16. Show that if F is of bounded variation in $[a, b]$, then:

(a) $\int_a^b |F'(x)|\, dx \le T_F(a, b)$.

(b) $\int_a^b |F'(x)|\, dx = T_F(a, b)$ if and only if F is absolutely continuous.

As a result of (b), the formula $L = \int_a^b |z'(t)|\, dt$ for the length of a rectifiable curve parametrized by z holds if and only if z is absolutely continuous.

证明. (a) 因为 $F \in BV([a,b])$，所以 F 可以写成 $F(x) = P_F(a,x) + F(a) - N_F(a,b)$，其中 $P_F(a,x)$ 和 $N_F(a,x)$ 递增，因此它们几乎处处可导，而且有：

$$F'(x) = P_F'(a,x) - N_F'(a,x).$$

根据 Lebesgue 定理的推论可知 P_F' 和 N_F' 可积，从而 $|F'|$ 可积，而且根据 Lebesgue 定理的推论给出的不等式可知：

$$\begin{aligned}\int_a^b |F'(x)|\mathrm{d}x &\le \int_a^b |P_F'(a,x)|\mathrm{d}x + \int_a^b |N_F'(a,x)|\mathrm{d}x \\ &= \int_a^b P_F'(a,x)\mathrm{d}x + \int_a^b N_F'(a,x)\mathrm{d}x \le P_F(a,b) + N_F(a,b) = T_F(a,b).\end{aligned}$$

(b) 若 F 绝对连续，则由绝对连续函数成立 Newton-Leibniz 公式，对任何 $[a,b]$ 的分划 Δ，都成立：

$$\begin{aligned}\sum_{\Delta} |F(x_j) - F(x_{j-1})| &= \sum_{\Delta} \Big|\int_{x_{j-1}}^{t_j} F'(x)\mathrm{d}x\Big| \\ &\le \sum_{\Delta} \int_{x_{j-1}}^{x_j} |F'(x)|\mathrm{d}x \\ &= \int_a^b |F'(x)|\mathrm{d}x.\end{aligned}$$

取上确界可得

$$T_F(a,b) \le \int_a^b |F'(x)|\mathrm{d}x.$$

因为绝对连续蕴含有界变差，所以根据 (a) 的结果可知反向不等式成立，因此等式

$$T_F(a,b) = \int_a^b |F'(x)|\mathrm{d}x$$

成立.

反过来，如果上述等式成立，我们去说明 F 是绝对连续的. 事实上，我们可以断言对任何 $y \in [a,b]$，都成立：

$$\int_a^x |F'(y)|\mathrm{d}y = T_F(a,x).$$

事实上，根据 $F \in BV([a,b])$ 以及 (a) 的结果可知：

$$\int_a^x |F'(y)|\mathrm{d}y \le T_F(a,x), \quad \int_x^b |F'(y)|\mathrm{d}y \le T_F(x,b).$$

但是

$$\int_a^x |F'(y)|\mathrm{d}y + \int_x^b |F'(y)|\mathrm{d}y = \int_a^b |F'(y)|\mathrm{d}y = T_F(a,b) = T_F(a,x) + T_F(x,b).$$

所以，以上两个不等号均取等，作为一个直接推论，$[a,b]$ 的任何子区间 $[a_k, b_k]$ 上都成立

$$\int_{a_k}^{b_k} |F'(y)|\mathrm{d}y = T_F(a_k, b_k).$$

我们计划利用可积函数积分的绝对连续性来证明 F 绝对连续. 因为 $F \in BV([a,b])$，所以 $T_F(a,b) < \infty$，因此 $F'(x) \in L^1_{loc}([a,b])$，根据可积函数积分的绝对连续性，对任何 $\varepsilon > 0$，存在 $\delta > 0$，使得对任何可测集 $E \subset [a,b]$ 满足 $m(E) < \delta$，都有：

$$\int_E |F'| < \varepsilon.$$

因此，对任何 $[a,b]$ 互不相交的有限子区间族 $\{(a_k,b_k)\}_{k=1}^N$，只要 $\sum_{k=1}^N (b_k - a_k) < \delta$，就有：

$$\sum_{k=1}^N |F(b_k) - F(a_k)| \le \sum_{k=1}^N T_F(a_k,b_k) = \sum_{k=1}^N \int_{a_k}^{b_k} |F'| = \int_{\bigcup_{k=1}^N (a_k,b_k)} |F'| < \varepsilon.$$

所以 F 绝对连续. □

Exercise 19: Show that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, then

(a) f maps sets of measure zero to sets of measure zero.

