A brief note on String Theory

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1 World Volume Action

For a relativistic particle, we describe it as a 1D manifold called worldline which embedded in the D dimension spacetime. $X^\mu(\tau)$ is our parametrization of the embedding, and $\tau$ is the only coordinate of the 1D worldline and $\mu$ takes from 1 to $D$ as the “place” of the particle. Alternatively we can view $X^\mu(\tau)$ as a bunch of fields lives on a 1D manifold, then they enjoy a symmetry under certain Lie group in the field manifold. This is almost the same to the Yang-Mills theory, though the symmetry is the diffeomorphism.

The action of a relativistic particle in D spacetime takes the form

$$S = -m \int d\tau \sqrt{-g}$$  \hspace{1cm} (1.1)

with

$$g_{\tau\tau} = G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu$$  \hspace{1cm} (1.2)

the induced metric get from the embedding and the D dimensional metric $G_{\mu\nu}$ with convention ($-, +, +, +, ...$).

This action is basically the length of the 1D worldline, and people usually call it the proper time. $m$ describe the hardness of distort it from a straight line, and people usually call it mass.
A well known way of linearize the above action

\[ S = -\frac{1}{2} \int d\tau \sqrt{-g} (g^{\tau\tau} \partial_\tau X^\mu \partial_\tau X_\mu + m^2) \]  

(1.3)

While we paid the price that we should view both \(X^\mu(\tau)\) and \(g(\tau)\) dynamic value and deserve Euler equations. Note the we achieve another goal of describe a massless particle while the above action cannot.

Now we generalize this idea to a 2D submanifold, which has a name worldsheet. We parameterize it as \(X^\mu(\sigma^a)\), with \(a\) takes 1 and 2.

The Nambu-Goto Action:

\[ S = -T \int d^2\sigma \sqrt{-g} \]  

(1.4)

\[ g = \det g_{ab} = \det G_{\mu\nu} \partial_a X^{\mu} \partial_b X^{\nu} \]  

(1.5)

\(T = \frac{1}{2\pi\alpha'}\) the tension of string, \(\alpha'\) the “universal Regge slope”. We choose \([X] = -1\) and \([\sigma] = 0\), then \([\alpha'] = 2\), basically the square of the string scale \(\ell_s\).

We can perform the linearization again and get the Polyakov Action:

\[ S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} \ g^{ab} \partial_\sigma X^a \partial_\sigma X^b G_{\mu\nu} \]  

(1.6)

Note that \(g_{ab}\) is no longer the induced metric but a dynamic tensor field lives on the worldsheet. The \(X\)’s are viewed as D scalar fields lives on the worldsheet.

The Polyakov Action enjoys below symmetries:

- We choose flat \(G_{\mu\nu}\), the Poincaré invariance for D-dimensional field manifold.
- Diffeomorphism invariance for 2-dimensional worldsheet.
- 2-dimensional Weyl invariance:

\[ X'(\sigma) = X(\sigma) \]  

(1.7)

\[ g'_{ab} = e^{2\omega(\sigma)} g_{ab} \]  

(1.8)

Obviously, we can make use of the diff. and Weyl invariance of the worldsheet to set \(g_{ab} = \eta_{ab}\) as a gauge so that the worldsheet has Minkowski coordinates.

The diff. and Weyl invariance of 2D worldsheet makes it a Conformal Field Theory see Chapter 3.

2 Light-cone Quantization

The EoMs of the Polyakov Action:
\[ T_{ab} = -4\pi \sqrt{-g} \frac{\delta S}{\delta g_{ab}} = -\frac{1}{\alpha'} (\partial^a X^\mu \partial^b X_\mu - \frac{1}{2} g^{ab} \partial^c X^\mu \partial_c X_\mu) = 0 \quad (2.1) \]

\[ \frac{\delta S}{\delta X^\mu} = \partial_b \left[ \sqrt{-g} g^{ab} \partial_b X_\mu \right] = \sqrt{-g} \nabla^a \nabla_a X_\mu = 0 \quad (2.2) \]

Note that when taking the later variation, suppose for the string \(-\infty < \tau \equiv \sigma^1 < \infty\) and \(0 < \sigma \equiv \sigma^2 < \ell\).

\[ \delta S = \frac{1}{2\pi \alpha'} \int_{-\infty}^{\infty} d\tau \int_{0}^{\ell} d\sigma \sqrt{-g} \delta X^\mu \nabla^\mu X_\mu - \frac{1}{2\pi \alpha'} \int_{-\infty}^{\infty} d\tau \sqrt{-g} \delta X^\mu \partial^\sigma X_\mu \bigg|_{\sigma=0} \quad (2.3) \]

There are choices to vanish the boundary term.

First we can choose any field the same at \(\sigma = 0\) and \(\sigma = \ell\), and this is actually not a real boundary condition because we can then view the world sheet as a cylinder: \(\sigma \sim \sigma + \ell\). We will always use the convention \(\ell = 2\pi\).

We call this kind of string the Closed String.

Second we can choose
\[ \partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau, \ell) = 0 \quad (2.4) \]

which is called the Neumann boundary conditions.

Or
\[ \delta X^\mu(\tau, 0) = \delta X^\mu(\tau, \ell) = 0 \quad (2.5) \]

which is called the Dirichlet boundary conditions. There 2 cases can be mixed up, which means that for some of \(\mu\)‘s the boundary satisfy the Neumann boundary conditions and the other \(\mu\)‘s satisfy the Dirichlet boundary conditions.

We call this kind of string the Open String.

Consider a flat background \(G_{\mu\nu} = \eta_{\mu\nu}\), with the flat choice of worldsheet metric, further we use lightcone coordinate \(\sigma^\pm = \tau \pm \sigma\), the EoM becomes

\[ \partial_+ \partial_- X^\mu = 0 \quad (2.6) \]
\[ \partial_+ X^\mu \partial_- X_\mu = \partial_- X^\mu \partial_+ X_\mu = 0 \quad (2.7) \]

We have left moving and right moving ansatz
\[ X^\mu(\sigma^+, \sigma^-) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \quad (2.8) \]

For closed string, considering the periodic condition, the mode expansion of classical solution

\[ X^\mu_L(\sigma^+) = \frac{1}{2} x^\mu + \sqrt{\frac{\alpha'}{2}} \alpha^\mu_0 \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\sigma^+} \quad (2.9) \]
\[ X^\mu_R(\sigma^-) = \frac{1}{2} x^\mu + \sqrt{\frac{\alpha'}{2}} \alpha^\mu_0 \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\sigma^-} \quad (2.10) \]
Here \( x^\mu \) is the “position” of the string, and \( p^\mu = \sqrt{\frac{2}{\alpha}} \alpha_0^\mu = \sqrt{\frac{2}{\alpha}} \tilde{\alpha}_0^\mu \) is the momentum of the string. Other modes can be viewed as the oscillation of the string. We see clearly that string has both picture of particle-like and field-like.

The vanish of energy-momentum tensor can be written as

\[
\partial_- X^\mu \partial_- X_\mu = \frac{\alpha'}{2} \sum_{m,n} \alpha_m \alpha_{n-m} e^{-i n \sigma^-} = 0 \tag{2.11}
\]

\[
\partial_+ X^\mu \partial_+ X_\mu = \frac{\alpha'}{2} \sum_{m,n} \tilde{\alpha}_m \tilde{\alpha}_{n-m} e^{-i n \sigma^-} = 0 \tag{2.12}
\]

so that

\[
L_n = \frac{1}{2} \sum_m \alpha_{n-m} \alpha_m = 0 \tag{2.14}
\]

\[
\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \tilde{\alpha}_m = 0 \tag{2.15}
\]

Since \( L, \tilde{L} \) here as constraints, If we quantize the string there are non-physical degrees of freedom. This can also be easily seen from Eq (1.6) that due to the signature of \( \eta_{\mu\nu} \), the 0 field have the wrong signature of Lagrangian so it is actually a ghost. Instead of perform problematic covariant canonical quantization, we only quantiza physical degree of freedom.

BRST quantization: a systematic covariant way to quantize string, see Section 4.

Introduce spacetime lightcone coordinates

\[
X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^{D-1}) \tag{2.16}
\]

\[
X^i \quad i = 1, \ldots, D - 2 \tag{2.17}
\]

Lightcone gauge is achieved by \( \sigma^+ \) and \( \sigma^- \) reparameterization separately :

\[
X^+ = x^+ + \alpha' p^+ \tau \tag{2.19}
\]

The constraints Eq (2.7) of \( X^- \) are

\[
\partial_+ X^-_L = \frac{1}{p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i \tag{2.20}
\]

\[
\partial_- X^-_R = \frac{1}{p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i \tag{2.21}
\]

with mode expansion \( p^-, \alpha^-_n, \tilde{\alpha}^-_n \) can be fixed, for example

\[
\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left( \frac{1}{2} \alpha'^2 p^2 + \sum_{n \neq 0} \alpha_n^i \alpha_{-n}^i \right) = (\alpha \rightarrow \tilde{\alpha}) \tag{2.22}
\]
then define the mass of oscillation state on string

\[ M^2 \equiv 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_i \sum_{n>0} \alpha^i_{-n} \alpha^i_n = (\alpha \rightarrow \tilde{\alpha}) \] (2.23)

Now introduce the algebra

\[
\begin{align*}
[x^i, p^j] &= i \delta^{ij}, & [x^-, p^+] &= -i \\
[\alpha^i_n, \alpha^j_m] &= [\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] = n \delta^{ij} \delta_{n+m,0} 
\end{align*}
\] (2.24, 2.25)

with \( \alpha_n \sim \sqrt{n} a^\dagger_n \) as the annihilation operators and \( \alpha_{-n} \sim \sqrt{n} a_n \) as the creation operators.

the vacuum state of the Hilbert states \(|0; p\rangle\) are viewed as eigenstate of operator \(p^\mu\), while \(\alpha^i_n |0; p\rangle = 0\).

the number operators of oscillation

\[ N = \sum_{i=1}^{D-2} \sum_{n>0} \alpha^i_{-n} \alpha^i_n, \quad \tilde{N} = (\alpha \rightarrow \tilde{\alpha}) \] (2.26)

the formula of \( \alpha \) and \( \tilde{\alpha} \) must be the same, this is called “level matching”.

If we consider the order of operators in \(M^2\), we naively choose

\[
M^2 = \frac{1}{2} \sum_{n<0} \alpha^i_{-n} \alpha^i_n + \frac{1}{2} \sum_{n>0} \alpha^i_{-n} \alpha^i_n \\
= \sum_{n>0} \alpha^i_{-n} \alpha^i_n + \frac{D-2}{2} \sum_{n>0} n \\
= \frac{4}{\alpha'} (N - \frac{D-2}{24}) \quad \text{or} \quad \frac{4}{\alpha'} (\tilde{N} - \frac{D-2}{24})
\] (2.27, 2.28, 2.29)

Where we naively used the regularization \(\sum n = -\frac{1}{12}\).

The Tachyon corresponding to the ground state with problematic mass \(M^2 = -\frac{D-2}{6\alpha'}\), and it is viewed as a problem of bosonic string which won’t happen in superstring theory.

The first excited states are \(\alpha^i_{-1} \alpha^j_{-1} |0; p\rangle\), while \(i, j\) are the index of representation \(SO(D-2)\). This can be the little group of a massless particle under the \(SO(1, D-1)\) Poincaré group.

So

\[ \frac{4}{\alpha'} (1 - \frac{D-2}{24}) = 0 \Rightarrow D = 26 \] (2.30)

We derived the spacetime dimension of bosonic string theory for the first time.
The case of the open string are almost same. The mode expansion of openstring are almost the same as Eq (2.9).

\[
X^\mu_L (\sigma^+ ) = \frac{1}{2} x^\mu + \sqrt{\frac{\alpha'}{2}} \alpha^\mu_0 \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{- i n \sigma^+} \tag{2.31}
\]

\[
X^\mu_R (\sigma^- ) = \frac{1}{2} x^\mu + \sqrt{\frac{\alpha'}{2}} \tilde{\alpha}^\mu_0 \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{- i n \sigma^-} \tag{2.32}
\]

But here \( p^\mu = \frac{1}{\sqrt{2 \alpha'}} \alpha^\mu_0 = \frac{1}{\sqrt{2 \alpha'}} \tilde{\alpha}^\mu_0 \).

