第二十九讲（2023.6.16）

Def $(X_1\times X_2,\ \mathcal{M}_1\otimes\mathcal{M}_2,\ \mu_1\times\mu_2)$

$E\subset X_1\times X_2$

对 $x\in X_1$，$E_x \overset{\text{def}}{=} \{y\in X_2 : (x,y)\in E\}$

称为 E 的 x-切片.

对 $y\in X_2$，$E^y \overset{\text{def}}{=} \{x\in X_1,\ (x,y)\in E\}$

称为 E 的 y-切片.

$f_x(y) \overset{\text{def}}{=} f(x,y)$，$\quad y\in X_2$

$f^y(x) \overset{\text{def}}{=} f(x,y)$，$\quad x\in X_1$

Prop.

(i) 设 $E\in\mathcal{M}_1\otimes\mathcal{M}_2$，则

$\forall x\in X_1$，$E_x\in\mathcal{M}_2$

$\forall y\in X_2$，$E^y\in\mathcal{M}_1$

(ii) f $\mathcal{M}_1\otimes\mathcal{M}_2$ 可测 $\Rightarrow$ f_x $\mathcal{M}_2$-可测

f^y $\mathcal{M}_1$ 可测

Thm (Tonelli)

设 $(X_1, \mathcal{M}_1, \mu_1)$, $(X_2, \mathcal{M}_2, \mu_2)$ 均 σ-有限

$f \in L^+(X_1 \times X_2)$

(i) $x \longmapsto \int_{X_2} f_x \, d\mu_2 \in L^+(X_1)$

$y \longmapsto \int_{X_1} f^y \, d\mu_1 \in L^+(X_2)$

(ii) $$\int_{X_1 \times X_2} f \, d(\mu_1 \times \mu_2) = \int_{X_1} \left[\int_{X_2} f(x,y) \, d\mu_2(y) \right] d\mu_1(x)$$
$$= \int_{X_2} \left[\int_{X_1} f(x,y) \, d\mu_1(x) \right] d\mu_2(y)$$

Thm (Fubini)

设 $(X_1, \mathcal{M}_1, \mu_1)$, $(X_2, \mathcal{M}_2, \mu_2)$ 均 σ-有限

$f \in L^1(X_1 \times X_2, \mu_1 \times \mu_2)$

(i) $f_x \in L^1(X_2, \mu_2)$ for μ_1-a.e. $x \in X_1$

$f^y \in L^1(X_1, \mu_1)$ for μ_2-a.e. $y \in X_2$

(ii) $x \longmapsto \int_{X_2} f_x \, d\mu_2 \in L^1(X_1, \mu_1)$

$$y \mapsto \int_{X_1} f^y \, d\mu_1 \in L^1(X_2, \mu_2)$$

(iii) 同 Tonelli (ii)

<u>Thm</u> 设 $(X_1, \mathcal{M}_1, \mu_1)$, $(X_2, \mathcal{M}_2, \mu_2)$ σ-有限

$E \in \mathcal{M}_1 \otimes \mathcal{M}_2$

(i) $x \mapsto \mu_2(E_x)$ $\mathcal{M}_1$-可测

$y \mapsto \mu_1(E^y)$ $\mathcal{M}_2$-可测

(ii) $$(\mu_1 \times \mu_2)(E) = \int_{X_1} \mu_2(E_x) \, d\mu_1$$

$$= \int_{X_2} \mu_1(E^y) \, d\mu_2$$

单调类引理 (Monotone Class Lemma)

<u>Def</u> 如果 $\mathcal{F} \subset 2^X$ 满足

$\mathcal{F} \ni E_k \nearrow E \implies E \in \mathcal{F}$

$\mathcal{F} \ni E_k \searrow E \implies E \in \mathcal{F}$

则称 $\mathcal{F}$ 是一个单调类

例：σ-代数是单调类

Thm (MCL)

$\mathcal{A}$ — X 上代数

$\mathcal{A}$ 生成的单调类 $= \sigma(\mathcal{A})$

Pf 令 $\mathcal{M} \overset{def}{=} \sigma(\mathcal{A})$

$\mathcal{F} \overset{def}{=} \mathcal{A}$ 生成的单调类

$\Longrightarrow \mathcal{F} \subset \mathcal{M}$ ($\because$ σ-代数是单调类)

只需证：

Claim $\mathcal{F}$ 是 σ-代数.

