第二十四讲 (2023.5.31)

Def 如果 $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. 对 $[a,b]$ 中任意有限个互不相交的开区间 $\{(a_k, b_k)\}_{k=1}^{N}$, 只要

$$\sum_{k=1}^{N} (b_k - a_k) < \delta,$$

就有

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \varepsilon$$

则称 f 在 $[a,b]$ 上绝对连续.

$$AC[a,b] \overset{\text{def}}{=\!=} \{ [a,b] \text{上绝对连续函数} \}$$

例: C-L 函数 $\notin AC[a,b]$

$$\underbrace{\text{Lipschitz 连续}}_{\text{i.e. } \exists L>0 \text{ s.t.}} \Longrightarrow \text{绝对连续}.$$

$$|f(x) - f(x')| \leq L|x - x'|, \quad \forall x, x' \in [a,b]$$

Prop 1 $AC[a,b]$ 是向量空间

Prop 2 $AC[a,b] \subset BV[a,b]$

Pf $\forall f \in AC[a,b]$

取 $\varepsilon = 1$, $\exists \delta > 0$, s.t. $\forall (a_k, b_k) \subset [a,b]$, $k = 1, 2, \cdots, N$ with $\sum_{k=1}^{N} (b_k - a_k) < \delta$, 有

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < 1 \qquad (*)$$

令

$$J \overset{def}{=} \left[\frac{b-a}{\delta} \right] + 1$$

$P: a = t_0 < t_1 < \cdots < t_J = b$,

$$t_j \overset{def}{=} a + \frac{b-a}{J} j, \quad j = 0, 1, 2, \cdots, J$$

$$t_j - t_{j-1} < \delta \underset{(*)}{\Longrightarrow} V_{t_{j-1}}^{t_j}(f) \leq 1.$$

$$\Longrightarrow V_a^b(f) = \sum_{j=1}^{J} V_{t_{j-1}}^{t_j}(f) \leq J$$

Cor: $f \in AC[a,b] \Longrightarrow \begin{cases} f \text{ a.e. 可微} \\ f' \in L^1[a,b] \end{cases}$

Prop 3. $f \in AC[a,b] \Longrightarrow f$ 把 $[a,b]$ 中任何零测集映为零测集 (HW, Ex. 19)

Prop 4 $\left.\begin{array}{l} f \in L^1[a,b] \\ F(x) \overset{\text{def}}{=} \int_a^x f(t)\,dt \end{array}\right\} \Longrightarrow F \in AC[a,b].$

Pf $f \in L^1[a,b]$

$\Longrightarrow \forall \varepsilon > 0,\ \exists \delta > 0,$ s.t.

$$\int_E |f|\,dm < \varepsilon, \quad \forall E \subset [a,b],\ m(E) < \delta$$

(积分的绝对连续性)

$\forall (a_k, b_k) \subset [a,b],\ k = 1, 2, \dots, N$ 互不相交,

with $\sum_{k=1}^{N} (b_k - a_k) < \delta$,

$$\Longrightarrow \sum_{k=1}^{N} |F(b_k) - F(a_k)|$$

$$\leq \sum_{k=1}^{N} \int_{a_k}^{b_k} |f(t)|\,dt$$

$$= \int_{\biguplus_{k=1}^{N}(a_k, b_k)} |f|\,dm < \varepsilon$$

Prop 5 $f \in AC[a,b] \Longrightarrow x \mapsto V_a^x(f) \in AC[a,b]$

且 $V_a^x(f) = \int_a^x |f'(t)|\,dt$

(HW. Ex.16)

__Thm 1__ $\left.\begin{array}{l} f\in AC[a,b] \\ f'=0 \ \ a.e. \end{array}\right\} \Longrightarrow f=\text{const}.$

Pf 假设 $f\neq \text{const}.$

$\Longrightarrow \exists\, c\in(a,b]$ s.t. $f(c)\neq f(a)$

令

$$\varepsilon_0 \overset{\text{def}}{=} \frac{1}{3}|f(c)-f(a)|$$

$\overset{AC}{\Longrightarrow} \exists\, \delta_0>0$, s.t. $\forall (a_j,b_j)\subset(a,c)$,

