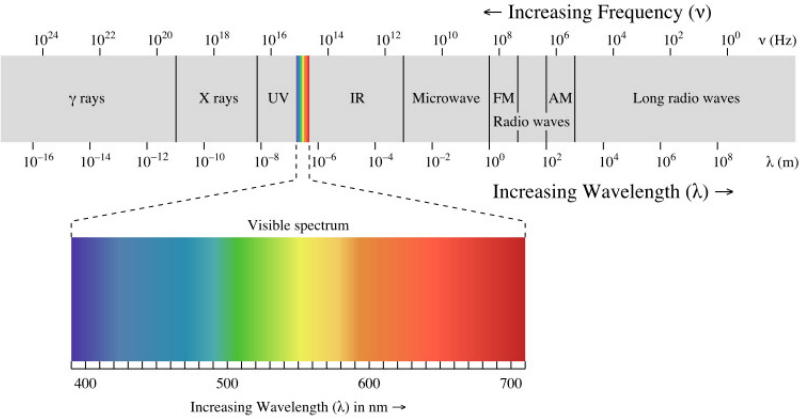
$$\begin{aligned} E &= -\boldsymbol{\mu}_A \cdot \mathbf{B} = -g_A \mu_N m_A B \\ \Delta E &= g_A \mu_N B \end{aligned} \tag{53}$$

where $m_A = -I_A, -I_A + 1, \cdots, I_A - 1, I_A$. If the magnetic field oscillates with a high frequency which satisfies

$$2\pi\nu = \Delta E \to \nu = \frac{g_A \mu_N}{2\pi} B \tag{54}$$

there is a strong resonance absorption or emission. This phenomenon is called nuclear magnetic resonance (NMR), and the frequency ν is called resonance frequency. This energy is at about 60-1000 MHz in the range of VHF and UHF in television broadcasts.


Static magnetic field B_0
Amplified signal vs time
Signal vs frequency
RF excitation to raise proton spins to upper level.
Proton relaxation signal received
time
Fourier Transform
frequency
Proton NMR signal at only one frequency because of the constant magnetic field.
RF excitation broad-banded enough to cover resonance at all field values
Added gradient magnetic field B_G
Proton relaxation signal received
time
Fourier Transform
frequency
Proton NMR frequency varies with position because of the gradient magnetic field.


← Increasing Frequency (ν)
10^24 10^22 10^20 10^18 10^16 10^14 10^12 10^10 10^8 10^6 10^4 10^2 10^0 ν (Hz)
γ rays
X rays
UV
IR
Microwave
FM
AM
Radio waves
Long radio waves
10^−16 10^−14 10^−12 10^−10 10^−8 10^−6 10^−4 10^−2 10^0 10^2 10^4 10^6 10^8 λ (m)
Increasing Wavelength (λ) →
Visible spectrum
400
500
600
700
Increasing Wavelength (λ) in nm →

2.3 Spin and magnetic moment (16)

- F. Bloch and E. M. Purcell, resonance phenomenon for protons, Nobel Prize in Physics, 1952
- R. Ernst, Nobel Prize in Chemistry, 1991
- K. Wüthrich, Nobel Prize in Chemistry, 2002
- P.C. Lauterbur and P. Mansfield, Nobel Prize in Physiology or Medicine, 2003

2.4 Parity (1)

Parity is one of the property of the wave function for a particle under spatial reversion $\mathbf{r} \to -\mathbf{r}$,

$$P\psi(\mathbf{r}) = \psi(-\mathbf{r}),$$

where P is the parity operator satisfying $P^2 = 1$. $P\psi(\mathbf{r}) = \pm\psi(\mathbf{r})$, corresponding to the even or odd parity. For a particle moving in a central potential, the wave function

$$\psi(r, \theta, \phi) = R(r) Y_{LM}(\theta, \phi),$$

where $Y_{LM}(\theta, \phi)$ are spherical harmonics. Under spatial reversion $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$, $Y_{LM}(\theta, \phi)$ transform as