(b) f maps measurable sets to measurable sets.

Solution.

(a) Suppose $E \subset \mathbb{R}$ has measure zero. Let $\epsilon > 0$. By absolute continuity, $\exists \delta > 0$ such that $\sum |b_j - a_j| < \delta \Rightarrow \sum |f(b_j) - f(a_j)| < \epsilon$. Since $m(E) = 0$, there is an open set $U \supset E$ with $m(U) < \delta$. Every open subset of $\mathbb{R}$ is a countable disjoint union of open intervals, so

$$U = \bigcup_{j=1}^{\infty} (a_j, b_j) \text{ with } \sum_{j=1}^{\infty} (b_j - a_j) < \delta.$$

For each $j = 1, 2, \ldots,$, let $m_j, M_j \in [a_j, b_j]$ be values of x with

$$f(m_j) = \min_{x \in [a,b]} f(x) \text{ and } f(M_j) = \max_{x \in [a,b]} f(x).$$

Such m_j and M_j must exist because f is continuous and $[a_j, b_j]$ is compact. Then

$$f(U) \subset \bigcup_{j=1}^{\infty} [f(m_j), f(M_j)].$$

But $|M_j - m_j| \leq |b_j - a_j|$ so

$$\sum_{j=1}^{\infty} |M_j - m_j| < \delta \Rightarrow \sum_{j=1}^{\infty} |f(M_j) - f(m_j)| < \epsilon.$$

Hence $f(E)$ is a subset of a set of measure less that ϵ. This is true for all ϵ, so $f(E)$ has measure zero.

(b) Let $E \subset \mathbb{R}$ be measurable. Then $E = F \cup N$ where F is F_σ and N has measure zero. Since closed subsets of $\mathbb{R}$ are σ-compact, F is σ-compact. But then $f(F)$ is also σ-compact since f is continuous. Then $f(E) = f(F) \cup f(N)$ is a union of an F_σ set and a set of measure zero. Hence $f(E)$ is measurable.

□

Exercise 20: This exercise deals with functions F that are absolutely continuous on $[a, b]$ and are increasing. Let $A = F(a)$ and $B = F(b)$.

(a) There exists such an F that is in addition strictly increasing, but such that $F'(x) = 0$ on a set of positive measure.

(b) The F in (a) can be chosen so that there is a measurable subset $E \subset [A, B]$, $m(E) = 0$, so that $F^{-1}(E)$ is not measurable.

(c) Prove, however, that for any increasing absolutely continuous F, and E a measurable subset of $[A, B]$, the set $F^{-1}(E) \cap \{F'(x) > 0\}$ is measurable.

Solution.

(a) Let

$$F(x) = \int_a^x \delta_C(x)dx$$

where $C \subset [a, b]$ is a Cantor set of positive measure and $\delta_C(x)$ is the distance from x to C. Note that $\delta_C(x) \geq 0$ with equality iff $x \in C$. Since δ_C is continuous, this integral is well-defined, even in the Riemann sense. Moreover, F is absolutely continuous by the absolute continuity of integration of L^1 functions. As shown in problem 9, $F'(x)$ exists and equals zero a.e. in C, hence on a set of positive measure. However, F is strictly increasing: Suppose $a \leq x < y \leq b$. Since C contains no interval, some point, and therefore some interval, between x and y belongs to C^C. The integral of δ_C over this interval will be positive, so $F(y) > F(x)$.

(b) The same function from part (a) does the trick. Since F is increasing, it maps disjoint open intervals to disjoint open intervals. Let $U = [a, b] \setminus C$. Since U is open, we can write

$$U = \bigcup_{j=1}^{\infty}(a_j, b_j)$$

where the intervals (a_j, b_j) are disjoint. Then

$$F(U) = \bigcup_{j=1}^{\infty}(F(a_j), F(b_j))$$

and

$$m(F(U)) = \sum_{j=1}^{\infty} F((b_j) - F(a_j)).$$

But

$$B - A = F(b) - F(a) = \int_a^b \delta(x)dx = \int_U \delta(x)dx = \sum_{j=1}^{\infty}(F(b_j) - F(a_j))$$

since $\delta = 0$ on C so $\int_C \delta(x)dx = 0$. Thus $m(F(U)) = m(F([a, b]))$, so that $m(F(C)) = 0$. This implies that $m(F(S)) = 0$ for any subset $S \subset C$. But since C has positive measure, it has a non-measurable subset. Then if $E = F(S)$, $m(E) = 0$ so E is measurable, but $F^{-1}(E) = S$ is not measurable.

