HW

1. $Mf \le f^* \le 2^n Mf$. The left "$\le$" is trivial.

$$f^*(x) = \sup_{B \ni x} \frac{1}{|B|}\int_B |f(x)| \le \sup_{B\ni x} \frac{2^n}{|B(x, \mathrm{diam}(B))|}\int_{B(x,\mathrm{diam}(B))} |f(x)| \le 2^n Mf(x)$$

2. $f^*((a,\infty))$ is open $\Rightarrow$ f^* measurable

3. $\exists r$, $\int_{B(0,r)} |f(x)|\,dx > 0$, $f^*(x) \ge \frac{1}{|B(0, \max(r,|x|))|}\int_{B(0,r)} |f(x)|\,dx$

$$\ge \frac{C_n \int_{B(0,r)} |f(x)|\,dx}{\max(r,|x|)^n} \ge \left(\frac{C_n}{r^n}\int_{B(0,r)} |f(x)|\,dx\right)\cdot\frac{1}{|x|^n}.$$

$$\int_{B(0,1)^c} |f^*| \ge c\cdot\int_{B(0,1)^c} |x|^{-n}\,dx = \infty$$

$$m\{x \mid f^*(x) > \alpha\} \le m\{x \mid 2^n Mf > \alpha\} = m\{x \mid Mf > \tfrac{\alpha}{2^n}\} \le \frac{C}{\alpha}\|f\|_1$$

$$m\{x \mid f^*(x) > \alpha\} \ge m\{x \mid \tfrac{C}{|x|^n} > \alpha\} = |B(0, (\tfrac{c}{\alpha})^{\frac{1}{n}})| = \frac{C'}{\alpha}$$

Some properties of involution

Theorem:

$f\in L^1(\mathbb{R}^n)$, $g\in L^\infty(\mathbb{R}^n) \Rightarrow f*g(x) = \int f(x-y)g(y)\,dy$ is (uniformly) continuous.

Pf: $|f*g(x_1) - f*g(x_2)| \le \|g\|_\infty \int_{\mathbb{R}^n} |f(x_1-y) - f(x_2-y)|\,dy \to 0$ as $|x_1-x_2|\to 0$

because of the average continuity.

Cor (Steinhaus)

$m(A)>0$, then $0\in \mathrm{Int}(A-A)$. It suffices to prove the case $m(A)<\infty$.

Pf: $f=\chi_A$, $g=\chi_{-A}$, $f*g(x) = \int \chi_A(x-y)\chi_{-A}(y)\,dy$ continuous

$\chi_A(x-y)\chi_A(y)\ne 0 \Leftrightarrow y\in -A,\ x-y\in A \Rightarrow y\in -A,\ x\in A-A$

$\Rightarrow$ If $x\notin A-A$, $f*g(x) = 0$.

$f*g(0) = \int \chi_A(-y)\chi_{-A}(y)\,dy = m(A) > 0 \Rightarrow \exists\,\delta>0$, $f*g(x)>0$, $\forall x\in B_\delta$

$\Rightarrow B_\delta \subset A-A$

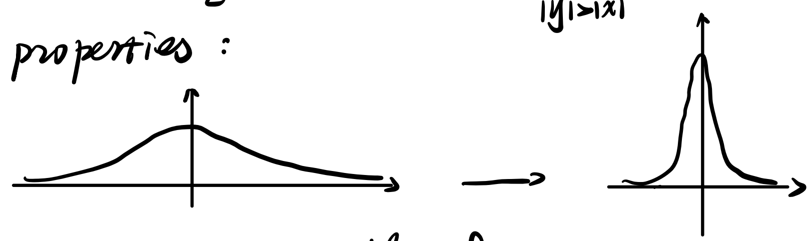
A fundamental lemma in Harmonic Analysis.

Theorem:

$\varphi \in L^1(\mathbb{R}^n)$, $\int \varphi = 1$, $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$, $\psi(x) = \sup_{|y| \geq |x|} |\varphi(y)|$

We have the following properties:

(1) $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) = 1$



$\psi(x)$ is a radial, positive, measurable function

(2) If $\psi(x) \in L^1(\mathbb{R}^n)$, then we have

$$\sup_\varepsilon |f * \varphi_\varepsilon(x)| \leq A\, Mf(x), \quad A = \|\psi\|_1$$

where $Mf(x) = \sup_{r>0} \frac{1}{|B_r|}\int_{B_r} |f(x-y)|\, dy$ is the H-L maximal function

(3) $\forall f \in L^p$, $1 \leq p \leq \infty$, $\lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x) = f(x)$ a.e.