对 $E \in \mathcal{F}$，令

$$\mathcal{F}_E \overset{def}{=} \{ F \in \mathcal{F} : E \setminus F,\ F \setminus E,\ E \cap F \in \mathcal{F} \}$$

1° $\phi, E \in \mathcal{F}_E$

2° $E \in \mathcal{F}_F \iff F \in \mathcal{F}_E$

3° $\mathcal{F}_E$ 是单调类 (∵ $\mathcal{F}$ 是)

4° $E \in \mathcal{A} \Longrightarrow \mathcal{A} \subset \mathcal{F}_E$

$(\forall F \in \mathcal{A},\ E \setminus F,\ F \setminus E,\ E \cap F \in \mathcal{A} \subset \mathcal{F})$

5° $E \in \mathcal{A} \Longrightarrow \mathcal{F} \subset \mathcal{F}_E$

$(\text{by } 4^\circ\ \mathcal{A} \subset \mathcal{F}_E \xRightarrow{\text{by } 3^\circ} \mathcal{F} \subset \mathcal{F}_E)$

6° $E \in \mathcal{F} \Longrightarrow \mathcal{F} \subset \mathcal{F}_E$

$E \in \mathcal{F} \Longrightarrow E \in \mathcal{F}_A,\ \forall A \in \mathcal{A}$ (by 5°)

$\Longrightarrow A \in \mathcal{F}_E,\ \forall A \in \mathcal{A}$ (by 2°)

$\Longleftrightarrow \mathcal{A} \subset \mathcal{F}_E$

$\xRightarrow{\text{by } 3^\circ} \mathcal{F} \subset \mathcal{F}_E$

7° $\mathcal{F}$ 是代数.

$\forall E, F \in \mathcal{F}$

$\xRightarrow{\text{by } 6^\circ} E \in \mathcal{F} \subset \mathcal{F}_F$

$\Longrightarrow E \setminus F,\ E \cap F \in \mathcal{F}$

8° $\mathcal{F}$ 是 σ-代数

设 $A_k \in \mathcal{F}$, $k=1,2\cdots$

令 $A \overset{def}{=} \bigcup_{k=1}^{\infty} A_k$

$B_n \overset{def}{=} \bigcup_{k=1}^{n} A_k$

$\overset{\text{by } 7°}{\Longrightarrow} B_n \in \mathcal{F}$

$\overset{B_n \nearrow A}{\Longrightarrow} A \in \mathcal{F}$

<u>Pf of Thm</u>

<u>Step 1</u> 先假设 μ_1, μ_2 都是有限测度

令

$$\mathcal{F} \overset{def}{=} \{E \in \mathcal{M}_1 \otimes \mathcal{M}_2 : E \text{ 满足 (i), (ii)}\}$$

<u>claim</u> $\mathcal{F} = \mathcal{M}_1 \otimes \mathcal{M}_2$

1° $\mathcal{M}_1 \times \mathcal{M}_2 \subset \mathcal{F}$

设 $E = A \times B$, $A \in \mathcal{M}_1$, $B \in \mathcal{M}_2$

$$\Longrightarrow E_x = \begin{cases} B, & \text{if } x \in A \\ \phi, & \text{if } x \notin A \end{cases}$$

$$E^y = \begin{cases} A & \text{if } y \in B \\ \phi & \text{if } y \notin B. \end{cases}$$

$$\Longrightarrow \quad \mu_2(E_x) = \mu_2(B)\,\chi_A(x)$$

$$\mu_1(E^y) = \mu_1(A)\,\chi_B(y)$$

$\Longrightarrow$ (i) 成立

$$\int_{X_1} \mu_2(E_x)\, d\mu_1(x) = \mu_2(B) \int_{X_1} \chi_A \, d\mu_1$$

$$= \mu_2(B)\,\mu_1(A)$$

$$= (\mu_1 \times \mu_2)(E)$$

同理, $(\mu_1 \times \mu_2)(E) = \int_{X_2} \mu_1(E^y)\, d\mu_2(y)$

2° 令

$$\mathcal{A} \overset{\text{def}}{=} \{\mathcal{M}_1 \times \mathcal{M}_2 \text{ 中矩形之有限不交并}\}$$

by 1° 和可加性 $\Longrightarrow$ $\mathcal{A} \subset \mathcal{F}$

且 $\mathcal{A}$ 是 $X_1 \times X_2$ 上代数.