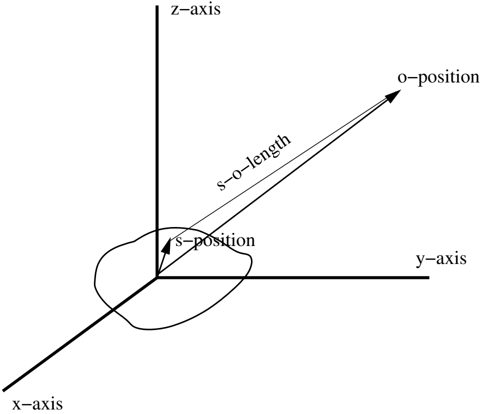
$$Y_{LM}(\theta, \phi) \to Y_{LM}(\pi - \theta, \phi + \pi) = (-1)^L Y_{LM}(\theta, \phi)$$

So the parity is $(-1)^L$. The orbital parity of a single nucleon in the central potential is $(-1)^L$. The intrinsic parity of nucleons is $+1$.

2.4 Parity (2)

Suppose a nucleon moves in the potential formed by other nucleons, we can obtain its wave function and then its parity. If we know the wave function of each nucleon we could determine the parity of the nucleus by the product of the parities of all nucleons. But in practice this is impossible. Like the nuclear spin, we regard the parity as an overall property of the nucleus. The nuclear parity can be measured by the decay products of the nucleus. We can denote the parity of a nucleus by I^P where I is its spin and P is its parity.

Figure : Multipole expansion.



2.5 Electric multipole moment (2)

The charge distribution of any charged systems can be described by electric multipole moments. The lowest multipole moment is monopole moment, followed by the dipole and quadrupole moments. Consider the electric potential from an electric source $\rho(\mathbf{r}')$,

$$\phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{55}$$

which satisfies Poisson equation

$$\nabla^2\phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}) \tag{56}$$

because

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{57}$$

We define $r = |\mathbf{r}|$. If $r \gg r'$, we can expand

$$\begin{aligned} \frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r} - r_i' \,\nabla_i \frac{1}{|\mathbf{r}-\mathbf{r}'|}\bigg|_{r'=0} + \frac{1}{2} r_i' r_j' \,\nabla_i \nabla_j \frac{1}{|\mathbf{r}-\mathbf{r}'|}\bigg|_{r'=0} + \cdots \\ &\approx \frac{1}{r} + r_i' \frac{r_i}{r^3} + \frac{1}{2} r_i' r_j' \frac{3 r_i r_j - r^2 \delta_{ij}}{r^5} \end{aligned} \quad (58)$$

where we have used

$$\nabla_i \frac{1}{r} = -\frac{r_i}{r^3}, \; \nabla_i \nabla_j \frac{1}{r} = -\nabla_j \frac{r_i}{r^3} = -r_i \nabla_j \frac{1}{r^3} - \frac{1}{r^3} \nabla_j r_i = \frac{3 r_i r_j - r^2 \delta_{ij}}{r^5} \quad (59)$$

Then the potential in Eq. (55) becomes

$$\phi(\mathbf{r}) = \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q}{r} + \frac{r_i D_i}{r^3} + \frac{1}{2}\frac{r_i r_j Q_{ij}}{r^5} \quad (60)$$

where

$$\begin{aligned} Q &= \int d^3r' \rho(\mathbf{r}') \\ D_i &= \int d^3r' r_i' \rho(\mathbf{r}') \\ Q_{ij} &= \int d^3r' [3r_i' r_j' - r'^2 \delta_{ij}] \rho(\mathbf{r}') \end{aligned} \quad (61)$$

The parity of the electric multipole moment is given by $(-1)^L$, the the magnetic mulipole moment is given by $(-1)^{L+1}$, where L is the order of the moment. In quantum mechanics, the moment can be obtained by,

$$\left\langle \hat{O} \right\rangle = \int d^3\mathbf{r}\psi^*\hat{O}\psi \sim \int d^3\mathbf{r}\hat{O}|\psi|^2 \tag{62}$$

where $\hat{O}$ is the moment operator with specific parity. For electric dipole, the operator is $\hat{O} = \mathbf{r}$; for the magnetic moment, the operator is $\hat{O} = -i\mathbf{r} \times \nabla$.