Pf: (2) $|f| * \psi_\varepsilon(x) \geq |f * \varphi_\varepsilon(x)|$. It suffices to prove $|f| * \psi(x) \leq \|\psi\|_1 Mf(x)$

for $\psi \in L^1$, $\psi \geq 0$, radial, descending.

Step 1. $\psi = \sum b_j \chi_{R_j}$, $R_j = B_j \setminus B_{j-1}$, $B_j = \{x \in \mathbb{R}^n \mid |x| \leq j\}$, $b_1 > b_2 > \cdots > b_m$

$$\Rightarrow \psi = \sum_{j=1}^m (b_j - b_{j+1})\chi_{B_j} = \sum_{j=1}^m a_j \chi_{B_j}, \quad \|\psi\|_1 = \sum a_j |B_j|$$

$$|f| * \psi(x) = \int |f|(x-y)\,\psi(y)\, dy = \sum_{j=1}^m a_j \int_{B_j} |f|(x-y)\, dy \leq \sum_j a_j |B_j|\, Mf(x)$$

$$= \|\psi\|_1\, Mf(x)$$

Step 2. For general ψ with the preceding properties, we approximate it by simple, radial, positive, decreasing ψ_k s.t. $\psi_k \uparrow \psi$ a.e.

$\|\psi_k\|_1 \leq \|\psi\|_1$.

$$|f| * \psi(x) \leq \lim \int |f|(x-y)\,\psi_k(y)\, dy \leq \lim \|\psi_k\|_1\, Mf(x) \leq \|\psi\|_1\, Mf(x)$$

Step 3. Replace ψ by ψ_ε, we obtain for each ε,

$$|f| * \psi_\varepsilon(x) \leq \|\psi_\varepsilon\|_1\, Mf(x) = \|\psi\|_1\, Mf(x)$$

Step 1. If $p<\infty$, we have $f*\varphi_\varepsilon \xrightarrow{L^p} f$.

Since C_c^∞ is dense in L^p, consider a sequence $f_k \in C_c^\infty$, $f_k \to f$ in L^p.

$$\left(\int |f*\varphi_\varepsilon(x) - f_k*\varphi_\varepsilon(x)|^p dx\right)^{1/p} = \left(\int \left|\int (f(x-y)-f_k(x-y))\varphi_\varepsilon(y)dy\right|^p dx\right)^{1/p}$$

$$\le \int \left(\int |f(x-y)-f_k(x-y)|^p dx\right)^{\frac{1}{p}} |\varphi_\varepsilon(y)| dy = \|f-f_k\|_p \cdot \|\varphi_\varepsilon(y)\|_1$$

↑ Minkowski Inequality

$$\left(\int_X \left(\int_Y |f(x,y)| d\mu(y)\right)^p d\nu(x)\right)^{1/p} \le \int_Y \left(\int_X |f(x,y)|^p d\mu(x)\right)^{\frac{1}{p}} d\mu(y)$$

$\varphi_\varepsilon(y) = \frac{1}{\varepsilon^n}\varphi(\frac{y}{\varepsilon})$

$$\|f_k*\varphi_\varepsilon - f_k\|_p \le \left(\int \left(\int |f_k(x-y)-f_k(x)| \varphi_\varepsilon(y) dy\right)^p dx\right)^{1/p}$$

$$= \left(\int \left(\int |f_k(x-\varepsilon z) - f_k(x)| \varphi(z) dz\right)^p dx\right)^{1/p}$$

$$\le \int \left(\int |f_k(x-\varepsilon z) - f_k(x)|^p dx\right)^{\frac{1}{p}} |\varphi(z)| dz$$

$$g_k(z) = \left(\int |f_k(x-\varepsilon z) - f_k(x)|^p dx\right)^{\frac{1}{p}} |\varphi(z)|$$

$$\le \left(\left(\int |f_k(x-\varepsilon z)|^p dx\right)^{1/p} + \left(\int |f_k(x)|^p dx\right)^{1/p}\right) |\varphi(z)| \le 2\|f\|_p |\varphi(z)| \in L^1$$

By DCT, $\lim_{\varepsilon\to 0} \|f_k*\varphi_\varepsilon - f_k\|_p = \int \lim_{\varepsilon\to 0} \|f_k(\cdot-\varepsilon z) - f_k(\cdot)\|_p |\varphi(z)| dz = 0$

$$\|f*\varphi_\varepsilon - f\|_p \le \|f*\varphi_\varepsilon - f_k*\varphi_\varepsilon\|_p + \|f_k*\varphi_\varepsilon - f_k\|_p + \|f_k - f\|_p$$

$\forall \sigma>0$, $\exists k$, $\|f-f_k\|_p \le \min\{\frac{\sigma}{3\|\varphi\|_1}, \frac{\sigma}{3}\}$, $\|f*\varphi_\varepsilon - f_k*\varphi_\varepsilon\|_p \le \frac{\sigma}{3}$

$\exists \delta$, $\forall \varepsilon<\delta$, $\|f_k*\varphi_\varepsilon - f_k\|_p < \frac{\sigma}{3}$ $\Rightarrow \|f*\varphi_\varepsilon - f\|_p < \sigma$.

