
第五章 现代谱估计

5.6 最大熵谱估计

一.问题的提出

传统的谱估计方法存在缺点的原因：

人为地假定观察到的数据以外的数据为零



导致自相关估计也是有限长



而实际情况是没有观察到的数据不一定为零

如何解决？

注意到没有观察到的数据和已观察到的数据之间是有关
系的。因此，我们有可能根据已观察到的数据推测未观察
到的数据。

如何推测？ 增加信息量

二.信息和熵

信息量：事件**A**以概率**P_A**出现所携带的信息量

$$I_A = -\log_r P_A, \begin{cases} r = 2, \text{比特} \\ r = e, \text{奈特} \\ r = 10, \text{哈特} \end{cases}$$

平均信息量（熵）：信息源发送一组彼此独立的不同的消息**X_j**, (**j=1,2,...,n**), 概率为**P_j**,

每一个消息： $I_j = -\log_{10} p_j$

平均信息量（熵）：

$$H = E[I_j] = -\sum_{j=1}^n p_j \log_{10} p_j$$

例：零均值高斯平稳随机序列 $x(n)$ 的熵

$H = \frac{1}{2} \log_{10} \det[\mathbf{R}_x]_{M+1}$
 $\mathbf{R}_x$ 是 $x(n)$ 的自相关阵。

熵率 (Entropy rate)

$$h = \lim_{M \rightarrow \infty} \frac{H}{M+1} = \lim_{M \rightarrow \infty} \frac{1}{2} \log_{10} \det[\mathbf{R}_x]^{\frac{1}{M+1}}$$

若 $x(n)$ 的功率谱 $S_x(f)$ 限制在 $-f_c \leq f \leq f_c$ 范围内, 有:

$$h = \frac{1}{2} \ln(2 f_c) + \frac{1}{2 f_c} \int_{-f_c}^{f_c} \ln[S_x(f)] df$$

(上述结论对一般平稳序列也是适应的)

三.最大熵谱估计原理

$x(n)$ 的频谱范围

$$[-f_c, f_c], f_s = 1/T, f_c = f_s/2$$

$$\begin{cases} R_x(m) = \int_{-f_c}^{f_c} S_x(f) e^{j2\pi fmT} df \\ S_x(f) = T \sum_{m=-\infty}^{\infty} R_x(m) e^{-j2\pi fmT} \end{cases}$$

问题：已知 $R_x(0), \dots, R_x(M) \rightarrow$ 估计 $S_x(f) \Rightarrow \hat{S}_x(f)$

方法：按最大熵原理，在已知

$$R_x(m) = \int_{-f_c}^{f_c} S_x(f) e^{j2\pi fmT} df, m = 0, 1, \dots, M$$

的条件下，估计 $R_x(m), m \geq M+1$, 使得

$$S_x(f) = T \sum_{m=-\infty}^{\infty} R_x(m) e^{-j2\pi fmT}$$

时对应的熵率
最大，或使

$$h = \frac{1}{2} \ln(2 f_c) + \frac{1}{2 f_c} \int_{-f_c}^{f_c} \ln[S_x(f)] df$$

$$H = \frac{1}{2} \log_{10} \det[\mathbf{R}_x]$$

最大。

按熵率最大的方法：利用**Lagrangian**乘数法解此有约束优化问题。得：

$$\hat{S}_x(f) = \frac{1}{\sum_{m=-M}^M c(m)e^{-j2\pi fmT}}$$

$$R_x(m) = \int_{-f_c}^{f_c} S_x(f) e^{j2\pi fmT} df, m = 0, 1, \dots, M$$

其中， $c(m)$ 是**Lagrangian**乘数，可由约束条件求得，并  代入上式可得：

$$\hat{S}_x(f) = \frac{\sigma^2}{\left| 1 + \sum_{m=1}^M a(m)e^{-j2\pi fmT} \right|^2}$$