While the Neumann boundary conditions \( \partial_\sigma X^a = 0, \ a = 0, ..., p \) require

\[
\alpha^a_n = \tilde{\alpha}^a_n \tag{3.33}
\]

and the Dirichlet boundary conditions \( X^I = c^I, \ I = p + 1, ..., D \) require

\[
dx^I = c^I, \ p^I = 0, \ \alpha^a_n = -\tilde{\alpha}^a_n \tag{3.34}
\]

Above requirement means that there are only one set of oscillators contrast to open string. Again, the spacetime lightcone is chosen as

\[
X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^p) \tag{3.35}
\]

\[
M^2 = \frac{1}{\alpha'} \left( \sum_{a=1}^{p-1} \sum_{n>0} \alpha^a_{-n} \alpha^a_n + \sum_{I} \sum_{n>0} \alpha^I_{-n} \alpha^I_n - C \right) \tag{3.36}
\]

the require of the reduced symmetry \( SO(1,p) \times SO(D - P - 1) \) lead us to the result \( D = 26 \) and \( C = 1 \) again.

The first states of open string:

\[
\alpha_{-1}^a \mid 0; p \rangle \ a = 0, ..., p - 1 \tag{3.37}
\]

These states transform under the \( SO(1,p) \) group, and actually this is the photon.

\[
\alpha_{-1}^I \mid 0; p \rangle \ I = p + 1, ..., D \tag{3.38}
\]

These states transform under the \( SO(D - P - 1) \) group, and actually they are the scalar fields describe the fluctuations of the brane in the transverse direction.

3 Conformal Field Theory

Conventions: the action

\[
S = \frac{1}{2 \pi \alpha'} \int d^2 z \ \partial X^\mu \bar{\partial} X_\mu \tag{3.1}
\]
the classical EoMs
\[ \partial \bar{\partial} X^\mu (z, \bar{z}) = 0 \] (3.2)

Expectation values under path integral
\[ \langle F[X] \rangle = \int DX e^{-S} F[X] \] (3.3)

When writing operator equations, usually we do not write \( \langle \ \rangle \) for convenience, which means
\[ F = G \Leftrightarrow \langle (F - G) \rangle = 0 \] (3.4)

Introduce normal ordering defined as:
\[ : A := A - \langle A \rangle \] (3.5)

For the free CFT in a 2D manifold without boundary, the propagator is
\[ \langle X^\mu (z_1, \bar{z}_1) X^\nu (z_2, \bar{z}_2) \rangle = -\frac{\alpha'}{2} \eta^{\mu\nu} \log |z_{12}|^2 \] (3.6)

Together with the Wick theorem, we have a compact expression:
\[ : F := F - \text{contractions} = \exp \left\{ \frac{\alpha'}{4} \int d^2 z_1 d^2 z_2 \log |z_{12}|^2 \delta \frac{\delta}{\delta X^\mu (z_1, \bar{z}_1)} \delta \frac{\delta}{\delta X^\nu (z_2, \bar{z}_2)} \right\} F \] (3.7)

Similarly
\[ : FG := : F :: G : - \text{cross contractions} \] (3.8)
\[ = \exp \left\{ \frac{\alpha'}{4} \int d^2 z_1 d^2 z_2 \log |z_{12}|^2 \delta \frac{\delta}{\delta X^\mu (z_1, \bar{z}_1)} \delta \frac{\delta}{\delta X^\mu (z_2, \bar{z}_2)} \right\} : F :: G : \] (3.9)

A tool in CFT for calculating expectation values of local operators is so called operator product expansion (OPE).

In operator language, we assume there are relations such as
\[ O_i (\sigma_1) O_j (\sigma_2) = \sum_k c^k_{ij} (\sigma_1 - \sigma_2) O_k (\sigma_2) \] (3.10)

What we want is the asymptotic behaviour when one of the local operator approach another. Which is similar to the power series expansion.

Definition of Conformal Transformation: \( \sigma^a \rightarrow \tilde{\sigma}^a \) such that
\[ g_{ab} \frac{\partial \sigma^a}{\partial \tilde{\sigma}^c} \frac{\partial \sigma^b}{\partial \tilde{\sigma}^d} = \Lambda (\sigma) g_{cd} \] (3.11)

but here we leave the components of the metric invariant. In other words, this is a reparameterization plus a Weyl transformation which leaves the components of the metric not
changed. For the circumstance we are interested in, 2D worldsheet case, any holomorphic change of coordinates satisfies this requirement.

For a Weyl invariant theory, while we change $\delta g_{ab} = \zeta_{ab}$ as a local scaling,

$$\delta S = \int \mathcal{D}^2\sigma \frac{\delta S}{\delta g_{ab}} = -\frac{1}{4\pi} \int \mathcal{D}^2\sigma \zeta^a \mathcal{T}^a$$  \tag{3.12}$$

So we find that for a Weyl invariant theory, $\mathcal{T}^a = 0$

which can be written as $T_{z\bar{z}} = 0$

The conservation of the energy-momentum tensor can be written as

$$\bar{\partial} T_{zz} = 0 \quad \partial T_{z\bar{z}} = 0$$  \tag{3.13}$$

So we use a simplified notation as holomorphic functions

$$T_{zz} = T(z) \quad T_{z\bar{z}} = \bar{T}(\bar{z})$$  \tag{3.14}$$

For a general conformal transformation $\delta \sigma^a = \zeta^a$, which correspond to a change to the metric $\delta g_{ab} = \partial_a \zeta_b + \partial_b \zeta_a$, the conserved current takes the form

$$J_a = \zeta^a T_{ab}$$  \tag{3.15}$$

and the energy-momentum is defined in the meaning of normal ordered. For example, for the scalar theory

$$T_{ab} = -\frac{1}{\alpha'} : \left( \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \delta_{ab} \partial^c X^\mu \partial_c X_\mu \right) :$$  \tag{3.16}$$

The current is defined by the Noether theorem, while the energy-momentum tensor defined proportional to the functional derivative of the action over the metric.

Again a noteion

$$J(z) = \zeta(z) T(z) = J_z \quad \bar{J}(\bar{z}) = \bar{\zeta}(\bar{z}) \bar{T}(\bar{z}) = J_{\bar{z}}$$  \tag{3.17}$$

Consider a general field transformation $X'(\sigma) = X(\sigma) + \rho(\sigma) \delta X(\sigma)$, with the assumption of both the measure and the action invariant.

$$\int \mathcal{D}X' e^{-S[X']} O' = \int \mathcal{D}X e^{-S[X]} - \frac{1}{2\pi} \int \partial^a \rho \partial_a \rho (O + \rho \delta O)$$  \tag{3.18}$$

which at leading order gives

$$\rho \delta O(\sigma_0, \bar{\sigma}_0) + \frac{1}{2\pi} \int \partial^a \rho J_a = 0 \quad \tag{3.19}$$

Where $O(\sigma_0, \bar{\sigma}_0)$ is a local insertion in the expectation, and the general case of more operators is similar.
Now the tricky part comes that $\rho(\sigma)$ are arbitrary. If we choose the region where $\rho \neq 0$ don’t include $\sigma_0$, we only got the trivial Ward identity. And if we choose a region $D$ where $\rho = 1$ include $\sigma_0$, using the Stokes’ theorem

$$\delta O(\sigma_0, \overline{\sigma}_0) = \frac{i}{2\pi} \int_{\partial D} (J dz - \overline{J} d\overline{z}) O(\sigma_0, \overline{\sigma}_0) = -\text{Res}_{\overline{z}_0} J(\overline{z}) O(\overline{z}_0, \overline{\overline{z}_0}) - \overline{\text{Res}_{\overline{z}_0} \overline{J}(\overline{z}) O(\overline{z}_0, \overline{\overline{z}_0})}$$

(3.20)

This result shows that a transformation of an operator can be read from the OPE between the energy-momentum tensor and this operator.

Consider a transformation $z' = z'(z)$, choose a basis of operators which are eigenstates under this transformation (they are call the primary fields)

$$O'(z', \overline{z'}) = (\frac{\partial z'}{\partial z})^{-\frac{h}{2}} (\frac{\partial \overline{z'}}{\partial \overline{z}})^{-\frac{\tilde{h}}{2}} O(z, \overline{z})$$

(3.21)

according to the Ward identity, this is equivalent to

$$T(z) O(z_0, \overline{z}_0) = ... + \frac{h}{z} O(z_0, \overline{z}_0) + \frac{1}{z} \partial O(z_0, \overline{z}_0) + ...$$

(3.22)

and so for $\overline{T}$. Here $(h, \tilde{h})$ are called the weights of $O$.

some results of free $X^\mu$ field:

$X^\mu$: (0, 0), $\partial X^\mu$: (1, 0), $\overline{\partial} X^\mu$: (0, 1), $\partial^2 X^\mu$: (2, 0),

e$^{ik \cdot X}$: $\left(\frac{a_k^2}{4}, \frac{a_i^2}{4}\right)$, $(\Pi_i \partial m_i X^\mu_i) (\Pi_j \overline{\partial} n_i X^\mu_j) e^{ik \cdot X}$: $\left(\frac{a_k^2}{4} + \sum m_i, \frac{a_i^2}{4} + \sum n_i\right)$.

For D free scalar field, using the Eq (3.8), we got the TT OPE:

$$T(z_1) T(z_2) = \frac{D}{2z_{12}^4} + \frac{2 T(z_2)}{z_{12}^2} + \frac{\partial T(z_2)}{z_{12}} + \text{nonsingular}$$

(3.23)

and generally

$$T(z_1) T(z_2) = \frac{c}{2z_{12}^4} + \frac{2 T(z_2)}{z_{12}^2} + \frac{\partial T(z_2)}{z_{12}} + \text{nonsingular}$$

(3.24)

where $c$ is the central charge. Obviously each scalar field has $c = 1$ and weights (2, 0).

Consider a CFT on a cylinder: $w = \sigma + i\tau$. Then we made a map to a plane $z = e^{-iw}$.

The Laurent expansion of the energy-momentum tensor

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \quad \overline{T}(\overline{z}) = \sum_{m=-\infty}^{\infty} \frac{\overline{L}_m}{\overline{z}^{m+2}}$$

(3.25)

We call $L$ and $\overline{L}$ the Virasoro generators.

$$[L_m, \overline{L}_n] = \oint \left(\frac{dz}{2\pi i} \frac{dw}{2\pi i} - \frac{dw}{2\pi i} \frac{dz}{2\pi i}\right) z^{m+1} w^{n+1} T(z) T(w)$$

(3.26)
Note that the operators are always time-ordered, so we define this as the Residue which comes from an observation of the integral contours.

\[
[L_m, L_n] = \oint \frac{dw}{2\pi i} \text{Res}_{z=w} \left( z^{m+1} w^{n+1} T(z) T(w) \right)
\]

\[
= \oint \frac{dw}{2\pi i} \text{Res}_{z=w} z^m w^n \left( \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} \frac{\partial T(w)}{z-w} + \text{nonsingular} \right)
\]

\[
= \oint \frac{dw}{2\pi i} w^{n+1} \left( w^{m+1} \partial T(w) + 2(m+1)w^m T(w) + \frac{c}{12} m(m^2-1) w^{m-2} \right)
\]

\[
= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}
\]

We call this the Virasoro algebra. This comes from the central extension of the natural algebra of the conformal transformation generators (see [1], Chapter 1).

Note that here \( L_{-1}, L_0 \) and \( L_1 \) form a closed algebra: \( SL(2, \mathbb{C}) \), the global conformal transformations.

Things becomes a little different when there are boundaries. For the open string case, just as we showed by light-cone quantization, there are just half modes. Here if we map a open string to the upper complex plane, considering the Neumann boundary requirement \( T_{ab} n^a b = 0 \) at real axis, we can extend the definition as \( T(z) = \bar{T}(\bar{z}) \) to the lower plane so that \( T(z) \) is holomorphic in the whole plane. The result is we have only one set of generators \( L_m \).

For a conformal theory on a cylinder, using our usual way to map the cylinder to a plane, we always map the infinite past \( \tau \to -\infty \) to the origin of the plane. If we consider the states lives on the cylinder, which is already decided by a path integral from a state at some \( \tau_0 \), we find that any states correspond to an operator at the origin.

\[
\Psi_f [X_f(\sigma), r_f] = \int \mathcal{D}X_i \int_{X_{r_f}=X_f}^{X_{r_i}=X_i} \mathcal{D}X e^{-S[X]} \Psi_i [X_i(\sigma), r_i]
\]

\[
= \int_{X(r)=X_f}^{X(r)=X_i} \mathcal{D}X e^{-S[X]} \mathcal{O}(z=0)
\]

So in our later path integral ways of quantizing string, we can always describe a particle state which lives on a cylinder by a vertex operator.