[Hint: (a) Let $F(x) = \int_a^x \chi_K(x)\,dx$, where K is the complement of a Cantor-like set C of positive measure. For (b), note that $F(C)$ is a set of measure zero. Finally, for (c) prove first that $m(\mathcal{O}) = \int_{F^{-1}(\mathcal{O})} F'(x)\,dx$ for any open set $\mathcal{O}$.]

1(c)

Hint 的证明.

Claim: $m(O) = \int_{F^{-1}(O)} F'(x)\,dx$ 下证

① 若 $O = (a,b)$. $\int_{F^{-1}(a)}^{F^{-1}(b)} F'(x)\,dx = F(F^{-1}(b)) - F(F^{-1}(a)) = b-a$.

② 若 $O = \bigcup_i (a_i, b_i)$ 线性相加即可证明此 Hint. $\Rightarrow m(O) = \int_{F^{-1}(O)} F'(x)\,dx$ for any open O.

只需证明 $F^{-1}(E) \cap \{F'(x) > \frac{1}{n}\}$ 可测 (可测集的可数极限可测).

$E = \bigcap O_i \setminus Z \Rightarrow$ 只需证明 $F^{-1}(Z) \cap \{F' > \frac{1}{n}\}$ 可测.

Hint $\Rightarrow$ 实际是零测的.

$\forall \varepsilon > 0$, $\exists\, O \supset Z$.

$\varepsilon > m(O) = \int_{F^{-1}(O)} F'(x)\,dx \geq \int_{F^{-1}(O) \cap \{F'(x) > \frac{1}{n}\}} F'(x)\,dx \geq \frac{1}{n}\, m(F^{-1}(Z) \cap F'(x) > \frac{1}{n})$ 是零测的.

$\Rightarrow m(F^{-1}(Z) \cap F'(x) > \frac{1}{n}) < n\varepsilon$. $\forall \varepsilon > 0$

$\Rightarrow m(F^{-1}(Z) \cap F'(x) > \frac{1}{n}) = 0$

21. Let F be absolutely continuous and increasing on $[a, b]$ with $F(a) = A$ and $F(b) = B$. Suppose f is any measurable function on $[A, B]$.

(a) Show that $f(F(x))F'(x)$ is measurable on $[a, b]$. Note: $f(F(x))$ need not be measurable by Exercise 20 (b).

(b) Prove the change of variable formula: If f is integrable on $[A, B]$, then so is $f(F(x))F'(x)$, and

$$\int_A^B f(y)\,dy = \int_a^b f(F(x))F'(x)\,dx.$$

[Hint: Start with the identity $m(\mathcal{O}) = \int_{F^{-1}(\mathcal{O})} F'(x)\,dx$ used in (c) of Exercise 20 above.]

证明. (a) 事实上，我们需要**题 20** 作为引理. 首先对于开集 $O\subset[A,B]$，成立 $m(O)=\int_{F^{-1}(O)}F'(x)\mathrm{d}x$. 这是因为，$F$ 是绝对连续函数，所以 O 的原像集 $F^{-1}(O)$ 也是开集，所以 $F^{-1}(O)$ 写成可数互不相交的开区间的并：$F^{-1}(O)=\bigsqcup_{k\geq 1}(a_k,b_k)$. 因为 F 连续且单调递增，所以对每个 (a_k,b_k)，$F(a_k,b_k)=(F(a_k),F(b_k)):=(A_k,B_k)$，因此 $O=\bigcup_{k\geq 1}(A_k,B_k)$. 而且，由单调递增可知，每个 (A_k,B_k) 至多在端点处相交，并且 $F(a_k)=A_k$，$F(b_k)=B_k$. 根据绝对连续函数的 Newton-Leibniz 公式可知：

$$m(O)=\sum_{k=1}^{\infty}(B_k-A_k)=\sum_{k=1}^{\infty}(F(b_k)-F(a_k))=\sum_{k=1}^{\infty}\int_{a_k}^{b_k}F'(x)\mathrm{d}x=\int_{\bigcup_k(a_k,b_k)}F'(x)\mathrm{d}x=\int_{F^{-1}(O)}F'$$