其中 $a(m)$ 可由**Yule-Walker**方程，由**M+1**个自相关函数求取：

$$\sum_{k=0}^M a_k R_x(m-k) = \begin{cases} \frac{2f_c}{|g(0)|^2} = 2f_c\sigma^2, m=0 \\ 0, m=1, 2, \dots, M \end{cases}, \sigma^2 = \frac{1}{|g(0)|^2}$$

四.最大熵自相关外推

$$H = \frac{1}{2} \log_{10} \det[\mathbf{R}_x]$$

$$\mathbf{R}_x(0), \dots, \mathbf{R}_x(M) \xrightarrow{\text{green arrow}} \mathbf{R}_x(M+1) \rightarrow \mathbf{R}_x(M+2) \rightarrow \dots$$

$$\frac{\partial \det[\mathbf{R}_x(M+1)]}{\partial R_x(M+1)} = 0, \frac{\partial \det[\mathbf{R}_x(M+2)]}{\partial R_x(M+2)} = 0, \dots$$

$$[\mathbf{R}_x(M+1)] = \begin{bmatrix} R_x(0) & R_x(1) & \dots & R_x(M+1) \\ R_x(1) & R_x(0) & \dots & R_x(M) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(0) \end{bmatrix}$$

$$\frac{\partial \det[\mathbf{R}_x(M+1)]}{\partial R_x(M+1)} = 0 \Rightarrow \det \begin{bmatrix} R_x(1) & R_x(0) & \dots & R_x(M-1) \\ R_x(2) & R_x(1) & \dots & R_x(M-2) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(1) \end{bmatrix} = 0$$

$$\begin{aligned}
& \det[\mathbf{R}_x(M+1)] \\
&= \det \begin{bmatrix} R_x(0) & R_x(1) & \dots & 0 \\ R_x(1) & R_x(0) & \dots & R_x(M) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(0) \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & \dots & R_x(M+1) \\ R_x(1) & R_x(0) & \dots & R_x(M) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(0) \end{bmatrix} \\
& \frac{\partial}{\partial R_x(M+1)} (*) = (-1)^{M+3} \det \begin{bmatrix} R_x(1) & R_x(0) & \dots & R_x(M-1) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(1) \end{bmatrix} \\
& \quad + (-1)^{M+3+M+2} R_x(M+1) \det \begin{bmatrix} R_x(0) & R_x(1) & \dots & R_x(M-1) \\ \dots & \dots & \dots & \dots \\ R_x(M-1) & R_x(M-2) & \dots & R_x(0) \end{bmatrix}
\end{aligned}$$

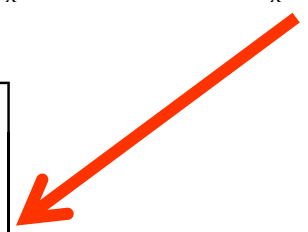
$$\det[\mathbf{R}_x(M+1)]$$

$$= (-1)^{M+3} \det \begin{bmatrix} R_x(1) & R_x(2) & \dots & 0 \\ R_x(0) & R_x(1) & \dots & R_x(M) \\ \dots & \dots & \dots & \dots \\ R_x(M-1) & R_x(M-2) & \dots & R_x(1) \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & \dots & R_x(M+1) \\ R_x(1) & R_x(0) & \dots & R_x(M) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(0) \end{bmatrix}$$

$$\frac{\partial}{\partial R_x(M+1)} (*) = (-1)^{M+3} \det \begin{bmatrix} R_x(1) & R_x(0) & \dots & R_x(M-1) \\ \dots & \dots & \dots & R_x(M-2) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(1) \end{bmatrix}$$

$$+ (-1)^{M+3} \det \begin{bmatrix} 0 & R_x(0) & \dots & R_x(M-1) \\ \dots & \dots & \dots & \dots \\ 0 & R_x(M-1) & \dots & R_x(0) \\ R_x(M+1) & R_x(M) & \dots & R_x(1) \end{bmatrix}$$