2.5 Electric multipole moment (6)

The parity of the wave function does not influence the result, but the parity of the moment does. For $\hat{O}$ with odd parity, the integral is vanishing. So we conclude that all electric/magnetic moments of the odd/even order are vanishing. Nucleus does not have eletric dipole moment and magnetic quadrupole moment.

If the nuclear is a sphere, we can clearly see that the dipole and quadruple moments D_i and Q_{ij} are zero from the definition. We assume that the nucleus is ellipsoid, it has rotational symmetry along z-axis, the length of the z-axis is $2c$ and the radius in xy-plane is a. The equation for the ellipsoid is:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \tag{63}$$

We can parametrize the ellipsoid coodinates as $x = a\xi \sin\theta \cos\phi$, $y = a\xi \sin\theta \sin\phi$, $z = c\xi \cos\theta$, where $\xi \leq 1$. In terms of (ξ, θ, ϕ), the volume element becomes

$$d^3r = dxdydz = a^2 cd\xi d\theta d\phi \xi^2 \sin\theta \tag{64}$$

.

Then the quadrupole moment is diagonal $Q_{ij} = Q_i \delta_{ij}$. Normally the quadrupole moment is defined by $Q \equiv Q_3$ and given by

$$\begin{aligned} Q &= \int d^3 r (3r_3^2 - r^2)\rho(\mathbf{r}) = \int d^3 r (2r_3^2 - r_1^2 - r_2^2)\rho(\mathbf{r}) \\ &= 2\frac{Z}{V}\left(\frac{1}{5}c^2 V - \frac{1}{5}a^2 V\right) = \frac{2}{5}Z(c^2 - a^2) \end{aligned} \quad (65)$$

We have used

$$\begin{aligned} \int d^3 r r_3^2 &= a^2 c^3 \int_0^1 d\xi \xi^4 \int d\theta \cos^2\theta \sin\theta \int d\phi \\ &= \frac{4\pi}{15} a^2 c^3 = \frac{1}{5} V c^2 \\ \int d^3 r r_1^2 &= a^4 c \int_0^1 d\xi \xi^4 \int d\theta \sin^3\theta \int d\phi \cos^2\phi \\ &= \frac{4\pi}{15} a^4 c = \frac{1}{5} V a^2 \end{aligned}$$

Note that $Q_1 = Q_2 = -Q/2$. For spherical shape, the quadruple moment is vanishing, $Q = 0$; for prolate or cigar-like shape, it is positive, $Q > 0$; for a oblate (discus-like) shape, it is negative, $Q < 0$. So the quadruple part of the potential is

$$\phi_4(\mathbf{r}) \equiv \frac{1}{2}\frac{r_i r_j Q_{ij}}{r^5} = -\frac{1}{4}\frac{(r_1^2 + r_2^2 - 2r_3^2)Q}{r^5} \qquad (66)$$

The deviation of the nuclear shape from a sphere is characterized by $\varepsilon \equiv \Delta R/R$ with R the radius of the sphere with the same volume as the ellipsoid, then we have $c = R(1+\varepsilon)$ and $a = R/\sqrt{1+\varepsilon}$ given by equating two volumes $\frac{4\pi}{3}R^3 = \frac{4\pi}{3}a^2 c$. Inserting a and c back into Q in Eq. (65), we get

$$Q \approx \frac{6}{5}ZR^2\varepsilon \approx \frac{6}{5}Zr_0^2 A^{2/3}\varepsilon \tag{67}$$