我们下面说明把开集 O 换成 G_δ 集 G 后结论依然成立. 对于 G_δ 集 $G=\bigcap_{i\geq 1}G_i$，记 $O_j=\bigcap_{i=1}^{j}G_i$ 是开集，所以

$$m(O_j)=\int_{F^{-1}(O_j)}F'(x)\mathrm{d}x=\int_{\mathbb{R}^d}F'(x)\chi_{F^{-1}(O_j)}(x)\mathrm{d}x.$$

观察到 $\chi_{F^{-1}(O_j)}$ 逐点收敛到 $\chi_{F^{-1}(G)}$，而且，O_j 是一列单调递减收敛到 G 的集列，因此，对上式取极限并由控制收敛定理知

$$m(G)=\int_{\mathbb{R}^d}F'(x)\chi_{F^{-1}(G)}(x)\mathrm{d}x=\int_{F^{-1}(G)}F'(x)\mathrm{d}x.$$

对于可测集 E，存在一个 G_δ 集记为 G，$G\supset E$ 而且 $G-E$ 是零测集. 记 $Z=\{F'(x)=0\}$，此时集合 $F^{-1}(E)\cap\{F'>0\}$ 以表示为 $F^{-1}(G\cup(G-E))\cap([a,b]\backslash Z)$.

我们先证明 Z 是一个零测集. 事实上，因为 $\{0\}$ 是一个 G_δ 集，所以：

$$0=m(\{0\})=\int_{F^{-1}(\{0\})}F'(x)\mathrm{d}x\Rightarrow F'(x)=0,\text{a.e. }x\in Z.$$

所以，只能是 Z 本身是零测集. 因此 $[a,b]\backslash Z$ 是可测集.

进一步地，写 $F^{-1}(G\cup(G-E))=F^{-1}(G)\cup F^{-1}(G-E)$. 因为 $G-E$ 是零测集，所以存在 $U\supset G-E$ 是 G_δ 集，使得 $m(U-(G-E))=0$，所以 $m(U)=m(G-E)+m(U-(G-E))=0$，同样可得

$$0=m(U)=\int_{F^{-1}(U)}F'(x)\mathrm{d}x\Rightarrow F'(x)=0,\text{a.e. }x\in F^{-1}(U).$$

所以 $m(F^{-1}(U)\cap([a,b]\backslash Z))=0$，所以 $m(F^{-1}(G-E))\cap([a,b]\backslash Z)=0$. 又因为 F 连续 $\Rightarrow F$ 可测 $\Rightarrow F^{-1}(G)$ 是可测集，所以

$$F^{-1}(E)\cap\{F'>0\}=F^{-1}(G\cup(G-E))\cap([a,b]\backslash Z)=[F^{-1}(G)\cap([a,b]\backslash Z)]\cup[F^{-1}(G-E)\cap([a,b]\backslash Z)]$$

是可测集.

回到本题，我们记 $G=(f\circ F)\cdot F'$，我们只需证明对任何 $t>0$，都有 $\{G>t\}$ 是可测集. 事实上，我们有如下的分解：

$$\{G>t\}=\bigcup_{\substack{r>0\\ r\in\mathbb{Q}}}\left\{(f\circ F)>\frac{t}{r}\right\}\cap\{F'>r\}=\bigcup_{\substack{r>0\\ r\in\mathbb{Q}}}F^{-1}(f^{-1}(t/r,+\infty))\cap\{F'>0\}\cap\{F'>r\}.$$

根据引理的结果并结合 F' 可测 $\Rightarrow \{F' > r\}$ 可测，由此可知 $\{G > t\}$ 是可测的. 所以 $f(F(x))F'(x)$ 在 $[a,b]$ 上可测.

(b) 根据 $m(G) = \int_{F^{-1}(O)} F'(x)\mathrm{d}x$ 立刻知换元公式对 G_δ 集上的特征函数成立，进而对 $[a,b]$ 的可测子集上的特征函数成立，进而对简单函数成立. 对于 $f \in L^1_{loc}([a,b])$，将 f 分解为正部和负部 f^+ 和 f^-，则 $f^+, f^- \in L^1_{loc}([a,b])$. 对于每个 f^+，它是非负、有界且具有有限测度支撑的可测函数，存在非负、递增简单函数逼近 $\{\varphi_n\}_{n\geq 1}$，使得 φ_n 逐点收敛到 f^+，由单调收敛定理可知

$$\lim_n \int_a^b \varphi_n(x)\mathrm{d}x = \int_a^b \lim_n \varphi_n(x)\mathrm{d}x = \int_a^b f^+(x)\mathrm{d}x.$$