$$\begin{aligned}
& \partial \det[\mathbf{R}_x(M+1)] / \partial R_x(M+1) \\
&= (-1)^{M+3} \det \begin{bmatrix} R_x(1) & R_x(2) & \dots & 0 \\ R_x(0) & R_x(1) & \dots & R_x(M) \\ \dots & \dots & \dots & \dots \\ R_x(M-1) & R_x(M-2) & \dots & R_x(1) \end{bmatrix} \\
&+ (-1)^{M+3} \det \begin{bmatrix} R_x(1) & R_x(0) & \dots & R_x(M-1) \\ \dots & \dots & \dots & R_x(M-2) \\ \dots & \dots & \dots & \dots \\ R_x(M+1) & R_x(M) & \dots & R_x(1) \end{bmatrix} \\
&+ (-1)^{M+3} \det \begin{bmatrix} 0 & R_x(0) & \dots & R_x(M-1) \\ \dots & \dots & \dots & \dots \\ 0 & R_x(M-1) & \dots & R_x(0) \\ R_x(M+1) & R_x(M) & \dots & R_x(1) \end{bmatrix}
\end{aligned}$$

$(-1)^{M+3} \det \begin{bmatrix} R_x(1) & R_x(0) & \dots & R_x(M-1) \\ R_x(2) & R_x(1) & \dots & R_x(M-2) \\ \dots & \dots & \dots & \dots \\ 0 & R_x(M) & \dots & R_x(1) \end{bmatrix}$


$$\hat{S}_x(f) = \frac{1}{\sum_{m=-M}^M c(m)e^{-j2\pi fmT}}$$

五.最大熵谱估计的解

$$h = \frac{1}{2} \ln(2f_c) + \frac{1}{2f_c} \int_{-f_c}^{f_c} \ln[S_x(f)] df$$

$$S_x(f) = T \sum_{m=-\infty}^{\infty} R_x(m) e^{-j2\pi fmT}$$

$$\begin{aligned} \frac{\partial h}{\partial R_x(m)} &= \frac{\partial h}{\partial S_x(f)} \frac{\partial S_x(f)}{\partial R_x(m)} = \frac{1}{2f_c} \int_{-f_c}^{f_c} \frac{1}{S_x(f)} \frac{\partial S_x(f)}{\partial R_x(m)} df \\ &= \frac{1}{2f_c} \int_{-f_c}^{f_c} \frac{1}{S_x(f)} T e^{-j2\pi fmT} df = 0, |m| \geq M + 1 \end{aligned}$$

$$c_m = \int_{-f_c}^{f_c} \frac{1}{S_x(f)} e^{j2\pi fmT} df = 0, |m| \geq M + 1$$

$$\frac{1}{S_x(f)} = \sum_{m=-M}^M c_m e^{-j2\pi fmT}, c_m = c_{-m} \quad \hat{S}_x(f) = \frac{1}{\sum_{m=-M}^M c_m e^{-j2\pi fmT}}$$

$$\text{代入约束条件: } R_x(m) = \int_{-f_c}^{f_c} S_x(f) e^{j2\pi fmT} df, m = 0, 1, \dots, M$$

$$= \int_{-f_c}^{f_c} \frac{e^{j2\pi fmT}}{\sum_{n=-M}^M c_n e^{-j2\pi fnT}} df, m = 0, 1, \dots, M$$

$$\text{令 } z = e^{j2\pi fT}, df = \frac{f_c}{j\pi} \left(\frac{dz}{z} \right), f_c = \frac{f_s}{2} = \frac{1}{2T}$$

$$R_x(m) = \frac{f_c}{j\pi} \oint_{u.c} \frac{z^{m-1}}{\sum_{n=-M}^M c_n z^{-n}} dz, m = 0, 1, \dots, M$$