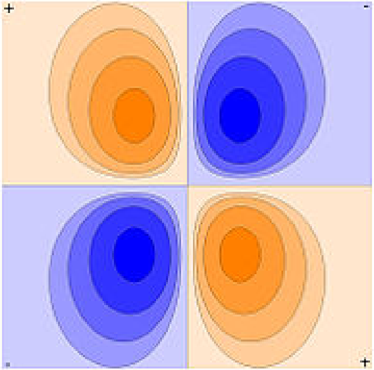
The value of ε can be obtained by using the above formula and by measuring Q in experiments. The electric quadruple moment can be measured by the violation of the separation rule in atomic hyperfine spectra. It can also be measured by the resonant absorption from the interaction between the nuclear electric quadruple and electons outside the nucleus.

Table 1 shows the electric quadruple moments of some nuclides. The unit is barn which is $10^{-24}\,\mathrm{cm}^2$. Usually the quadruple moment is about one tenth of electron-barn (eb) for nuclides with $A < 150$ until it reaches about 2 for $A > 150$.

Table : Some values of nuclear electric quadruple moments. Data from V. S. Shirley, Table of Isotopes, Wiley, New York, 1978, Appendix VII.

Nuclide	$^{2}\mathrm{H}$	$^{17}\mathrm{O}$	$^{63}\mathrm{Co}$	$^{63}\mathrm{Cu}$
Q(eb)	$+2.88\times10^{-3}$	-2.578×10^{-2}	$+0.40$	-0.209
Nuclide	$^{133}\mathrm{Cs}$	$^{161}\mathrm{Dy}$	$^{176}\mathrm{Lu}$	$^{209}\mathrm{Bi}$
Q(eb)	-3×10^{-3}	$+2.4$	$+8.0$	-0.37

Figure : Electric quadrupole moment.



2.6 The mass formula and binding energy (1)

A nucleus is a bound state of protons and neutrons. The binding energy of a nucleus is defined as

$$B(Z,A) = ZM_H + (A-Z)M_n - M(Z,A) \quad (68)$$

where M_H is the mass of the hydrogen. A nucleus can be regarded as an incompressible liquid drop which reflects the satuaration property of the nuclear force. According to the liquid drop model, the binding energy can be expressed by the Weizsäcker's formula,

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C Z^2 A^{-1/3} - a_{sym} I^2 A + s a_P A^{-1/2} \quad (69)$$

where $I = (N-Z)/A$. Here $a_V \approx 15.75$ MeV is the volume energy, $a_S \approx 17.8$ MeV the surface energy, $a_C \approx 0.71$ MeV the Coulomb energy, $a_{sym} \approx 23.3$ MeV the symmetric energy, $a_P \approx 12\ \mathrm{MeV}$ the pairing energy.

2.6 The mass formula and binding energy (2)

The sign of the surface energy is negative because the binding force of the nucleons in the surface is weakened compared to those inside the volume. The Coulomb energy comes from the static electric energy which is repulsive, so it is to decrease the binding energy. The symmetric energy is a quantum effect. For the pairing energy, the coefficient s is given by

$$s = \begin{cases} 1, & \text{even} - \text{even nuclei} \\ 0 & \text{odd A} \\ -1 & \text{odd} - \text{odd nuclei} \end{cases} \tag{70}$$

For the even-even nuclei are more stable becuase of the pairing of nucleons.

Table : The binding energies per nucleon for some nuclei.