Again we write down the mode expansion, but of imaginary time coordinate \( w = \sigma + i\tau \), and complex plane coordinate \( z = e^{-iw} \).

\[
X^\mu = x^\mu - i\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left( \alpha_n e^{inw} + \tilde{\alpha}_n e^{in\bar{w}} \right)
\]
\[ x^\mu - i \frac{\alpha'}{2} p^\mu \log |z|^2 + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left( \frac{\alpha_\mu_n}{z^n} + \tilde{\alpha}_\mu_n \right) \] (3.34)

thus we have

\[ \alpha_{-m} = \sqrt{\frac{2}{\alpha'}} \int \frac{dz}{2\pi} \partial X^\mu(z) \sim \sqrt{\frac{2}{\alpha'} (m-1)!} \partial^m X^\mu(0) \] (3.35)

and similar to \( \tilde{\alpha}_{-m} \)

so the correspondence can be written as

\[ |0, k\rangle \sim e^{ik \cdot X(0)} : \] (3.36)

which comes from

\[ p^\mu |0, k\rangle \sim \int \frac{dz}{2\pi} \partial X^\mu(z) : e^{ik \cdot X(0)} : = - \int \frac{dz}{2\pi} \frac{iz \alpha' k^\mu}{2} \sim k^\mu : e^{ip \cdot X(0)} : \] (3.37)

and similarly

\[ \alpha^\mu_{-m} |0, k\rangle \sim \partial^m X^\mu(0) : e^{ip \cdot X(0)} : \] (3.38)

For later use we want some diff. \times Weyl invariant vertex operators. They should take the form of an integral of local vertex operator to acquire diff. invariance. Then to offset the transformation of \( d^2 z \), we require the integrand weights \((1, 1)\).

Then for a state

\[ (\Pi_i \alpha_{-m_i}^{\mu_i}) \left( \Pi_j \tilde{\alpha}_{-n_j}^{\nu_j} \right) |0, k\rangle \sim : \int d^2 z (\Pi_i \partial^{m_i} X^{\mu_i}) (\Pi_j \tilde{\partial}^{n_i} X^{\nu_i}) e^{ik \cdot X} : \] (3.39)

conformal invariance require

\[ \frac{\alpha' k^2}{4} + \sum_i m_i = 1 \] (3.40)

\[ \frac{\alpha' k^2}{4} + \sum_i n_i = 1 \] (3.41)

this totally agree with Eq (2.29). So here conformal means on-shell for a vertex operator!

And for open string vertex operator things are similar. One significant difference is that open string vertex operator live on the boundary.

\[ : \int_{\partial M} d\tau (\Pi_i \partial^{m_i} X^{\mu_i}) e^{ik \cdot X} : \] (3.42)

again, conformal invariance require

\[ \alpha' k^2 + \sum_i m_i = 1 \] (3.43)

the on-shell condition of open string.
4 The Polyakov Path Integral

Consider the worldsheet embedding in the flat spacetime $X(\sigma)$ and the dynamical field $g(\sigma)$ as quantum fields living on the 2D worldsheet $\mathcal{M}$. From now on we basically only consider the Path Integral way of quantizing string.

After the Wick rotation (see [2] page 82-83), we have

$$Z = \int \mathcal{D}X \mathcal{D}g \ e^{-S[X,g]} \quad (4.1)$$

$$S = \frac{1}{2\pi} \int_{\mathcal{M}} d^2 \sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu \quad (4.2)$$

Now since there are tremendous overcounting in the field configurations due to the diff.×Weyl symmetry. Actually we should choose a slice which only intersects with each equivalence class once. This slice can be chosen by a gauge fixing condition. What we should mention is that we should times a appropriate Jacobian to the functional integrand. It is the standard Faddeev-Popov process.

A shorthand for a diff.×Weyl transformation: $\zeta : g \rightarrow \hat{g}^\zeta$.

$$1 = \Delta_{FP}(g) \int \mathcal{D}\zeta \delta(g - \hat{g}^\zeta) \quad (4.3)$$

As mentioned above, the can always set $\hat{g}_{ab} = \delta_{ab}$ at least locally with some diff.×Weyl transformation.

then

$$Z = \frac{1}{V_{\text{gauge}}} \int \mathcal{D}X \mathcal{D}\zeta \mathcal{D}g \ e^{-S[X,g]} \quad (4.4)$$

$$= \frac{1}{V_{\text{gauge}}} \int \mathcal{D}X \mathcal{D}\zeta \Delta_{FP}(g) \delta(g - \hat{g}^\zeta) e^{-S[X,g]} \quad (4.5)$$

$$= \frac{1}{V_{\text{gauge}}} \int \mathcal{D}X \mathcal{D}\zeta \Delta_{FP}(\hat{g}^\zeta) e^{-S[X,\hat{g}^\zeta]} \quad (4.6)$$

$$= \frac{1}{V_{\text{gauge}}} \int \mathcal{D}X \mathcal{D}\zeta \Delta_{FP}(\hat{g}) e^{-S[X,\hat{g}]} \quad (4.7)$$

$$= \int \mathcal{D}X \Delta_{FP}(\hat{g}) e^{-S[X,\hat{g}]} \quad (4.8)$$

Where we require the gauge invariance of the measure, the delta functional and the action. Note that $\Delta_{FP}$ is also gauge invariant.

Now calculate the Faddeev-Popov determinate.

$$g^\zeta_{ab}(\sigma') = e^{2\omega(\sigma)} \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} g_{cd}(\sigma) \quad (4.9)$$
so that

\[
\delta g_{ab} = 2\delta \omega g_{ab} - \nabla_a \delta \sigma_b - \nabla_b \delta \sigma_a
\]

\[
= (2\delta \omega - \nabla_c \delta \sigma^c)g_{ab} - (\nabla_a \delta \sigma_b + \nabla_b \delta \sigma_a - \nabla_c \delta \sigma^c g_{ab})
\] (4.11)

Then using the integral form of the delta functional

\[
\Delta_{FP}^{-1} = \int \mathcal{D}\delta \omega \mathcal{D}\delta \sigma \mathcal{D}\beta \exp \left\{ 2\pi i \int d^2 \sigma \sqrt{g} \beta^{ab} \left[ -2\delta \omega - \nabla_c \delta \sigma^c \right] g_{ab} + (\nabla_a \delta \sigma_b + \nabla_b \delta \sigma_a - \nabla_c \delta \sigma^c g_{ab}) \right\}
\]

\[
= \int \mathcal{D}\delta \sigma \mathcal{D}\beta \mathcal{D}\epsilon \exp \left\{ 2\pi i \int d^2 \sigma \sqrt{g} \beta^{ab} (\nabla_a \delta \sigma_b + \nabla_b \delta \sigma_a - \nabla_c \delta \sigma^c g_{ab}) \right\}
\] (4.13)

Above we integrate out $\delta \omega$ which force $\beta$ to be traceless. A usual way to reverse the minus power of the determinant is changing the boson fields to the fermion fields. Together with a field normalization,

\[
\Delta_{FP} = \int \mathcal{D}b \mathcal{D}c \exp \left\{ \frac{1}{4\pi} \int d^2 \sigma \sqrt{g} \beta^{ab} (\nabla_a c_b + \nabla_b c_a - \nabla_d c^d g_{ab}) \right\}
\] (4.15)

As usual we add a ghost action.

After choosing the flat worldsheet metric, the total action include $bc$ ghosts is

\[
S = \frac{1}{2\pi} \int d^2z \left( 2\partial X^\mu \partial X_\mu + b \partial c + \bar{b} \partial \bar{c} \right)
\] (4.16)

It is obvious that we can do the mode expansion of field $b$ and $c$ as before.

\[
b(z) = \sum_{n=-\infty}^{\infty} \frac{b_n}{z^{n+\lambda_b}}, \quad c(z) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+\lambda_c}}
\] (4.17)

Note that the conformal dimensions are $\lambda_b = 2$, $\lambda_c = -1$.

Then the OPE of $bc$ ghosts are $c(z)b(w) = 1/(z-w) + ...$, which in other words is \(\{b_m, c_n\} = \delta_{m+n,0}\).

The energy momentum tensor has an additional terms comes from $bc$ ghost:

\[
T_X = -2 : \partial X \cdot \partial X : , \quad T_g = -2 : b \partial c : + : c \partial b :
\] (4.18)

$T_g$ gives a central charge $c_g = -26$ in the Virasoro algebra in terms of $L_0^g$. And we want to forbid the conformal anomaly:

\[
\langle T^a_a \rangle = -\frac{1}{12} (c + c_g) R
\] (4.19)

where $R$ is the curvature scalar. So we need other fields with central charge $c = 26$, such as 26 $X$ scalars. (for details see [2] section 2.5 and 3.4)

Now we want to show how to perform the BRST quantization, which quantize the string in a modern covariant way.
Even we used the conformal symmetry to set the metric to be flat, there is still a symmetry left: the BRST symmetry
\[ \delta X^\mu = \eta c \partial X^\mu, \quad \delta c = \eta c \partial c, \quad \delta b = \eta (TX + T_g) \]  
Here \( \eta \) is a constant infinitesimal Grassmann parameter.

At this time, the corresponding BRST charge can be obtained from the conserved current which comes from the Noether theorem.

\[ Q = \oint \frac{dz}{2\pi i} (cT_X + :bc\partial c:) \]  
This charge generate the BRST transformation of fields: e.g. \( \{Q, b\} = T \) as above.

In terms of modes it appears as:
\[ Q = \sum_{m=-\infty}^{\infty} (L_m^X - \delta_{m,0})c_m - \frac{1}{2} \sum_{m,n=-\infty}^{\infty} (m-n) c_{-m} c_{-n} b_{m+n} : \]  
It can be proved that (though it is not quite easy) \( Q \) is nilpotent
\[ Q^2 = 0 \]  
As \( Q \) represent an residual gauge symmetry, we want the physical states not changed by \( Q \). As we did in topology, if \( Q |\phi\rangle = 0 \) we call \( |\phi\rangle \) BRST-closed, and if \( |\varphi\rangle = Q |\chi\rangle \) we call \( |\varphi\rangle \) BRST-exact. The physical states is the equivalence class of BRST-closed states, and two stats are equivalent if their only difference is a BRST-exact state.

Because of the ghost zero modes, the vacuum is degenerate: \( |\downarrow\rangle \) and \( |\uparrow\rangle \) which satisfy \( c_0 |\downarrow\rangle = |\uparrow\rangle \), \( b_0 |\uparrow\rangle = |\downarrow\rangle \). Our choice is that for a physical state, \( b_n |\phi\rangle = 0 \), where \( n \geq 0 \), which means that |\phi\rangle is some \( \alpha \) oscillators act on |\downarrow\rangle, and do not contain any c oscillators. Then the Virasoro constraints follows from \( Q |\phi\rangle = 0 \).

\[ \{Q, b_n\} = (L_n^X - \delta_{n,0}) |\phi\rangle = 0, \quad n \geq 0 \]  
This is exactly what we need for a covariant way to quantize string. Here the \( n = 0 \) requirement gives the mass shell condition of particle states.

Now comes to the interactions in string theory. As we mentioned above, the particle states lives on the cylinder of the closed string. The way in string theory to introduce interaction is choosing a manifold which has the corresponding boundary of the in and out quantum states. Of course there are many manifolds satisfy this requirement, and we just add them with different powers of some perturbation coupling constant. However there can be a much beautiful way to do this in the action, although this is still not the non-perturbative definition.

\[ Z = \int \mathcal{D}X \mathcal{D}g \ e^{-S[X,g]} \]  
\[ S = \frac{1}{4\pi\alpha'} \int_M d^2 \sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu + \lambda \chi \]
Here the second and the third part of the action is a term called the Gauss-Bonnet term. In 2D this term is not dynamical but pure topological according to the Gauss-Bonnet Theorem.

\[
\chi = \frac{1}{4\pi} \int_M d^2\sigma \sqrt{g} R + \frac{1}{2\pi} \int_{\partial M} ds \kappa \tag{4.27}
\]

\(\chi\) is the Euler number of the compact manifold, \(R\) the curvature scalar of the 2D worldsheet metric and \(\kappa\) the geodesic curvature. Additional number of \(\chi\) see [2] page 84.

A quick counting of the Euler number:

\[
\chi = 2 - 2g - b - c \tag{4.28}
\]

Constructing from a 2-sphere, \(g\) is the number of handles: the genus, \(b\) is the number of cutting holes and \(c\) the number of cross-caps. Introduce cross-cap for unoriented manifold: cut a hole and identify diametrically opposite points.

Examples: \((g, b, c) =\) sphere: \((0, 0, 0)\), torus: \((1, 0, 0)\), disk: \((0, 1, 0)\), annulus: \((0, 2, 0)\), the Mobius strip: \((0, 1, 1)\), the Klein bottle: \((0, 0, 2)\).

A direct result: the topological perturbation coupling constants are

\[
g_{\text{Closed}} \sim g_{\text{Open}}^2 \sim e^\lambda \tag{4.29}
\]

There are another way to add interactions, as we usually do in the Lagrangian field theory: add higher order terms in the Lagrangian which correspond to the interacting vertices. Of course we can add them here, but a big problem is to find terms that are Weyl invariant not only classically but also at quantum level. We will see that actually we use this requirement to write the Lagrangian.