另一方面，

$$\int_a^b \varphi_n(x)\mathrm{d}x = \int_A^B \varphi_n(F(x))F'(x)\mathrm{d}x.$$

所以取 $n \to \infty$ 可知

$$\int_a^b f^+(x)\mathrm{d}x = \int_A^B f^+(F(x))F'(x)\mathrm{d}x.$$

同理可知

$$\int_a^b f^-(x)\mathrm{d}x = \int_A^B f^-(F(x))F'(x)\mathrm{d}x.$$

两式相减可知 $f(F(x))F'(x)$ 可积，而且换元公式成立. □

Exercise 22: Suppose that F and G are absolutely continuous on $[a,b]$. Show that their product FG is also absolutely continuous. This has the following consequences.

(a) Whenever F and G are absolutely continuous in $[a,b]$,

$$\int_a^b F'(x)G(x)dx = -\int_a^b F(x)G'(x)dx + [F(x)G(x)]_a^b.$$

(b) Let F be absolutely continuous in $[-\pi,\pi]$ with $F(\pi) = F(-\pi)$. Show that if

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{-inx}dx,$$

such that $F(x) \sim \sum a_n e^{inx}$, then

$$F'(x) \sim \sum in a_n e^{inx}.$$

(c) What happens if $F(-\pi) \neq F(\pi)$?

Proof. Since F and G are absolutely continuous, they are continuous and therefore bounded on the compact interval $[a,b]$. Suppose $|F|, |G| \leq M$ on this interval. Now given $\epsilon > 0$, we can choose $\delta > 0$ such that $\sum |b_j - a_j| < \delta \Rightarrow \sum |F(b_j) - F(a_j) < \frac{\epsilon}{M}$ and $\sum |G(b_j) - G(a_j)| < \frac{\epsilon}{2M}$. Then

$$\begin{aligned}
&\sum |F(b_j)G(b_j) - F(a_j)G(a_j)| \\
&\quad = \sum \frac{1}{2} |(F(b_j - F(a_j))(G(b_j) + G(a_j)) + (F(b_j) + F(a_j))(G(b_j) - G(a_j))| \\
&\quad \leq \frac{1}{2} \left(\sum |F(b_j) - F(a_j)||G(b_j) + G(a_j)| + \sum |F(b_j) + F(a_j)||G(b_j) - G(a_j)| \right) \\
&\quad \leq \frac{1}{2} \left(\sum (2M)|F(b_j) - F(a_j)| + \sum (2M)|G(b_j) - G(a_j)| \right) \\
&\quad \leq \frac{1}{2} \left(2M \cdot \frac{\epsilon}{2M} + 2M \cdot \frac{\epsilon}{2M} \right) = \epsilon.
\end{aligned}$$

This proves that FG is absolutely continuous on $[a,b]$. We now turn to the consequences of this:

(a) Since FG is absolutely continuous, it's differentiable almost everywhere. By elementary calculus, $(FG)' = F'G + FG'$ at any point where all three derivatives exist, which is almost everywhere. Integrating both sides and subtracting $\int FG'$ yields

$$\int_a^b F'(x)G(x)dx = -\int_a^b F(x)G'(x)dx + \int_a^b (FG)'(x)dx.$$

Since FG is absolutely continuous, this implies

$$\int_a^b F'(x)G(x)dx = -\int_a^b F(x)G'(x)dx + [F(x)G(x)]_a^b.$$

(b) It would be nice if the problem would actually define this for us, but I'm assuming that the $\sim$ here means "is represented by" as opposed to any kind of statement about whether the function actually converges to its Fourier series or not. Then suppose b_n are the Fourier coefficients of F', so by definition

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} F'(x) e^{-inx} dx.$$

Using part (a), we have

$$b_n = -\frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)(-ine^{-inx})dx + [F(x)e^{-inx}]_a^b = in\frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)e^{-inx}dx = ina_n.$$

(c) Then all bets are off. As one example, consider $F(x) = x$ which is clearly absolutely continuous on $[-\pi, \pi]$. Then

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi} \left(\frac{xe^{-inx}}{-in} + \frac{e^{-inx}}{n^2} \right) \Bigg|_{-\pi}^{\pi} = \frac{2i}{n}(-1)^n$$

for $n \neq 0$, and $a_0 = \int_{-\pi}^{\pi} xdx = 0$. However, $F'(x) = 1$ which has Fourier coefficients $b_0 = 1$ and $b_n = 0$ for $n \neq 0$.