	$^{3}_{1}\mathrm{H}$	$^{4}_{2}\mathrm{He}$	$^{5}_{2}\mathrm{He}$	$^{6}_{3}\mathrm{Li}$	$^{7}_{3}\mathrm{Li}$
B/A (MeV)	2.8273	7.0739	5.4811	5.3323	5.6063

	$^{8}_{4}\mathrm{Be}$	$^{9}_{4}\mathrm{Be}$	$^{10}_{5}\mathrm{B}$	$^{11}_{5}\mathrm{B}$	$^{12}_{6}\mathrm{C}$
B/A (MeV)	7.0624	6.4628	6.4751	6.9277	7.6801

	$^{13}_{6}\mathrm{C}$	$^{14}_{7}\mathrm{N}$	$^{15}_{7}\mathrm{N}$	$^{16}_{8}\mathrm{O}$	$^{17}_{8}\mathrm{O}$
B/A (MeV)	7.4698	7.4756	7.6995	7.9762	7.7507

	$^{18}_{9}\mathrm{F}$	$^{19}_{9}\mathrm{F}$	$^{20}_{10}\mathrm{Ne}$	$^{21}_{10}\mathrm{Ne}$
B/A (MeV)	7.6316	7.7790	8.0322	7.9717

2.6 The mass formula and binding energy (4)

- The volume energy is due to the short distance and satuation properties of nuclear force. If nuclear force is between any pair of nucleons, the volume term would be proportional to $A(A-1)/2 \sim A^2$.
- The surface term is like the surface tension in liquid since a nucleus is like a liquid droplet. The larger the droplet's surface, the less stable the droplet is. So the surface term is to reduce the binding energy.
- The Coulomb term is from Coulomb energy of a charged sphere.

- With the constant charge density $\rho_c = \frac{Q}{V}$ where $V = \frac{4\pi}{3}R^3$ and $Q = Ze$, the electric potential inside a nucleus is

$$\begin{aligned} \phi(r) &= \frac{Q}{R} + \frac{Q}{2R^3}(R^2 - r^2) \\ E_c &= \frac{1}{2}\int dV \rho\phi(r) = \frac{3}{5}\frac{Z^2e^2}{R} \approx \frac{3e^2}{5r_0}\frac{Z^2}{A^{1/3}} \end{aligned} \quad (71)$$

Therefore $a_C = \frac{3\alpha}{5r_0} \approx 0.71$ MeV with $\alpha = e^2 = 1/137$ $= 1.44\ \mathrm{fm} \cdot \mathrm{MeV}$ being the fine structure constant and $r_0 \approx 1.25$ fm.

2.6 The mass formula and binding energy (6)

The Fermi gas model for nuclei. We consider a square box potential

$$V(x,y,z)=\left\{\begin{array}{c} 0,\ 0<x,y,z<L, \\ \infty,\ \mathrm{otherwise} \end{array}\right. \tag{72}$$

One particle wave function under the periodic condition is

$$\begin{aligned} \psi &\sim \sin\frac{\pi n_x x}{L}\sin\frac{\pi n_y y}{L}\sin\frac{\pi n_z z}{L} \\ E &= \frac{1}{2m}(k_x^2+k_y^2+k_z^2)=\frac{1}{2mL^2}(n_x^2+n_y^2+n_z^2)\pi^2 \end{aligned} \tag{73}$$

where n_x, n_y, n_z are positive integers.

2.6 The mass formula and binding energy (7)

The number of state can be calculated by the inequality

$$n_x^2 + n_y^2 + n_z^2 \quad \leq \quad 2mE\frac{L^2}{\pi^2} \equiv R^2 \tag{74}$$

So the number of state is

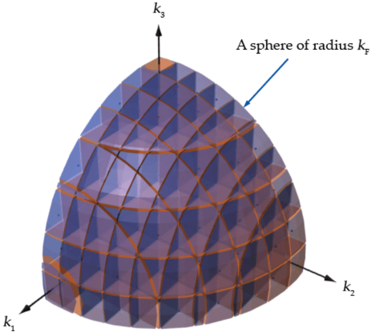
$$N \quad = \quad \frac{1}{8}\cdot\frac{4}{3}\pi R^3 = \frac{4}{3}\pi(2mE)^{3/2}\frac{L^3}{(2\pi)^3} \equiv \frac{4}{3}\pi k_F^3\frac{V}{(2\pi)^3} \tag{75}$$

where Fermi momentum is defined as $k_F \equiv (2mE)^{1/2}$. The unit of the phase space volume is $h^3 = \hbar^3(2\pi)^3$, then we have the number of state is

$$dN = \frac{Vd^3\mathbf{k}}{(2\pi)^3} \tag{76}$$

2.6 The mass formula and binding energy (8)

Figure : Fermi sphere.