A natural generalization to the curved spacetime:

\[
S = \frac{1}{4\pi\alpha'} \int d^2\sigma \; \partial^a X^\mu \partial_a X^\nu G_{\mu\nu}(X) \tag{4.30}
\]

Where the curved spacetime metric \(G_{\mu\nu}(X)\) can be viewed as a function of the fields \(X\), so the exact form of \(G_{\mu\nu}(X)\) is also part of the Lagrangian of the theory.

Amazingly, the conformal requirement lead to (calculations see [3] 7.1.1)

\[
\beta_{\mu\nu} = \mu \frac{\partial G_{\mu\nu}}{\partial \mu} \sim R_{\mu\nu}(X) = 0 \tag{4.31}
\]

at leading order. This is exactly the Einstein equation in vacuum.

So understood in the 2D field theory way, we can put the general solution of Einstein equation in the Lagrangian. We know it can be a superposition of the gravitational wave modes.

\[
Z = \int D\mathcal{X} Dg \exp\left\{-\frac{1}{4\pi\alpha'} \int d^2\sigma \; \partial^a X^\mu \partial_a X^\nu (\eta_{\mu\nu} + h_{\mu\nu}(X))\right\} \tag{4.32}
\]

\[
= \int D\mathcal{X} Dg \exp\left\{-\frac{1}{4\pi\alpha'} \int d^2\sigma \; \partial^a X^\mu \partial_a X^\nu \eta_{\mu\nu}\right\} e^{-V} \tag{4.33}
\]
while
\[ V = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial^\alpha X^\mu \partial_a X^\nu h_{\mu\nu}(X) = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial^\alpha X^\mu \partial_a X^\nu e^{ip\cdot X} \zeta_{\mu\nu} \] (4.34)

which is just the vertex operator of graviton.

So actually we construct the Lagrangian from the vertex operators of the particle states. Simply the coherent state of gravitons combine together as a classical field which appears in the Lagrangian as the spacetime background.

A further generalization is to also include the anti-symmetric and diagonal representations.

\[ S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left( \partial_a X^\mu \partial_b X^\nu g^{ab} G_{\mu\nu}(X) + i\partial_a X^\mu \partial_b X^\nu e^{ab} B_{\mu\nu}(X) + \alpha' R(g) \Phi(X) \right) \] (4.35)

It is amazing that the expectation value of \( \Phi_0 \) can replace the role of the perturbation coupling constant \( \lambda \) we mentioned above.

The beta functions
\[ \beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H^{\mu\lambda\kappa} + O(\alpha'^2) \] (4.36)
\[ \beta^B_{\mu\nu} = -\frac{\alpha}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu} + O(\alpha'^2) \] (4.37)
\[ \beta^{\Phi} = \frac{D - 26}{6} - \frac{\alpha}{2} \nabla^\lambda \nabla_\lambda \Phi + \alpha' \nabla^\mu \Phi \nabla_\mu \Phi - \frac{\alpha'}{24} H_{\mu\lambda\kappa} H^{\mu\lambda\kappa} + O(\alpha'^2) \] (4.38)

Where \( H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \), which is invariant under the gauge transformation:
\( \delta B_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu \).

They can be the EoMs of the low energy effective action
\[ S = \frac{1}{2\kappa_0^2} \int d^D X \sqrt{-G} e^{-2\Phi} \left( R - \frac{1}{12} H^{\mu\lambda\kappa} H_{\mu\lambda\kappa} + 4\partial^\mu \Phi \partial_\mu \Phi \right) \] (4.39)

This action is in the string frame, and can be transformed to the Einstein frame ([3] section 7.3.1).

5 The String Amplitude

The leading interaction in string theory is the tree-level amplitude. As we mentioned, tree-level amplitude correspond to sphere for closed string and disk for open string.

Consider closed string first, the n-point amplitude is given by the correlation function of the vertex operators on the worldsheet.

\[ A^{(n)}(p_1, \ldots p_n) = \sum_{\text{topologies}} \int \frac{D X D g}{\text{Volume}} e^{S - \lambda X} V_1(p_1) \ldots V_n(p_n) \] (5.1)
\( A^{(n)}_{\text{closed tree}}(p_1, \ldots p_n) = e^{-2\lambda} \int \frac{\mathcal{D}X\mathcal{D}g}{\text{Volume}} e^{-S} V_1(p_1) \ldots V_n(p_n) \) (5.2)

For bosonic string, \( V(p) = \int d^2z : \partial^m \partial^\dagger \partial^m \partial^\dagger X \ldots \exp \{ ipX \} : \leftrightarrow \alpha^- \ldots \alpha^- |0, p\rangle \).

Here we neglect the bc ghosts, for a detailed discussion, see [2] Chapter 5.

For the simpler amplitudes for tachyon, the path integral is nothing more than a Gaussian integral:

\[ A^{(n)}_{\text{tachyon tree}} \propto \int \prod_{i=1}^n d^2z_i \int \mathcal{D}X \exp \left( -\frac{1}{2\pi\alpha'} \int d^2z \partial X \cdot \bar{\partial} X \right) \exp \left( i \sum_{i=1}^n p_i \cdot X(z_i) \right) \] (5.3)

\[ \propto \int \prod_{i=1}^n d^2z_i \exp \left( \frac{1}{2} \int d^2w d^2w' J(w) \frac{1}{\partial \bar{\partial}} J(w') \right) \] (5.4)

Here

\[ J(z) = \sum_i p_i \delta(z - z_i), \quad \frac{1}{\partial \bar{\partial}} = \frac{\alpha'}{2} \log |z - z'|^2 \] (5.5)

\[ A^{(n)}_{\text{tachyon tree}} \propto \delta(\sum_i p_i) \int \prod_{i=1}^n d^2z_i \prod_{i<j} |z_i - z_j|^{\alpha' p_i \cdot p_j} \] (5.6)

Note that the delta function is simply the conservation of spacetime momentum, which comes from the integral of zero mode.

For the cases with higher order \( \alpha \) oscillators, we need to contract \( \partial X \)s to find out the result.

The diff\( \times \)Weyl symmetry allow us to view the 2-sphere as a complex plane with the north pole is described by the infinity. There is a remnant symmetry by the conformal killing group. It correspond to the non-singular transformations on sphere:

\[ L_{-1} = \partial_z, \quad L_0 = z \partial_z, \quad L_1 = z^2 \partial_z \] (5.7)

They all together combined into the group: \( PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2 \).

\[ z' = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad \text{identify } (a, b, c, d) \sim (-a, -b, -c, -d) \] (5.8)

It is well known that such a transformation can map any 3 points on the 2-sphere to 0, 1 and \( \infty \). Fixing 3 integrals by this symmetry is a great simplification to the calculation.

\[ z_1 = 0, \quad z_2 = 1, \quad z_3 = \infty \] (5.9)

\[ A^{(3)}_{\text{tachyon tree}} \sim \delta(\sum p_i) \] (5.10)
\[ z_1 = 0, z_2 = 1, z_3 = z, z_4 = \infty \] (5.11)

\[ A^{(4)}_{\text{techyon tree}} \sim \delta(\sum p_i) \int d^2z \, |z|^{\alpha' p_1 p_3} |1 - z|^{\alpha' p_2 p_3} \] (5.12)

\[ = 2\pi \delta(\sum p_i) \frac{\Gamma(-1 - \frac{\alpha'}{4} s) \Gamma(-1 - \frac{\alpha'}{4} t) \Gamma(-1 - \frac{\alpha'}{4} u)}{\Gamma(2 + \frac{\alpha'}{4} s) \Gamma(2 + \frac{\alpha'}{4} t) \Gamma(2 + \frac{\alpha'}{4} u)} \] (5.13)

where \( s, t \) and \( u \) the Mandelstam variables with \( s + t + u = 4M_{\text{techyon}}^2 = -\frac{16}{\alpha'} \). This 4pt techyon amplitude is called the Virasoro-Shapiro amplitude.

We can easily find out the triple crossing symmetry of a 4pt scalar scattering amplitude \( A(s, t) = A(s, u) = A(t, u) \). We know \( \Gamma(x) \) has poles at 0 and negative integers, so from a spacetime QFT perspective the intermediate on-shell states are at \( M^2 = \frac{4}{\alpha'} (n - 1) \), which is the same as the mass spectrum we got before.

Regge limit: \( s \to \infty, t \) fixed.

\[ A \to s^{2+\frac{\alpha'}{4}} t^{\frac{\alpha'}{4}} \frac{\Gamma(-1 - \frac{\alpha'}{4} t)}{\Gamma(2 + \frac{\alpha'}{4} t)} \] (5.14)

Regge limit: \( s \to \infty, t \) fixed. This amplitude has the so called Regge behavior:

\[ A \to s^{2+\frac{\alpha'}{4}} t^{\frac{\alpha'}{4}} \frac{\Gamma(-1 - \frac{\alpha'}{4} t)}{\Gamma(2 + \frac{\alpha'}{4} t)} \] (5.15)

Hard scattering limit: \( s, t \to \infty, s/t \sim 1 \). \( \Gamma(x) \sim e^{x \log x} \) as \( x \to \infty \).

\[ A \to e^{\exp \left( -\frac{\alpha'}{2} (s \log s + t \log t + u \log u) \right)} \] (5.16)

Now consider the tree-level amplitude of the open string. We consider the Neumann boundary on a disk.

The disk can be made by identifying \( z \sim 1/\bar{z} \) on the complex plane. Obviously by this definition the fundamental region can be \( |z| \leq 1 \). It is usually more convenient to map this region to the upper half plane by \( z' = i(1 + z)/(1 - z) \), identifying \( z' \sim z' \). The boundary now \( \text{Im} \, z = 0 \). The conformal killing group now is simply \( PSL(2, \mathbb{R}) \), for not changing the boundary.

Things different from the closed string case, because the Neumann boundary condition on the real axis. The result is that simply there is only one part of the 2 parts of modes (chiral and anti-chiral), as we see before. The propagator on the upper half plane is (by considering an image charge):

\[ \frac{1}{\partial \bar{\partial}} = -\frac{\alpha'}{2} \left( \log |z_1 - z_2|^2 + \log |z_1 - \bar{z}_2|^2 \right) \] (5.17)

Then similar to the closed string case
\[ A_{\text{techyon tree}}^{(n)} \propto \delta(\sum p_i) \int \prod_{i=1}^{n} dz_i \prod_{i<j} |z_i - z_j|^\alpha p_i p_j |z_i - \bar{z}_j|^\alpha p_i p_j \]  

(5.18)

\[ \propto \delta(\sum p_i) \int \prod_{i=1}^{n} dx_i \prod_{i<j} |x_i - x_j|^{2\alpha} p_i p_j \]  

(5.19)

As the vertex operator of open string is on the boundary, we finally got an extra 2 factor on the exponential. Note that here is a normal ordering issue as \( z_i = \bar{z}_i \) lead to divergence on the boundary. See [2] Page 175.

Making use the conformal killing group we can set 3 of the coordinates to 0, 1 and \( \infty \), leaving only one integral over \( x \). Something different to the closed string theory is that for open string amplitude, the ordering of the vertex operators are important. For 4pt amplitudes, there are 6 permutations, which equivalent to

\[ x_1 = 0, x_2 = 1, x_3 = \infty, x_4 = x \text{ from } -\infty \text{ to } 0, \text{ from } 0 \text{ to } 1, \text{ from } 1 \text{ to } \infty. \]

\[ x_1 = 0, x_3 = 1, x_2 = \infty, x_4 = x \text{ from } -\infty \text{ to } 0, \text{ from } 0 \text{ to } 1, \text{ from } 1 \text{ to } \infty. \]

\[ A_{\text{techyon tree}}^{(4)} \propto \delta(\sum p_i) \int_{-\infty}^{\infty} dx \left( |x|^{2\alpha} p_1 p_4 |1 - x|^{2\alpha} p_2 p_4 + |x|^{2\alpha} p_1 p_4 |1 - x|^{2\alpha} p_3 p_4 \right) \]

\[ = \delta(\sum p_i) \left( B(-\alpha^t - 1, -\alpha^t t - 1) + B(-\alpha^t s - 1, -\alpha^t u - 1) + B(-\alpha^t t - 1, -\alpha^t u - 1) \right) \]

where \( B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \)  

(5.20)

This amplitude is the Veneziano amplitude.

Now comes to the 1-loop amplitude in string theory. Here we will only talk about the closed string case on torus.

6 Toroidal Compactification

Kaluza-Klein unification: consider a \( D = d + 1 \) dimensional theory with \( x^d \) periodic: \( x^d \sim x^d + 2\pi R \), and \( x^\mu \) noncompact for \( \mu = 0,...,d-1 \).