∎

23. Let F be continuous on $[a,b]$. Show the following.

(a) Suppose $(D^+F)(x) \geq 0$ for every $x \in [a,b]$. Then F is increasing on $[a,b]$.

(b) If $F'(x)$ exists for every $x \in (a,b)$ and $|F'(x)| \leq M$, then $|F(x) - F(y)| \leq M|x-y|$ and F is absolutely continuous.

[Hint: For (a) it suffices to show that $F(b) - F(a) \geq 0$. Assume otherwise. Hence with $G_\epsilon(x) = F(x) - F(a) + \epsilon(x-a)$, for sufficiently small $\epsilon > 0$ we have $G_\epsilon(a) = 0$, but $G_\epsilon(b) < 0$. Now let $x_0 \in [a,b)$ be the greatest value of x_0 such that $G_\epsilon(x_0) \geq 0$. However, $(D^+G_\epsilon)(x_0) > 0$.]

证明. (a) 只需证明 $F(b) - F(a) \geq 0$（根据下面的证明过程可知实际上 a 和 b 都可以换成 $[a,b]$ 中的任何一个点）. 用反证法，假设不等式不成立，令 $G_\varepsilon(x) = F(x) - F(a) + \varepsilon(x-a)$，因此对充分小的 $\varepsilon > 0$ 成立 $G_\varepsilon(a) = 0$. 但是，$G_\varepsilon(b) < 0$. 令 x_0 是 G_ε 在 $[a,b]$ 上的最大值点，则对足够小的 $h > 0$，成立

$$\frac{G_\varepsilon(x_0+h) - G_\varepsilon(x_0)}{h} \leq 0.$$

于是：

$$\limsup_{\substack{h>0 \\ h\to 0}} \frac{G_\varepsilon(x_0+h) - G_\varepsilon(x_0)}{h} \leq 0.$$

但是这和 $D^+(G_\varepsilon) = D^+(F) + \varepsilon > 0$ 对每个 $x \in [a,b]$ 矛盾.

(b) 由微积分基本定理 $|F(x) - F(y)| = |\int_x^y F'(t)\mathrm{d}t| \leq M|x-y|$. 另外 $F(x) = F(a) + \int_a^x F'(y)\mathrm{d}y \Rightarrow F$ 绝对连续. □

27. (**Stein 中译本, P113, 题 32**) 令 $f:\mathbb{R}\to\mathbb{R}$. 证明 f 对某个 M 和所有 $x,y\in\mathbb{R}$ 满足 Lipschitz 条件：

$$|f(x)-f(y)|\leq M|x-y|.$$

当且仅当 f 满足：(i)f 绝对连续；(ii) 对 a.e.x，$|f'(x)|\leq M$.

证明. 先证明 $\Rightarrow$. 若能证明一致连续，根据

$$\frac{|f(x)-f(y)|}{|x-y|}\leq M\Rightarrow\limsup_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right|\leq M,$$

因此只要能说明 f' 几乎处处存在，就有 $|f'(x)|\leq M$. 而绝对连续 $\Rightarrow$ 有界变差 $\Rightarrow$ f' 几乎处处存在，所以只需要说明 f 绝对连续.

我们用定义说明. 对任何 $\varepsilon>0$，取 $\delta=\frac{\varepsilon}{M}$，对 $[a,b]$ 的子区间 $\{(a_k,b_k)\}_{k=1}^N$ 互不相交且满足 $\sum_{k=1}^N(b_k-a_k)<\delta$，都有：

$$\sum_{k=1}^N|f(b_k)-f(a_k)|\leq M\sum_{k=1}^N(b_k-a_k)<M\cdot\frac{\varepsilon}{M}=\varepsilon.$$

根据定义可知 f 绝对连续.

再证明 $\Leftarrow$，因为 f 绝对连续 $\Rightarrow$ f 有界变差，所以 f' 几乎处处存在且可积，且成立微积分基本定理：

$$f(x)-f(y)=\int_x^y f'(t)\mathrm{d}t.$$

根据积分的三角不等式：

$$|f(x)-f(y)|\leq\int_{(x,y)}|f'(t)|\mathrm{d}t\leq M|x-y|.$$

所以 f 满足系数为 M 的 Lipschitz 条件. □