2.6 The mass formula and binding energy (9)

The binding energy for nuclei, Eq. (69), can be described by the model of the fermion gas. The nuclear number density is related to the Fermi momentum k_F,

$$\rho = \frac{A}{V} = d_g \frac{1}{(2\pi)^3} \frac{4\pi}{3} k_F^3 = \frac{d_g}{6\pi^2} k_F^3 = \frac{2}{3\pi^2} k_F^3 \tag{77}$$

where $d_g = 4$ is the degeneracy factor from the number of the spin states (2) and the isospin states (2). Here we treat protons and neutrons as identical particles with different isospins. From $\rho = 0.16\ \mathrm{fm}^{-3}$, we can determine $k_F = 1.36\ \mathrm{fm}^{-1} = 268$ MeV. The corresponding kinetic energy is $\epsilon_F = k_F^2/(2m_N) \approx 38$ MeV. The average kinetic energy per nucleon is then

$$\bar{\epsilon} = d_g \frac{1}{2\pi^2 \rho} \int_0^{k_F} dk k^2 \frac{k^2}{2m_N} = \frac{3}{5} \epsilon_F \approx 23\ \mathrm{MeV} \tag{78}$$

2.6 The mass formula and binding energy (10)

When the numbers of protons and neutrons are not equal, the proton and neutron number densities are

$$\frac{Z}{V}=\frac{1}{3\pi^2}k_{F,p}^3=\frac{1}{2}\left(\frac{k_{F,p}}{k_F}\right)^3\frac{A}{V},\quad \frac{N}{V}=\frac{1}{3\pi^2}k_{F,n}^3=\frac{1}{2}\left(\frac{k_{F,n}}{k_F}\right)^3\frac{A}{V} \qquad (79)$$

where the degeneracy factors for protons and neutrons are the same $d_g=2$ accouting for two spin states. Then the Fermi momenta for the protons and neutrons are given by

$$k_{F,p}=k_F\left(\frac{2Z}{A}\right)^{1/3},\quad k_{F,n}=k_F\left(\frac{2N}{A}\right)^{1/3} \qquad (80)$$

2.6 The mass formula and binding energy (11)

The average kinetic energies are

$$\begin{aligned}
\bar{\epsilon}(I) &= \frac{1}{\pi^2\rho}\left(\int_0^{k_{F,p}} dk k^2 \frac{k^2}{2m_N} + \int_0^{k_{F,n}} dk k^2 \frac{k^2}{2m_N}\right) \\
&= \frac{3}{10}\epsilon_F\left[\left(\frac{2Z}{A}\right)^{5/3} + \left(\frac{2N}{A}\right)^{5/3}\right] \\
&= \frac{3}{10}\epsilon_F\left[(1-I)^{5/3} + (1+I)^{5/3}\right] \\
&\approx \frac{3}{5}\epsilon_F + \frac{1}{3}\epsilon_F I^2
\end{aligned} \tag{81}$$

where $I = (N-Z)/A$.

We can also obtain the surface energy after taking the boundary condition into account, the nucleon number element is

$$\begin{aligned} dA &= d_g \frac{L^3}{(2\pi)^3} 4\pi k^2 dk - 3 d_g \frac{L^2}{(2\pi)^2} 2\pi k dk \\ &= d_g \frac{V k^2}{2\pi^2} \left(1 - \frac{\pi}{2k}\frac{S}{V}\right) dk \end{aligned} \quad (82)$$

where the second term comes from three circle area corresponding to $k_1 = 0$ ($k_2^2 + k_3^2 \leq k_F^2$) or $k_2 = 0$ ($k_1^2 + k_3^2 \leq k_F^2$) or $k_3 = 0$ ($k_1^2 + k_2^2 \leq k_F^2$). Here $S = 6L^2$ and $V = L^3$ are the surface area and volume of the cube box with length L.