Parameterize the D dimensional metric:

\[ ds^2 = G_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + G_{dd}(dx^d + A_\mu dx^\mu)^2 \]

(6.1)

Assume \( G_{\mu\nu}, A_\mu \) and \( G_{dd} \) only depend on those noncompact coordinates. Then

\[ x'^d = x^d + \Lambda(x^\mu) \]

(6.2)

\[ A'_\mu = A_\mu - \partial_\mu \Lambda(x^\mu) \]

(6.3)
leaves the metric invariant. This shows that the $U(1)$ gauge can be viewed as comes from the compact space translation.

We can also consider the antisymmetric $B$ field and dilaton $\Phi$. parameterize $G_{dd} = e^{2\sigma}$, $\Phi_d = \Phi - \sigma/2$, and integrate out the compact dimension

$$S = \frac{1}{2\kappa_0^2} \int d^D X \sqrt{-G} e^{-2\Phi} \left( R_D - \frac{1}{12} H^{MNL} H_{MNL} + 4 \partial^\mu \Phi \partial_\mu \Phi \right)$$

$$= \frac{\pi R}{\kappa_0^2} \int d^d X \sqrt{-G} e^{-2\Phi_d} \left( R_d - \partial^\mu \sigma \partial_\mu \sigma + 4 \partial^\mu \Phi_d \partial_\mu \Phi_d - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F_{\mu\nu} - \frac{1}{12} \tilde{H}^{\mu\nu\lambda} \tilde{H}_{\mu\nu\lambda} + \frac{1}{4} e^{-2\sigma} H^{\mu\nu}_{d} H_{d\mu\nu} \right)$$

Where

$$\tilde{H}^{\mu\nu\lambda} = (\partial_\mu B_{\nu\lambda} - A_{\mu} H_{d\nu\lambda}) + \text{cyclic permutations}$$

$B_{MN}$ split into $B_{\mu\nu}$ and the second gauge field $B_{d\mu}$. The original $D$ dimensional anti-symmetric tensor gauge also split into the $d$ dimensional gauge and a $U(1)$ gauge. The $d$ dimensional gauge are fixed by

$$B'_{\nu\lambda} = B_{\nu\lambda} - \Lambda H_{d\nu\lambda}$$

Note that usually in field theory there are no fields charged under this gauge field unlike the Kaluza-Klein gauge field. But some states can be charged under it in string theory.

Now let’s see the effect of toroidal compactification on closed string theory.

For simplicity consider the case of a single periodic spacetime dimension, which means $X^{25} \sim X^{25} + 2\pi R$, now the spacetime indices $\mu, \nu$ takes from 0 to 24.

The effects include: First, the momentum in the compact dimension becomes discrete. This comes from the argue that $\exp(2\pi i R p^{25})$ translate a state once around the compact dimension should leave that state invariant, so we have:

$$p^{25} = \frac{n}{R}, \ n \in \mathbb{Z}$$

Second, there can be some novel states due to the compactification. A closed string may wind around the compact dimension, which means

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi R w, \ w \in \mathbb{Z}$$

The integer $w$ is called the winding number which indicates how many times the closed string surrounds the compact dimension. Strings with nonzero winding number are topological solitons.

The above result together with the mode expansion Eq (2.9), gives the result:

$$2\pi \alpha' p^{25} = X^{25}(\tau + 2\pi) - X^{25}(\tau) = 2\pi \sqrt{\frac{\alpha'}{2}}(\alpha_0 + \tilde{\alpha}_0)$$

$$2\pi R w = X^{25}(\sigma + 2\pi) - X^{25}(\sigma) = 2\pi \sqrt{\frac{\alpha'}{2}}(\alpha_0 - \tilde{\alpha}_0)$$
So we distinguish left hand and right hand momentum:

\[
p_L^{25} = \sqrt{\frac{2}{\alpha'} \alpha_0^{25}} = \frac{n}{R} + \frac{wR}{\alpha'} \tag{6.13}
\]
\[
p_R^{25} = \sqrt{\frac{2}{\alpha'} \tilde{\alpha}_0^{25}} = \frac{n}{R} - \frac{wR}{\alpha'} \tag{6.14}
\]

Then the mass tower becomes:

\[
M^2 = -\sum_{\mu=0}^{24} p_\mu p_\mu = (p_L^{25})^2 + \frac{4}{\alpha'} (\tilde{N} - 1) = (p_R^{25})^2 + \frac{4}{\alpha'} (N - 1) \tag{6.16}
\]

which is equivalent to:

\[
M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'} (N + \tilde{N} - 2) \tag{6.17}
\]

together with a new level matching requirement:

\[
N - \tilde{N} + n w = 0 \tag{6.18}
\]

We express the string states with new quantum numbers as \(|N; p^\mu; n; w\rangle\).

Now consider the first states:

- \(|0; p; 0; 0\rangle\) the tachyon is not changed.
- \(\alpha_{\mu}^{25} \tilde{\alpha}_{\nu}^{25} |0; p; 0; 0\rangle\) states of \(G_{\mu\nu}, B_{\mu\nu}\) and \(\Phi\) under \(SO(1, 24)\) group.
- \((\alpha_\mu^{25} \tilde{\alpha}_\nu^{25} + \alpha_\nu^{25} \tilde{\alpha}_\mu^{25}) |0; p; 0; 0\rangle\) and \((\alpha_\mu^{25} \tilde{\alpha}_\nu^{25} - \alpha_\nu^{25} \tilde{\alpha}_\mu^{25}) |0; p; 0; 0\rangle\) they are the Kaluza-Klein gauge vector \(G_{25\mu}\) and antisymmetric gauge vector \(B_{25\mu}\).
- \(\alpha_{\mu}^{25} \tilde{\alpha}_{\nu}^{25} |0; p; 0; 0\rangle\) the scalar \(\sigma\), the modulus for the radius of the compact dimension.

Note that the \(p^{25}\) is the conserved current of \(X^{25}\) translation or Kaluza-Klein gauge transformation, while \(w\) is the conserved current of the antisymmetric gauge transformation. (see [2] page 241)

The above states has \(n = 0\) and \(w = 0\), so naturally the \(i p^{25} X_{25}\) drops off and the vertex operators takes the form, for example

\[
(\alpha_\mu^{25} \tilde{\alpha}_{\nu}^{25} + \alpha_\nu^{25} \tilde{\alpha}_\mu^{25}) |0; p; 0; 0\rangle \sim: \int d^2 \sigma \left( \partial X^\mu \bar{\partial} X^{25} + \partial X^{25} \bar{\partial} X^\mu \right) e^{ip \cdot X} : \tag{6.19}
\]

Other states included are not massless so are suppressed, but there are a special radius \(R = \sqrt{\alpha'}\), which makes other states massless possible. They are

\[
\tilde{\alpha}^M |0; p; \pm 1; \pm 1\rangle \quad \alpha^M |0; p; \pm 1; \mp 1\rangle \\
|0; p; 0; \pm 2\rangle \quad |0; p; \pm 2; 0\rangle
\]

\(M : (\mu, 25)\) takes value from 0 to 25.

This states takes the \(i p^{25} X_{25} = i(p_L^{25} X_L^{25} + p_R^{25} X_R^{25})\) in the vertex operator, for example

\[
\tilde{\alpha}^\mu |0; p; 1; 1\rangle \sim: \bar{\partial} X^\mu e^{\frac{24}{\alpha'} X_L^{25}} e^{ip \cdot X} : \tag{6.20}
\]
Note that $\tilde{\omega}_\mu |0; p; \pm 1; \pm 1\rangle$ and $\tilde{\omega}_\mu |0; p; \pm 1; \mp 1\rangle$ are gauge vectors. They have internal momentum and winding number, so they are charged under Kaluza-Klein vector and antisymmetric vector. The only possible way for massless fields to be charged is that they form a bigger non-Abelian gauge theory. Here the modulus the enhanced gauge symmetry is $SU(2) \times SU(2)$ which can be characterized by the current algebra. (see [2] page 243)

$$j^1 = \cos \left( \frac{2}{\sqrt{\alpha'}} X_{25}^L \right) : j^2 = \sin \left( \frac{2}{\sqrt{\alpha'}} X_{25}^L \right) : j^3 = \frac{i}{\sqrt{\alpha'}} \partial X_{25}^L$$

(6.21)

$\tilde{j}^i$ has a similar definition. The vertex operators of the $3 \times 3$ scalars: $\alpha_{25}^L \alpha_{25}^R |0; p; 0; 0\rangle$, $\tilde{\alpha}_{25}^L |0; p; \pm 1; \pm 1\rangle$, $\tilde{\alpha}_{25}^R |0; p; \pm 1; \mp 1\rangle$, $|0; p; 0; \pm 2\rangle$, $|0; p; \pm 2; 0\rangle$ can be reconstructed as

$$j^i \tilde{j}^j e^{ip \cdot X}.$$ 

(6.22)

This reminds us a spontaneous symmetry breaking $SU(2) \times SU(2) \rightarrow U(1) \times U(1)

$$\mathcal{L} = -\frac{1}{4} Tr[F_{\mu\nu}F_{\mu\nu}] - \frac{1}{2} Tr[D_{\mu}\phi D_{\mu}\phi] + \kappa Det[\phi]$$

(6.23)

$$= -\frac{1}{4} Tr[F_{\mu\nu}F_{\mu\nu}] - \frac{1}{2} (D_{\mu}\phi)_{j} (D_{\mu}\phi)_{j} + \kappa \epsilon^{ijk} \epsilon^{mnl} \phi_{im} \phi_{jn} \phi_{kl}$$

(6.24)

where $\phi_{ij}$ the 9 scalars corresponding to the spacetime field. The explicit form can be checked by the amplitude of the worldsheet path integral. The classical stationary point conditions $\frac{\partial Det[\phi]}{\partial s} = 0$ after diagonalized by $SU(2) \times SU(2)$ transformation is $\phi_{11}\phi_{22}\phi_{33} = \phi_{11}\phi_{22} = \phi_{22}\phi_{33} = \phi_{11}\phi_{33} = 0$, so one of the diagonal components can be nonzero. Equivalently, the modulus $\sigma$ get a nonzero vacuum expectation, and $\sigma$ correspond to the changing the compact radius, shows that when $R$ moves away from the $SU(2) \times SU(2)$ radius, only 1 scalar is still massless while others acquire mass from this Higgs mechanism. For example, from Eq (6.17), extra gauge bosons acquire $M = \frac{|R^2 - \alpha'|}{\alpha'} \approx \frac{2}{\alpha'} |R - \sqrt{\alpha'}|$.

A very interesting result in toroidal compactification is an equivalence of different radius. It is obvious that the mass spectrum Eq (6.17) is invariant under:

$$R \rightarrow \frac{\alpha'}{R}, \quad n \leftrightarrow w$$

(6.25)

Notice that reversing $n$ and $w$ is equivalent to

$$p_{25}^L \rightarrow p_{25}^L, \quad p_{25}^R \rightarrow -p_{25}^R$$

(6.26)

Actually this equivalence extend to the CFT. If we do the transformation as

$$X^{25}(z, \bar{z}) = X_{25}^L(z) - X_{25}^R(\bar{z})$$

(6.27)

Then the new field $X^{25}$ has almost the same OPEs except some minus sign.
Above shows that different radius of the compact dimension can be the same theory, and this $\mathbb{Z}_2$ symmetry is called the T-duality.

There is a very elegant way to show the T-duality by path integral.

Reparameter the embedding by $X^{25}(\sigma) = R\phi(\sigma)$, and $\phi$ has periodicity $2\pi$. Then this component in the Polyakov action

$$S = \frac{R^2}{4\pi\alpha'} \int d^2\sigma \partial^a \phi \partial_a \phi$$

(6.28)

There is a symmetry $\phi \rightarrow \phi + \lambda$. We can make it local by introducing $A_a$ with transformation $A_a \rightarrow A_a - \partial_a \lambda$, and we replace $\partial_a \phi \rightarrow D_a \phi = \partial_a \phi + A_a$. Then we have to add a Lagrange multiplier.

$$S = \frac{R^2}{4\pi\alpha'} \int d^2\sigma D^a \phi D_a \phi + \frac{i}{2\pi} \int d^2\sigma \theta e^{ab} \partial_a A_b$$

(6.29)

then consider the correlation function

$$Z = \int \mathcal{D}\theta \mathcal{D}A \mathcal{D}\phi \exp\left\{ -\frac{R^2}{4\pi\alpha'} \int d^2\sigma D^a \phi D_a \phi - \frac{i}{2\pi} \int d^2\sigma \theta e^{ab} \partial_a A_b \right\}$$

(6.30)

$$= \int \mathcal{D}\theta \mathcal{D}A \exp\left\{ -\frac{R^2}{4\pi\alpha'} \int d^2\sigma A^a A_a + \frac{i}{2\pi} \int d^2\sigma \partial_a \theta e^{ab} A_b \right\}$$

(6.31)

$$= \int \mathcal{D}\theta \exp\left\{ -\frac{\alpha'}{4\pi R^2} \int d^2\sigma \partial^a \theta \partial_a \theta \right\}$$

(6.32)

Here the first equality comes from the gauge fixing $\phi = 0$, and the second equality is just a Gaussian integral.