$$\sum_{n=-M}^M c_n z^{-n} = G_M(z) G_M^* \left(\frac{1}{z^*} \right) = |g(0)|^2 A_M(z) A_M^* \left(\frac{1}{z^*} \right)$$

$$G_M(z) = \sum_{n=0}^M g(n) z^{-n}, G_M^* \left(\frac{1}{z^*} \right) = \sum_{n=0}^M g^*(n) z^n$$

$$R_x(m) = \frac{f_c}{j\pi} \oint_{u.c} \frac{z^{m-1}}{\sum_{n=-M}^M c_n z^{-n}} dz, m = 0, 1, \dots, M$$

$$A_M(z) = \sum_{k=0}^M a_k z^{-k}, A_M^*\left(\frac{1}{z^*}\right) = \sum_{k=0}^M a_k^* z^k, a_0 = 1$$

$$R_x(m) = \frac{f_c}{j\pi} \oint_{u.c} \frac{z^{m-1}}{G_M(z)G_M^*\left(\frac{1}{z^*}\right)} dz = \frac{1}{2\pi j} \oint_{u.c} \frac{2f_c}{G_M(z)G_M^*\left(\frac{1}{z^*}\right)} z^{m-1} dz$$

$$R_x(m) \longrightarrow \boxed{G_M(z)} \longrightarrow y(m) = R_x(m) * g(m)$$

$$y(m) = \sum_{k=0}^M g(k) R_x(m-k)$$

$$= \frac{f_c}{j\pi} \oint_{u.c} \frac{1}{G_M(z)G_M^*\left(\frac{1}{z^*}\right)} \bullet G_M(z) \bullet z^{m-1} dz = \frac{f_c}{j\pi} \oint_{u.c} \frac{1}{G_M^*\left(\frac{1}{z^*}\right)} \bullet z^{m-1} dz$$

$$A_M(z) = \sum_{k=0}^M a_k z^{-k}, A_M^*\left(\frac{1}{z^*}\right) = \sum_{k=0}^M a_k^* z^n$$

$$y(m) = \frac{f_c}{j\pi} \oint_{u.c} \frac{1}{G_M^*\left(\frac{1}{z^*}\right)} \bullet z^{m-1} dz = 2f_c \bullet \frac{1}{2\pi j} \oint_{u.c} \frac{z^m}{G_M^*\left(\frac{1}{z^*}\right)} \frac{dz}{z}$$

$$y(m) = \begin{cases} \frac{2f_c}{g^*(0)}, m=0 \\ 0, m=1, 2, \dots, M \end{cases}$$

$$G_M(z) = \sum_{n=0}^M g(n) z^{-n}, G_M^*\left(\frac{1}{z^*}\right) = \sum_{n=0}^M g^*(n) z^n$$

$$\sum_{k=0}^M g(k) R_x(m-k) = \begin{cases} \frac{2f_c}{g^*(0)}, m=0 \\ 0, m=1, 2, \dots, M \end{cases}$$

$$\sum_{k=0}^M a_k R_x(m-k) = \begin{cases} \frac{2f_c}{|g(0)|^2} = 2f_c \sigma^2, m=0 \\ 0, m=1, 2, \dots, M \end{cases}, \sigma^2 = \frac{1}{|g(0)|^2}$$

$$\therefore \hat{S}_x(f) = \frac{1}{\sum_{n=-M}^M c_n e^{-j2\pi fnT}} = \frac{1}{\sum_{n=-M}^M c_n z^{-n}} \bigg|_{z=e^{-j2\pi fT}} = \frac{1}{|g(0)|^2 A_M(z) A_M^*\left(\frac{1}{z^*}\right)} \bigg|_{z=e^{-j2\pi fT}}$$

$$\therefore \hat{S}_x(f) = \frac{\sigma^2}{\left| 1 + \sum_{n=1}^M a_n e^{-j2\pi fnT} \right|^2}$$