Due to the wave function proportinal to $\sin k_1 x \sin k_2 y \sin k_3 z$ all boundary states with $k_i = 0$ have to be excluded. The average kinetic energy is

$$\bar{\epsilon} = \frac{\int_0^{k_F} dA \frac{k^2}{2m_N}}{\int_0^{k_F} dA} = \frac{\frac{1}{2m_N}\left(\frac{V}{10\pi^2}k_F^5 - \frac{S}{16\pi}k_F^4\right)}{\frac{V}{6\pi^2}k_F^3 - \frac{S}{8\pi}k_F^2} \approx \frac{3}{5}\epsilon_F\left(1 + \frac{\pi S}{8Vk_F}\right) \quad (83)$$

where we have treated the surface energy as a perturbation. The surface energy is then

$$E_s \approx \frac{3}{5}\epsilon_F \frac{\pi SA}{8Vk_F} = \epsilon_F \frac{9\pi}{40 r_0 k_F} A^{2/3} \approx 16.5 A^{2/3} \text{ MeV} \quad (84)$$

2.6 The mass formula and binding energy (14)

The nuclear binding energies of all nuclei per nucleon are shown in Fig. 10, which is a benchmark for how tight the nucleons are bound in nuclei. From the first three terms of the binding energy in Eq. (69), we can estimate the most tightly bound nuclide by looking at the extrema point of binding energy per nucleon,

$$\frac{\partial}{\partial A}\left[\frac{B(Z,A)}{A}\right] \approx \frac{\partial}{\partial A}\left(-a_S A^{-1/3} - \frac{a_C}{4}A^{2/3}\right) = 0, \qquad (85)$$

which gives $A \approx \frac{2a_S}{a_C} \approx 51$ roughly in agreement with the atomic numbers of iron or nickel.

2.6 The mass formula and binding energy (15)

The binding energies per nucleon are largest for nuclei with mediate atomic number and reach maximum for iron nucleus $^{56}\mathrm{Fe}$. So the splitting of heavy nuclei into lighter ones or the merging of lighter nuclei into heavy ones can release substantial energy called nuclear energy by converting nuclear mass difference of initial and final state nuclei to kinetic energy, following Einstein's mass-energy formula $E = mc^2$. The splitting and merging processes are called nuclear fission and fusion respectively.

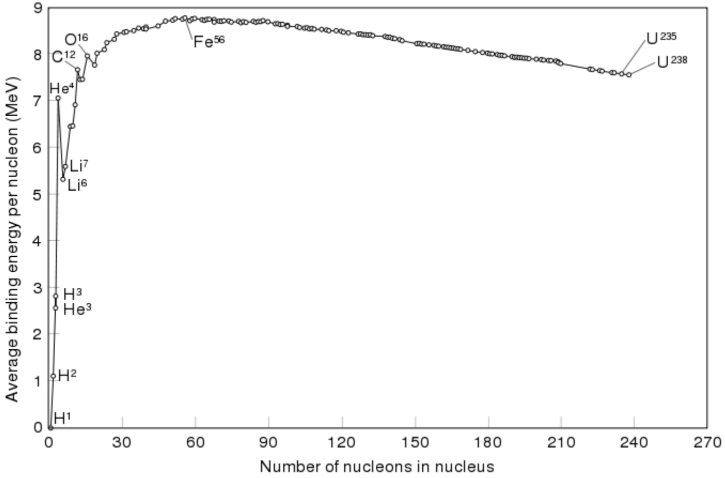


Figure : The nuclear binding energy. From http://en.wikipedia.org/wiki/Nuclear_energy.