Note that if the worldsheet is $\mathbb{R}^2$, then the above action is equivalent to Eq (6.28). If the worldsheet is topologically nontrivial, we need $\theta \sim \theta + 2\pi$. This result can be easily understand by the EoMs of $\phi$ and $A$.

Now let’s see what happens in open string theory.

Naively thinking, in open string theory there is no such thing like winding number. But now the compact dimension momentum becomes a little different. Consider one end point moving in a constant background field $A_{25} = -\frac{\theta}{2\pi R}$.

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial^\mu X^\nu \partial_\nu X_\mu + i \int d\tau A_\mu \partial_\tau X^\mu$$

(6.33)

Then the canonical momentum takes the form

$$p^{25} = i\partial_\tau X^{25} + A^{25}$$

(6.34)

and this momentum must satisfy $p_{25} = n/R$ for integer $n$, so

$$i\partial_\tau X^{25} = \frac{n}{R} + \frac{\theta}{2\pi R}, \quad n \in \mathbb{Z}$$

(6.35)
If we consider the $U(n)$ Chan-Paton factors at open string endpoints. $A_{25}$ are diagonalized as $A_{25}^{25} = -\frac{1}{2\pi R} diag(\theta_1, \ldots, \theta_n)$. Then state $|ij\rangle$ are charged $+1$ under $A_{ii}^{25}$ and $-1$ under $A_{jj}^{25}$, thus

$$i\partial_\tau X^{25} = \frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R}, \quad n \in \mathbb{Z} \quad (6.36)$$

the open string mass spectrum

$$M^2 = -\sum_{\mu=0}^{24} p^\mu p_\mu = (p^{25})^2 + \frac{1}{\alpha'}(N - 1) = \left(\frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N - 1) \quad (6.37)$$

Note that, for example, the gauge bosons with $l = 0$ and $N = 1$ acquire mass

$$M = \frac{|\theta_i - \theta_j|}{2\pi R} \quad (6.38)$$

So for general $\theta_i$ the symmetry group spontaneously from $U(n)$ to $U(1)^n$. The gauge field in the compact dimension are now scalar and are the usual so called Higgs field whose vacuum expectation gives gauge fields mass.

The T-duality in open string theory is a surprise this can answer a question we meet before: where is the Dirichlet boundary conditions comes from.

As in closed string theory, the embedding

$$X'^{25}(z, \bar{z}) = X_1^{25}(z) - X_2^{25}(\bar{z}) \quad (6.39)$$

then it is obvious that

$$i\partial_\sigma X^{25} = \partial_\tau X'^{25} \quad (6.40)$$

which shows that a Neumann boundary convert to a Dirichlet boundary.

Another surprise is that actually different endpoints of open strings share the same hyperplane. If there is no background $A_{25}$

$$X'^{25}(\pi) - X'^{25}(0) = \int_0^\pi d\sigma \, \partial_\sigma X^{25} = -i \int_0^\pi d\sigma \, \partial_\tau X'^{25} = -2\pi \alpha' \frac{n}{R} = -2\pi n R' \quad (6.41)$$

Since $X'^{25} \sim X^{25} + 2\pi R'$, we found that two ends lives on the same plane. Now in the T-dual case, $n$ converts from the number of wave period to the winding number.

A saying goes that all the end points of different open string still lie on the same plane. The reason is that there should be graviton exchange between any two open strings and so there must be a path connecting any two endpoints.

Then we consider the effect of $U(n)$ gauge field $A^{25}$.

$$X'^{25}(\pi) - X'^{25}(0) = -(2\pi n + \theta_i - \theta_j) R' \quad (6.42)$$

This can be interpreted as there are generally $n$ planes, each has a place

$$X^{25} = \theta_i R' = -2\pi \alpha' A_{ii}^{25} \quad (6.43)$$
The mode expansion of state $|ij\rangle$ becomes
\begin{equation}
X^{25} = \theta_i R' - i \frac{R'}{2\pi}(2\pi n + \theta_i - \theta_j)\log(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left( \frac{\alpha_n^{25}}{z^n} - \frac{\hat{\alpha}_n^{25}}{\bar{z}^n} \right) \tag{6.44}
\end{equation}

As we found above, there are boundary hypersurfaces and open strings lives on them. These hypersurfaces are called Dirichlet membrane, or D-brane for short. If a hyperplane is p+1 dimensional, it is called a Dp-brane.

When there is no such things as torodial compactification and Dirichlet boundary condition, the $26D$ spacetime itself is a D25-brane. When there are periodic compact dimension, we make use of T-duality and get a lower dimensional D-brane. The string living on it has the action
\begin{equation}
S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial^\mu X^A \partial_\mu X_A + i \int d\tau A_\mu \partial_\tau X^\mu \tag{6.45}
\end{equation}
where $A = 0, ..., 25$ and $\mu = 0, ..., 25 - k$.

Just what we mentioned in Chapter 4, the perturbation on background field $A^\mu$ is corresponding to the coherent state of a photon lives on the D-brane. And the requirement of conformal invariance gives the beta function which is proportional to the Maxwell Equation at leading order.

A D-brane itself is a dynamical object. For the $U(1)$ case the low energy action of a D-brane takes the form
\begin{equation}
S = -T_p \int d^{p+1}\xi \ e^{-\Phi} \sqrt{-\det (G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} \tag{6.46}
\end{equation}
where $\xi^a$, $a = 0, ..., p$ the coordinates on the D-brane, and $X^\mu(\xi)$, $\mu = 0, ..., 25$ the embedding.

\begin{equation}
G_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}, \quad B_{ab} = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu} \tag{6.47}
\end{equation}

This low energy action is understood from general resoning. Similar to the worldline and worldsheet, first consider just the metric and the embedding, the action is the worldvolume, integral over $\sqrt{-\det G_{\mu\nu}}$. The dilaton dependence comes from the $e^{-\Phi_0} \sim g_{C}^{-1}$, which corresponding to the topological perturbation. *****General 2D Action for open string? geodesic curvature?*****

The $F_{ab}$ dependence can be understood by using T-duality. For example, suppose there is a D-brane extended in $X^1$ and $X^2$ directions with a constant $F_{12}$ on it, and we can choose $A_2 = X^1 F_{12}$, then after taking the T-dual along 2-direction, $X'^2 = -2\pi\alpha' X^1 F_{12}$, so the D-brane tilts under a constant field strength. The tilt gives
\begin{equation}
\sqrt{1 + (2\pi\alpha' F_{12})^2} \tag{6.48}
\end{equation}
This is equivalent to the determinant in the Born-Infeld action.
Finally, the dependence of $B_{ab}$ is given by the reason that only the combination of $B_{ab}$ and $A_a$ is invariant under gauge transformation. Consider the open string action

$$i \frac{2\pi}{\alpha'} \int_M B + i \int_{\partial M} A$$

(6.49)

with gauge transformation

$$\delta B = d\Lambda, \quad \delta A = -\frac{1}{2\pi \alpha'} \Lambda$$

(6.50)

So the gauge invariant 2-form is

$$B + 2\pi \alpha' dA$$

(6.51)

Although there are some problems in finding a non-Abelian generalization of the DBI action, we know it at $\alpha'$ leading order.

$$S = -\left(2\pi \alpha'\right)^2 T_p \int d^{p+1}\xi \; Tr \left( \frac{1}{4} F^{ab} F_{ab} + \frac{1}{2} D^a \phi^M D_a \phi^M - \frac{1}{4} \sum_{M \neq N} \left[ \phi^M, \phi^N \right]^2 \right)$$

(6.52)

where $\phi^M = \frac{1}{2\pi \alpha'} X^M = -A^M$, $M = p+1, \ldots, 25$. $D_a \phi^M = \partial_a \phi^M + i \left[ A_a, \phi^M \right]$.

The last term means that at low energy these scalars are commute to each other, and so they are simultaneously diagonalizable. And when different D-branes have different places, for example the $U(2)$ case and two 24D branes.

$$A_a = \begin{pmatrix} A_{a11} & A_{a12} \\ A_{a21} & A_{a22} \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ 0 \\ 0 \phi_2 \end{pmatrix}$$

(6.53)

the term

$$\frac{1}{2} Tr \left[ A_a, \phi \right]^2 = - (\phi_1 - \phi_2)^2 |A_{a12}|^2$$

(6.54)

gives the mass

$$M = \frac{|X_1 - X_2|}{2\pi \alpha'}$$

(6.55)

The same as Eq (6.38).

7 The Ramond-Neveu-Schwarz String

The bosonic string has a few problems that we cannot consider it as a consistent theory, for example the existence of techyon. Another problem is that in order to construct the particle physics of this world, at least we need some fermions.

The Ramond-Neveu-Schwarz string has the simple idea that we can introduce some fermions $\psi^\mu$ on the worldsheet and the corresponding SUSY between $X^\mu$ and $\psi^\mu$ has $\mathcal{N} = 1$. As we done before, we can find out the only consistent spacetime dimension is $D = 10$, and after some operation, the RNS string actually gives a theory that has spacetime bosons and
fermions and they also enjoy the spacetime SUSY. What’s more is that there won’t be any techyons.

\[ S = -\frac{1}{2\pi} \int d^2\sigma \left( \partial^\alpha X^\mu \partial_\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right) \]  

(7.1)

Here we have: \( X^\mu \) as worldsheet scalar with \( c = 1 \) and spacetime vector, \( \psi^\mu \) as worldsheet fermion with \( c = 1/2 \) and spacetime vector.

The degrees of freedom of 2D fermions: \( 2^{[2/2]} = 2 \) complex for Dirac fermion, 2 real for Majorana fermion, 1 real for Majorana-Weyl fermion.

Here \( \psi^\mu \) has two real components

\[ \psi^\mu = \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix} \]  

(7.2)

\[ \bar{\psi} = \psi^\dagger \beta, \ \beta = i\rho^0 \]  

(7.3)

The 2D Dirac matrices:

\[ \{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta} \]  

(7.4)

\[ \rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

(7.5)

In light-cone coordinate, the action is

\[ S = \frac{1}{\pi} \int d^2\sigma \left( 2\partial_+ X \cdot \partial_- X + i\psi_- \cdot \partial_+ \psi_- + i\psi_+ \cdot \partial_- \psi_+ \right) \]  

(7.6)

The variations of the fermions gives:

\[ \delta S \sim \int d\tau \left( \psi_+ \delta \psi_+ - \psi_- \delta \psi_- \right) |_{\sigma=\pi} - \left( \psi_+ \delta \psi_+ - \psi_- \delta \psi_- \right) |_{\sigma=0} \]  

(7.7)

As a result, for open string there are 2 types of boundary conditions:

Ramond boundary conditions:

\[ \psi_+ |_{\sigma=0} = \psi_- |_{\sigma=0}, \ \psi_+ |_{\sigma=\pi} = \psi_- |_{\sigma=\pi} \]  

(7.8)

then the mode expansion:

\[ \psi_-^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in(\tau-\sigma)} \]  

(7.9)

\[ \psi_+^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-in(\tau+\sigma)} \]  

(7.10)

\[ \{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu} \delta_{m+n,0} \]  

(7.11)
Nuveu-Schwarz boundary conditions:

\[ \psi_+|_{\sigma=0} = \psi_-|_{\sigma=0}, \quad \psi_+|_{\sigma=\pi} = -\psi_-|_{\sigma=\pi} \quad (7.12) \]

then the mode expansion:

\[ \psi_-^\mu = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z}+1/2} b_r^\mu e^{-ir(\tau-\sigma)} \quad (7.13) \]

\[ \psi_+^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}+1/2} b_n^\mu e^{-ir(\tau+\sigma)} \quad (7.14) \]

\[ \{b_r^\mu, b_n^\nu\} = \eta^{\mu\nu} \delta_{r+n,0} \quad (7.15) \]

For closed string there are two sectors at the same time,

\[ \psi^\pm (\sigma) = \psi^\pm (\sigma + \pi) \quad \text{or} \quad \psi^\pm (\sigma) = -\psi^\pm (\sigma + \pi) \quad (7.16) \]

thus

\[ \psi_-^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \quad \text{or} \quad \psi_-^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}+1/2} b_n^\mu e^{-2ir(\tau-\sigma)} \quad (7.17) \]

\[ \psi_+^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \quad \text{or} \quad \psi_+^\mu = \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}+1/2} \tilde{b}_n^\mu e^{-2ir(\tau+\sigma)} \quad (7.18) \]

So there are 4 choice for closed RNS string: NS-NS, R-R, NS-R, R-NS.