$\Rightarrow f*\varphi_\varepsilon \to f$ in L^p.

Step 2. By Riesz Lemma, $\exists \varepsilon_k \to 0$, s.t. $f*\varphi_{\varepsilon_k} \to f$ a.e. We next prove, $\lim_{\varepsilon\to 0} f*\varphi_\varepsilon(x)$ exists a.e. To prove this kind of problem, the general method is to find a "proper" maximal function.

Denote for $f\in L^p$, $x\in\mathbb{R}^n$, $\Omega f(x) = |\limsup_{\varepsilon\to 0} f*\varphi_\varepsilon(x) - \liminf_{\varepsilon\to 0} f*\varphi_\varepsilon(x)|$

Fact: $\Omega(f_1+f_2) \le \Omega(f_1) + \Omega(f_2)$

If $f \in C_c(\mathbb{R}^n)$, $\lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x) = f(x)$, $\forall x$.

This is because $|f*\varphi_\varepsilon(x) - f(x)| \le \int |f(x-\varepsilon z) - f(x)| \cdot |\varphi(z)| dz \xrightarrow{DCT} 0$

For $f \in L^p$, $1 \le p < \infty$, we can decompose f into two parts.

$f = f_1 + f_2$. $f_1 \in C_c(\mathbb{R}^n)$, $\|f_2\|_p$ is arbitrarily small.

$\Omega f(x) \le \Omega f_1(x) + \Omega f_2(x) = \Omega f_2(x) = |\limsup f_2 * \varphi_\varepsilon(x) - \liminf f_2 * \varphi_\varepsilon(x)|$

$\le 2A\, Mf_2(x)$

$m\{x \mid \Omega f(x) > \varepsilon\} \le m\{x \mid \Omega f_2(x) > \frac{\varepsilon}{2}\} \le m\{x \mid Mf_2(x) > \frac{\varepsilon}{2A}\} \le C \cdot \left(\frac{\|f_2\|_p}{\varepsilon/2A}\right)^p$

$\|f_2\|_p \to 0 \Rightarrow m\{x \mid \Omega f(x) > \varepsilon\} = 0$, $\forall \varepsilon \Rightarrow \Omega f(x) = 0$ a.e.

Step 3. $p = \infty$, $f \in L^\infty$. We prove $\lim_{\varepsilon \to 0} f * \varphi_\varepsilon(x) = f(x)$ a.e. in $B_R = \{|x| \le R\}$

$f_1(x) = f(x) \cdot \chi_{B_{2R}}$, $f_2(x) = f(x) - f_1(x)$, $f_1 \in L^1$, then

$\lim_{\varepsilon \to 0} f_1 * \varphi_\varepsilon = f_1$ a.e. For $x \in B_R$, $|f_2 * \varphi_\varepsilon(x)| = |\int f_2(x-y)\varphi_\varepsilon(y) dy|$

$\le \|f\|_\infty \cdot |\int_{|y|>R} \varphi_\varepsilon(y) dy|$

$f_2(x-y)\varphi_\varepsilon(y) \ne 0 \Rightarrow |x-y| > 2R$ $|x| \le R \Rightarrow |y| > R$

$= \|f\|_\infty \cdot |\int_{|z| > \frac{R}{\varepsilon}} \varphi(z) dz| \to 0$ as $\varepsilon \to 0$

$\Rightarrow$ If $x \in B_R$, $f * \varphi_\varepsilon(x) = f_1 * \varphi_\varepsilon(x) + f_2 * \varphi_\varepsilon(x) \to f(x)$ a.e.

Rmk:

Similar to $m\{x \mid Mf(x) > \alpha\} \le C \frac{\|f(x)\|_1}{\alpha}$,

$\alpha |B_x| < \int_{B_x} |f(y)| dy \le \left(\int_{B_x} |f(y)|^p dy\right)^{\frac{1}{p}} \cdot |B_x|^{\frac{1}{q}} \Rightarrow |B_x| \le \frac{1}{\alpha^p}\left(\int_{B_x} |f(y)|^p\right)$

$m\{x \mid Mf(x) > \alpha\} \le \frac{c}{\alpha^p} \sum |B_{ij}| \le C \cdot \frac{\int_{\mathbb{R}^n} |f(y)|^p}{\alpha^p}$. $C \sim n, p$