3° $\mathcal{F}$ 是单调类.

设 $\mathcal{F} \ni E_k \nearrow E$

$$f_k(y) \overset{\text{def}}{=} \mu_1((E_k)^y), \qquad y \in X_2$$

$$\overset{E_k \in \mathcal{F}}{\Longrightarrow} \quad f_k \ \mathcal{M}_2\text{-可测}$$

$$E_k \nearrow E \quad \Longrightarrow \quad (E_k)^y \nearrow E^y$$

$$\Longrightarrow \quad f(y) \overset{\text{def}}{=} \mu_1(E^y) = \lim_{k\to\infty} \mu_1((E_k)^y)$$

$$\Longrightarrow \quad f \ \mathcal{M}_2\text{-可测} \ \underline{\text{且}} \ f_k \nearrow f$$

$$\begin{aligned}
\int_{X_2} \mu_1(E^y)\, d\mu_2(y) &= \int_{X_2} f(y)\, d\mu_2(y) \\
&\overset{\text{MCT}}{=} \lim_{k\to\infty} \int_{X_2} f_k(y)\, d\mu_2(y) \\
&= \lim_{k\to\infty} \int_{X_2} \mu_1((E_k)^y)\, d\mu_2(y) \\
&\overset{E_k \in \mathcal{F}}{=} \lim_{k\to\infty} \mu_1 \times \mu_2 (E_k) \\
&\overset{\text{测度连续性}}{=} \mu_1 \times \mu_2 (E)
\end{aligned}$$

$$\Longrightarrow \quad E \in \mathcal{F}$$

类似地，$\mathcal{F} \ni E_k \downarrow E \implies E \in \mathcal{F}$

（这里用到 μ_1, μ_2 是有限测度）

4° 由单调类定理，$\mathcal{M}_1 \otimes \mathcal{M}_2 \subset \mathcal{F}$

<u>Step 2</u> 一般情形

$$X_1 = \bigcup_{k=1}^{\infty} A_k \quad \text{with} \quad \mu(A_k) < \infty, \quad A_k \nearrow X_1$$

$$X_2 = \bigcup_{k=1}^{\infty} B_k \quad \text{with} \quad \mu(B_k) < \infty, \quad B_k \nearrow X_2$$

$$\implies X_1 \times X_2 = \bigcup_{k=1}^{\infty} (A_k \times B_k)$$

$\mathcal{M}_1 \cap A_k$ 是 A_k 上 σ-代数

$\mu_1^{(k)} \overset{def}{=} \mu_1 \big|_{\mathcal{M}_1 \cap A_k}$ 是有限测度.

$\mu_2^{(k)} \overset{def}{=} \mu_2 \big|_{\mathcal{M}_2 \cap B_k}$ - - - - .

对 $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$,

$$\left(E \cap (A_k \times B_k)\right)^y = \begin{cases} E^y \cap A_k & \text{if } y \in B_k \\ \phi & \text{if } y \notin B_k \end{cases}$$

By Step 1,

$$\mu_1 \times \mu_2 (E \cap (A_k \times B_k)) \longrightarrow (\mu_1 \times \mu_2)(E) \text{ as } k \to \infty$$

(由測度連續性)

$$= \int_{X_2} \mu_1(E^y \cap A_k) \chi_{B_k}(y)\, d\mu_2(y)$$

$$\to \int_{X_2} \mu_1(E^y)\, d\mu_2(y) \quad \text{as } k \to \infty$$

(by MCT)

例: $\displaystyle\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \overset{?}{=} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn}$

由 Riemann 重排定理，上式一般來說不成立.

Q: 何時成立?

(HW)