It should be mentioned that actually this is just the result as we set the worldsheet to be flat (both zweibein \(e\) and its susy partner the Rarita-Schwinger gauge field \(\chi\)), and as a result we only has global SUSY:

\[ \delta X^\mu = e\psi^\mu, \quad (7.19) \]

\[ \delta \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon \quad (7.20) \]

In bosonic string the underlying \(e\) gives the Virasoro constraints \(T_{++} = T_{--} = 0\),

\[ T_{++} = \partial_+ X_\mu \partial_+ X^\mu + \frac{i}{2} \psi_+^\mu \partial_+ \psi_{+\mu} \quad (7.21) \]

\[ T_{--} = \partial_- X_\mu \partial_- X^\mu + \frac{i}{2} \psi_-^\mu \partial_- \psi_{-\mu} \quad (7.22) \]

but here we have extra constraints \(J_+ = J_- = 0\):

\[ J_+ = \psi_+^\mu \partial_+ X_\mu \quad (7.23) \]

\[ J_- = \psi_-^\mu \partial_- X_\mu \quad (7.24) \]

\(J^\mu\) is the conserved current of supersymmetry:

\[ \partial_- J_+ = \partial_+ J_- = 0 \quad (7.25) \]
just as
\[ \partial_- T_{++} = \partial_+ T_{--} = 0 \]  
(7.26)

T and J can be written in modes:

\[ L_m = \frac{1}{\pi} \int_{-\pi}^{\pi} d\sigma e^{i m \sigma} T_{++} \]  
(7.27)

\[ L_m^X = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_{m+n} : \]  
(7.28)

For NS sector

\[ L_m^{\psi} = \frac{1}{2} \sum_{r \in \mathbb{Z} + 1/2} \left( r + \frac{m}{2} \right) : b_{-r} \cdot b_{m+r} : \]  
(7.29)

\[ G_r = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma e^{i r \sigma} J_+ = \sum_{r \in \mathbb{Z}} \alpha_{-n} \cdot b_{r+n} : \]  
(7.30)

For R sector

\[ L_m^{\psi} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left( n + \frac{m}{2} \right) : d_{-n} \cdot d_{m+n} : \]  
(7.31)

\[ F_m = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma e^{i m \sigma} J_+ = \sum_{r \in \mathbb{Z}} \alpha_{-n} \cdot b_{m+n} : \]  
(7.32)

Note that the integral here is from \(-\pi\) to \(\pi\) should be understand as:
e.g. \(T_{++}(\tau + \sigma) = T_{--}(\tau - \sigma)\) if \(-\pi \leq \sigma \leq 0.\)

The above super-virasoro generators satisfy the algebra:

\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{D}{8} m (m^2 - 1) \delta_{m+n,0} \]  
(7.33)

\[ [L_m, G_r] = \left( \frac{m}{2} - r \right) G_{m+r} \]  
(7.34)

\[ \{G_r, G_s\} = 2 L_{r+s} + \frac{D}{2} (\delta^2 - \frac{1}{4}) \delta_{m+n,0} \]  
(7.35)

\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{D}{8} m^3 \delta_{m+n,0} \]  
(7.37)

\[ [L_m, F_n] = \left( \frac{m}{2} - n \right) F_{m+n} \]  
(7.38)

\[ \{F_m, F_n\} = 2 L_{m+n} + \frac{D}{2} m^2 \delta_{m+n,0} \]  
(7.39)
We should have used the BRST quantization for a covariate quantization, but the process is completely similar to before.

There will be $\beta \gamma$ ghosts as the SUSY partner of $bc$ ghosts. They should be bosonic fields with conformal dimension $3/2$ and $-1/2$.

\[ \gamma(z)\beta(w) = \frac{1}{z-w} + \ldots, \beta(z)\gamma(w) = -\frac{1}{z-w} + \ldots \quad (7.41) \]

All these ghosts together gives $c_g = -15$, thus $D = 10$ for RNS string.

BRST charge $Q$ gives the constraints of physical states:

\[ G_r |\phi\rangle = 0, \ r > 0 \quad (7.42) \]
\[ (L_m - a_{NS} \delta_{m,0}) |\phi\rangle = 0, \ m \geq 0 \quad (7.43) \]

The critical string has $a_{NS} = 1/2$.

\[ F_n |\phi\rangle = 0, \ n \geq 0 \quad (7.44) \]
\[ (L_m - a_{R} \delta_{m,0}) |\phi\rangle = 0, \ m \geq 0 \quad (7.45) \]

The critical string has $a_{R} = 0$. As we will see, R-sector describe spacetime fermions, the $F_0$ constrain gives the Dirac-Ramond equation of that fermion.

\[ \left( p \cdot \Gamma + 2\sqrt{2} \sum_{n=1}^{\infty} (\alpha_{-n} \cdot d_n + d_{-n} \cdot \alpha_n) \right) |\phi\rangle = 0 \quad (7.46) \]

The $L_0$ requirement gives the mass shell condition of spacetime particles. The story is that due to $\zeta$ function regularization, each periodic boson gives $-1/24$ to the zero-point energy, each periodic fermion gives $1/24$, each antiperiodic boson gives $1/48$, and each antiperiodic fermion gives $-1/48$.

As what we have done for bosonic string, the reparameter invariance and residual SUSY allow us to choose the light-cone gauge

\[ X^+ = x^+ + p^+ \tau \quad (7.47) \]
\[ \psi^+ = 0 \quad (7.48) \]

Easily we can find for open string:

\[ \frac{1}{2} M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{r=1/2}^{\infty} rb_{-r}^i b_r^i - \frac{1}{2} \quad (7.49) \]
\[ \frac{1}{2} M^2 = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n=1}^{\infty} nd_{-n}^i d_n^i \quad (7.50) \]

If we look at the mass spectrum carefully we find that for R-sector, $d_{0}^i$ do not contribute to the energy, which means that not as the case in bosonic string, the ground state of R-sector is degenerate.
\[ \{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu} \]  

is some sort of the Dirac algebra

\[ \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \]  

(7.52)

So this states form the \( D = 10 \) Majorana fermions with \( 2^{10/2} = 32 \) real components. What should be mentioned is that the Dirac-Ramond equation act on the ground state is the massless Dirac equation, which eliminate half of the degrees of freedom (from \( \text{Spin}(1, 9) \) to \( \text{Spin}(8) \)).

There are still some problem, as there are still techyon in NS sector and the spacetime boson and fermion do not form a SUSY multiplet. So we need another operation which seems quite ad hoc, named GSO projection.

We define the so called G-parity:

For NS sector,

\[ G = (-)^{\sum_{i=1}^{\infty} b^i_r b^i_{r+1}} \]  

(7.53)

we require \( G |\phi\rangle_{NS} = |\phi\rangle_{NS} \).

For R sector,

\[ G = \Gamma_{11}(-)^{\sum_{n=1}^{\infty} d^i_n d^i_{n+1}} \]  

(7.54)

we require \( G |\phi\rangle_{R} = \pm |\phi\rangle_{R} \), there is a choice but when there is only one R sector this is purely a convention.

Basically this GSO progection means that there must be odd number of \( b^i_r \) excitations and on leave half of the R sector ground state. Then there won’t be a techyon. At each level the number of spacetime boson and fermion is the same now after GSO projection, which hint that they should be in a SUSY representation, and we will see this in GS formulism.

Now let’s consider the massless modes of closed RNS string.

typically IIA

\[ |-\rangle_{R} \otimes |+\rangle_{R} \]  

(7.55)

\[ \tilde{b}^i_{-1/2} |0\rangle_{NS} \otimes \tilde{b}^j_{1/2} |0\rangle_{NS} \]  

(7.56)

\[ \tilde{b}^i_{1/2} |0\rangle_{NS} \otimes |+\rangle_{R} \]  

(7.57)

\[ |-\rangle_{R} \otimes \tilde{b}^j_{-1/2} |0\rangle_{NS} \]  

(7.58)

typically IIB

\[ |+\rangle_{R} \otimes |+\rangle_{R} \]  

(7.59)

\[ \tilde{b}^i_{-1/2} |0\rangle_{NS} \otimes \tilde{b}^j_{-1/2} |0\rangle_{NS} \]  

(7.60)

\[ \tilde{b}^i_{1/2} |0\rangle_{NS} \otimes |+\rangle_{R} \]  

(7.61)

\[ |+\rangle_{R} \otimes \tilde{b}^j_{1/2} |0\rangle_{NS} \]  

(7.62)

They both form a \( \mathcal{N} = 2 \) supergravity multiplet.
NS-NS sector contain graviton(56), antisymmetric tensor(28) and dilaton(1).
NS-R and R-NS sectors both contain spin 3/2 gravitino(56) and spin 1/2 dilatino(8).
The type IIA R-R sector contain 1-form(8) and 3-form(56). The type IIB R-R sector contain 0-form(1), 2-form(28) and self-dual 4-form(35).

\[
\psi_a \chi_b = \sum_{n=0, \text{even}}^8 \gamma^{i_1,...,i_n}_{ab} A_{i_1,...,i_n} \text{(7.63)}
\]
\[
\psi_a \chi_{\dot{a}} = \sum_{n=0, \text{odd}}^{8} \gamma^{i_1,...,i_n}_{a\dot{a}} A_{i_1,...,i_n} \text{(7.64)}
\]

More about \textit{Spin}(8) representation \textit{8}_v, \textit{8}_s and \textit{8}_c will be in next section.

8 The Green-Schwarz String

We have showed the RNS string, which is in some sense quite easy to understand and has an elegant formulism. However, there is a problem that we cannot easily find out whether or not the spacetime partical states of RNS string is under some supersymmetry representation. We also do not want any artificial operations: the GSO projection.

The Green-Schwarz formulism gives an equivalent way to show such a superstring theory. The difference is that we add some worldsheet scalars but spacetime spinors to the bosonic string, and it is supersymmetric under spacetime \textit{N} SUSY. From the cancelation of conformal anomaly we know that we need 16 extra worldsheet scalars when \(D = 10\).

We describe such fields as

\[
\Theta^A a(\tau, \sigma) \text{(8.1)}
\]

Here \(A = 1, ..., N\) is the label of SUSY copies, and \(a = 1, ..., 2^{10/2}\) is the label of the \(D = 10\) Majorana spinors.

We should start from the spacetime SUSY worldline action. Such a theory describe massive supersymmetric point particles in type II theory.

The global SUSY transformation is

\[
\delta \Theta^A a = \epsilon^A a \text{(8.2)}
\]
\[
\delta X^\mu = \bar{\epsilon}^A \Gamma^\mu \Theta^A \text{(8.3)}
\]

Obviously, We can construct a ‘covariant’ derivative:

\[
\Pi_0^\mu = \frac{\partial}{\partial \tau} X^\mu - \bar{\Theta}^A \Gamma^\mu \frac{\partial}{\partial \tau} \Theta^A \text{(8.4)}
\]

Then we can write down a supersymmetric action:

\[
S = -m \int d\tau \sqrt{-\Pi_0 \cdot \Pi_0} \text{(8.5)}
\]
It gives EoMs:

\[ P^\mu = \frac{m}{\sqrt{-\Pi_0 \cdot \Pi_0}} \Pi_0^\mu \]  
(8.6)

\[ \partial_\tau P^\mu = 0 \]  
(8.7)

\[ P \cdot \Gamma \partial_\tau \Theta = 0 \]  
(8.8)

The third one we want is

\[ (P \cdot \Gamma + m \Gamma_{11}) \partial_\tau \Theta = 0 \]  
(8.9)

which only constrain half of the components. Thus we should modified the action as

\[ S = -m \int d\tau \left( \sqrt{-\Pi_0 \cdot \Pi_0} + \bar{\Theta} \Gamma_{11} \partial_\tau \Theta \right) \]  
(8.10)

Such a action has a local \( \kappa \) symmetry.

\[ \delta \bar{\Theta} = \bar{\kappa} P_- , \quad \delta X^\mu = -\bar{\kappa} P_- \Gamma^\mu \Theta \]  
(8.11)

Here \( \kappa(\tau) \) is an arbitrary Majorana spinor, and

\[ P_\pm = \frac{1}{2} \left( 1 \pm \frac{\Gamma \cdot \Pi_0}{\sqrt{-\Pi_0 \cdot \Pi_0}} \right) \]  
(8.12)

Such a symmetry is important for the reason that there are wrong number of on-shell propagating degrees of freedom. The \( \kappa \) symmetry shows that half of them are decoupled.

Now comes to the spacetime supersymmetric string. We want \( \mathcal{N} = 2 \) supersymmetry, so there are two fermionic \( \Theta^1 \) and \( \Theta^2 \). For type IIA theory they have opposite chirality, while for type IIB they have same chirality.

\[ S_1 = -\frac{1}{\pi} \int d^2\sigma \sqrt{-det (\Pi_\alpha \cdot \Pi_\beta)} \]  
(8.13)

\[ \Pi_\alpha^\mu = \partial_\alpha X^\mu - \bar{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A \]  
(8.14)

To make sure we have the \( \kappa \) symmetry, we need another term (see [4] section 5.2), the final action is:

\[ S_1 = -\frac{1}{2\pi} \int d^2\sigma \sqrt{-gg^{\alpha\beta} \Pi_\alpha \cdot \Pi_\beta} \]  
(8.15)

\[ S_2 = \frac{1}{\pi} \int d^2\sigma \epsilon^{\alpha\beta} \left[ \partial_\alpha X^\mu \left( \bar{\Theta}^1 \Gamma^\mu \partial_\beta \Theta^1 - \bar{\Theta}^2 \Gamma^\mu \partial_\beta \Theta^2 \right) - \Theta^1 \Gamma^\mu \partial_\alpha \Theta^1 \bar{\Theta}^2 \Gamma^\mu \partial_\beta \Theta^2 \right] \]  
(8.16)

\( \kappa \) symmetry:

\[ \delta \bar{\Theta}^1 = \bar{\kappa}^1 P_- , \quad \delta \bar{\Theta}^2 = \bar{\kappa}^2 P_+ , \quad \delta X^\mu = \bar{\Theta}^A \Gamma^\mu \delta \Theta^A \]  
(8.17)
The EoMs:

\[ \Pi_\alpha \cdot \Pi_\beta - \frac{1}{2} g_{\alpha\beta} h^{\gamma\delta} \Pi_\gamma \cdot \Pi_\delta = 0 \] (8.18)

\[ \partial_\alpha \left[ \sqrt{-g} \left( g^{\alpha\beta} \partial_\beta X^\mu - 2 P^\alpha_\beta \Theta^1 \Gamma_\mu \partial_\beta \Theta^1 - 2 P^\alpha_+ \Theta^2 \Gamma_\mu \partial_\beta \Theta^2 \right) \right] = 0 \] (8.19)

\[ \Gamma \cdot \Pi_\alpha P^\alpha_\beta \partial_\beta \Theta^1 = 0 \] (8.20)

\[ \Gamma \cdot \Pi_\alpha P^\alpha_+ \partial_\beta \Theta^2 = 0 \] (8.21)

\[ P^\alpha_\pm = \frac{1}{2} \left( h^{\alpha\beta} \pm \epsilon^{\alpha\beta} \right) \] (8.22)

It is obvious that all these EoMs are highly nonlinear, and it’s so hard to quantize the GS string in covariant way. The light-cone gauge can be achieved with \( \sigma^\pm \) reparameterization and \( \kappa \) symmetry

\[ X^+ = x^+ + p^+ \tau, \quad \Gamma^+ \Theta^A = 0 \] (8.23)

Note that here we gauged away half of the degrees of freedom of \( \Theta^A \), from 32 to 16, just as the requirement of Weyl anomaly.

The light-cone gauge correspond to \( SO(1, 9) \to SO(8) \). The covering group \( Spin(8) \) has 3 representation of dimension 8: \( 8_v, 8_s \) and \( 8_c \). We use the symbol \( S \) for the surviving components of \( \Theta \):

For type IIA theory: \( 8_v : S_1^a \) and \( 8_c : S_2^a \). For type IIB theory: \( 8_s : S_1^a \) and \( 8_s : S_2^a \)

Then the EoMs are simplified as:

\[ \partial_+ \partial_- X^i = 0, \quad \partial_+ S_1^a = 0, \quad \partial_- S_2^a = 0 \] (8.24)

Obviously we now have 8 bosonic and 8 fermionic propagating degrees of freedom, as the requirement of supersymmetry.

Now its quite easy to quantize such a theory:

Type I open string:

\[ S_1^a|_{\sigma=0} = S_2^a|_{\sigma=0}, \quad S_1^a|_{\sigma=\pi} = S_2^a|_{\sigma=\pi} \] (8.25)

Note that such a boundary condition require SUSY transformation \( \delta \Theta^A = \epsilon^A \), \( \epsilon^1 = \epsilon^2 \), so it only has \( \mathcal{N} = 1 \) SUSY.

\[ S_1^a = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} S_n^a e^{-in(\tau-\sigma)} \] (8.26)

\[ S_2^a = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} S_n^a e^{-in(\tau+\sigma)} \] (8.27)

\[ \{ S_m^a , S_n^b \} = \delta^{ab} \delta_{m+n,0} \] (8.28)
\[
\frac{1}{2} M^2 = \sum_{n=1}^{\infty} \left( \alpha_n^i \alpha_n^i + nS_n^a S_n^a \right)
\]  
(8.29)

Note that here \( a = 8 \times 1/24 + 8 \times (-1/24) = 0 \).

Such a theory has degenerate ground state with 8 bosonic states and 8 fermionic states:
\( 8_v : |i \rangle \) and \( 8_c : |\dot{a} \rangle \).

Type II closed string:
\[
S^{Aa} (\sigma, \tau) = S^{Aa} (\sigma + \pi, \tau)
\]  
(8.30)

\[
S^{1a} = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} S_n^a e^{-2i(n-\sigma)}
\]  
(8.31)

\[
S^{2a} = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \tilde{S}_n^a e^{-2i(n+\sigma)}
\]  
(8.32)

Type IIA:
\[
(8_v \oplus 8_s) \otimes (8_v \oplus 8_c)
\]  
(8.33)

\[
8_v \otimes 8_v = 1 \oplus 28 \oplus 35
\]  
(8.34)

\[
8_s \otimes 8_c = 8 \oplus 56
\]  
(8.35)

Type IIB:
\[
(8_v \oplus 8_s) \otimes (8_v \oplus 8_s)
\]  
(8.36)

\[
8_v \otimes 8_v = 1 \oplus 28 \oplus 35
\]  
(8.37)

\[
8_s \otimes 8_s = 1 \oplus 28 \oplus 35
\]  
(8.38)

All of this we have mentioned before.

9 The Heterotic String

We have find type IIA and type IIB theory with \( \mathcal{N} = 2 \) SUSY, and the type I theory we mentioned before require an orientifold projection ****** so only have \( \mathcal{N} = 1 \) SUSY. There is still another way to construct a \( \mathcal{N} = 1 \) SUSY theory, we call it the heterotic string. It provide an alternative way to construct superstring with \( SO(32) \) gauge group and it is also the only way to construct superstring with \( E_8 \times E_8 \) gauge symmetry.
The heterotic string comes from the idea that it is possible that the charge of Yang-Mills theory is carried only by the left movers. Here we only show the fermionic construction of the heterotic string, as it is equivalent to the bosonic construction of the heterotic string. Except from the $10 X^\mu$ fields, there are fermions in the left movers. It’s number 32 in order to cancel the $c_g = -26$ Weyl anomaly. The left mover is the normal RNS superstring(or GS string) with $D = 10$.

The $SO(32)$ heterotic string: $\lambda^A$ transform under the $SO(32)$ gauge group, and they are worldsheet fermion

$$S = \frac{1}{\pi} \int d^2 \sigma \left( 2 \partial_+ X^\mu \partial_- X_\mu + i \psi^\mu \partial_+ \psi_\mu + i \sum_{A=1}^{32} \lambda^A \partial_- \lambda^A \right) \quad (9.1)$$

The right movers is RNS string:

$$\frac{1}{8} M^2 = N \quad (9.2)$$

$$N_R = \sum_{n=1}^\infty \left( \alpha^i_{=-n} \alpha^i_n + n S^a_{-n} S^a_n \right) \quad (9.3)$$

The right movers:

Just like the $\psi^\mu$ in RNS string, $\lambda^A$ can be periodic or antiperiodic.

P sector

$$\lambda^A = \sum_{n \in \mathbb{Z}} \lambda^A_n e^{-2in(\tau + \sigma)} \quad (9.4)$$

$$\{ \lambda^A_m, \lambda^B_n \} = \delta^{AB} \delta_{m+n,0} \quad (9.5)$$

$$N_{L,P} = \sum_{n=1}^\infty (\tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + n \lambda^A_{-n} \lambda^A_n) \quad (9.6)$$

$$a_{L,P} = 8 \frac{1}{24} + 32 (-\frac{1}{24}) = -1 \quad (9.7)$$

A sector

$$\lambda^A = \sum_{r \in \mathbb{Z} + 1/2} \lambda^A_r e^{-2ir(\tau + \sigma)} \quad (9.8)$$

$$\{ \lambda^A_r, \lambda^B_s \} = \delta^{AB} \delta_{r+s,0} \quad (9.9)$$

$$N_{L,A} = \sum_{n=1}^\infty \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \sum_{r=1/2}^\infty r \lambda^A_r \lambda^A_r \quad (9.10)$$

$$a_{L,A} = 8 \frac{1}{24} + 32 \frac{1}{48} = 1 \quad (9.11)$$
The level matching condition requires
\[ N_R = N_L - a_L \]  
(9.12)

Now let’s consider the massless spectrum, obviously \( N_R = 0 \) and \( N_L = 1 \) in the A sector.

\[ (|i\rangle_R \oplus |a\rangle_R) \otimes (\tilde{\alpha}^{-1}_i |0\rangle_L \oplus \lambda^A_{1/2}\lambda^{-1/2}_{-1/2} |0\rangle_L) \]  
(9.13)

\(|i\rangle \otimes \tilde{\alpha}^{-1}_i |0\rangle_L\) is the graviton, antisymmetric tensor and dilaton. \(|a\rangle_R \otimes \tilde{\alpha}^{-1}_i |0\rangle_L\) is the gravitino and dilatino. These bosons and fermions form the \( \mathcal{N} = 1 \) supergravity multiplet.

Here we also need a GSO-like projection on the left mover:

\[ G = \bar{\lambda}_0 (\sum_{n=1}^{\infty} \lambda^A_n \lambda^A_n) \]  
(9.14)

Here \( \bar{\lambda}_0 = \lambda^1_0 \ldots \lambda^{32}_0 \) projects out half of the \( 2^{[32/2]} \) modes. We require \( G = 1 \) for physical states.

The \( E_8 \times E_8 \) heterotic string: \( \lambda^A \) transform under the \( SO(n) \times SO(32 - n) \) gauge group, and they are worldsheet fermion

\[ S = \frac{1}{\pi} \int d^2\sigma \left( 2 \partial_+ X^\mu \partial_- X_\mu + i \psi^\mu \partial_+ \psi_\mu + i \sum_{A=1}^{n} \lambda^A \partial_- \lambda^A + i \sum_{B=n+1}^{32} \lambda^B \partial_- \lambda^B \right) \]  
(9.15)

\[ a_{L,AP} = 8 \frac{1}{24} + n \frac{1}{48} + (32 - n)(-\frac{1}{24}) = \frac{n}{16} - 1 \]  
(9.16)

\[ a_{L,PA} = 8 \frac{1}{24} + (32 - n) \frac{1}{48} + n(-\frac{1}{24}) = 1 - \frac{n}{16} \]  
(9.17)

Due to level matching requirement they must be integer or half-integer, so \( n = 8, 16, 24 \).

The only one free of gauge anomaly is \( n = 16 \).

The massless modes:

AA sector

\[ \tilde{\alpha}^{-1}_i |0\rangle_L \oplus \lambda^A_{-1/2}\lambda^{-1/2}_{-1/2} |0\rangle_L \]  
(9.18)

Except for the \( \mathcal{N} = 1 \) supergravity multiplet, there are \( \mathcal{N} = 1 \) SYM multiplet:

\[ (120, 1), \ (16, 16), \ (1, 120) \]  
(9.19)

AP sector and PA sector

There are spacetime fermionic ground state, \( 2^{[16/2]} = 256 = 128 + 128' \).

\[ AP: (1, 128) \oplus (1, 128'), \ \ PA: (128, 1) \oplus (128', 1) \]  
(9.20)

The GSO-like projection of the \( E_8 \times E_8 \) heterotic string is

\[ G^{(1)} = \bar{\lambda}^{(1)}_0 (-) \sum_{n=1}^{\infty} \lambda^A_n \lambda^A_n \]  
(9.21)

\[ G^{(2)} = \bar{\lambda}^{(2)}_0 (-) \sum_{n=1}^{\infty} \lambda^B_n \lambda^B_n \]  
(9.22)
Here $\lambda_0^{(1)} = \lambda_0^1 \ldots \lambda_0^{16}$ and $\lambda_0^{(2)} = \lambda_0^{17} \ldots \lambda_0^{32}$. We require both $G^{(1)} = 1$ and $G^{(2)} = 1$.

After such projection what left in the massless modes are

$$((120, 1) \oplus (128, 1)) \oplus ((1, 120) \oplus (1, 128))$$

This is an irreducible representation of $E_8 \times E_8$, which hint that such a heterotic string theory gives $E_8 \times E_8$ gauge group.

References