$j=1,2,\cdots n$, 互不相交，只要

$$\sum_{j=1}^{n}(b_j-a_j) < \delta_0$$

就有

$$\sum_{j=1}^{n}|f(b_j)-f(a_j)| < \varepsilon_0$$

令

$$E \overset{\text{def}}{=} \{x\in(a,c) : f'(x)=0\}$$

$\forall x\in E$,

$\overset{f'(x)=0}{\Longrightarrow} \forall \eta>0$, $\exists\, h_x^{(k)}>0$ with $[x, x+h_x^{(k)}]\subset(a,c)$

s.t.

$$|f(x+h_x^{(k)})-f(x)| < \eta\, h_x^{(k)}, \qquad k=1,2,\cdots$$

$$\Rightarrow \quad \Gamma \overset{\text{def}}{=} \{[x, x+h_x^{(k)}]\}_{x\in E,\, k\in\mathbb{N}}$$ 是 E 的 Vitali 覆盖.

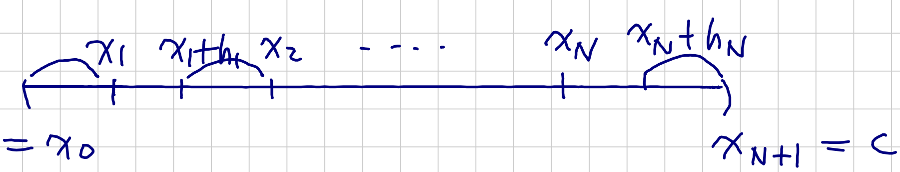
$$\overset{\text{Vitali}}{\Longrightarrow} \quad \exists\ [x_1, x_1+h_1], \cdots, [x_N, x_N+h_N] \in \Gamma,$$

互不相交, s.t.

$$m\Big((a,c)\setminus \bigcup_{k=1}^{N}[x_k, x_k+h_k]\Big) = m\Big(E\setminus \bigcup_{k=1}^{N}[x_k, x_k+h_k]\Big) < \delta_0$$

不妨设这些区间依次排列

$$a = x_0 < x_1 < x_1+h_1 < x_2 < \cdots < x_N+h_N < x_{N+1} = c$$



$$\Big((a,c)\setminus \bigcup_{k=1}^{N}[x_k, x_k+h_k] = (x_0,x_1)\cup\cdots\cup(x_N+h_N, x_{N+1}) \Big)$$

$$\Rightarrow \quad \sum_{k=0}^{N}[x_{k+1}-(x_k+h_k)] < \delta_0 \qquad (令\ h_0=0)$$

$$\overset{\text{AC}}{\Longrightarrow} \quad \sum_{k=0}^{N}|f(x_{k+1})-f(x_k+h_k)| < \varepsilon_0$$

$$\Rightarrow \quad 3\varepsilon_0 = |f(c) - f(a)|$$

$$\leq \sum_{k=0}^{N} |f(x_{k+1}) - f(x_k + h_k)| + \sum_{k=1}^{N} |f(x_k + h_k) - f(x_k)|$$

$$< \varepsilon_0 + \eta \sum_{k=1}^{N} h_k$$

$$< \varepsilon_0 + (b-a)\eta$$

$$\xrightarrow[\text{s.t. } (b-a)\eta < \varepsilon_0]{\text{取 } \eta} \quad 3\varepsilon_0 < 2\varepsilon_0, \quad 矛盾.$$

Thm 2 (i) $F \in AC[a,b] \Rightarrow F$ a.e. 可微,

$F' \in L^1[a,b]$ 且

$$F(x) - F(a) = \int_a^x F'(t)\,dt, \quad x \in [a,b]$$

(ii) $f \in L^1[a,b] \Rightarrow \exists F \in AC[a,b]$

s.t. $F' = f$ a.e.

事实上, $F(x) \stackrel{\text{def}}{=} \int_a^x f(t)\,dt$ 满足条件.

Rmk. $N-L \Leftrightarrow F \in AC[a,b]$

Pf: (i)

$F \in AC[a,b] \implies F$ a.e. 可微 且 $F' \in L^1[a,b]$

令 $G(x) \overset{\text{def}}{=} \int_a^x F'(t)\,dt$, $x \in [a,b]$

$\overset{\text{LDT}}{\implies} G' = F'$ a.e.

$\text{Prop 4} \implies G \in AC[a,b]$

$\implies F - G \in AC[a,b]$

且 $(F-G)' = 0$ a.e.

$\overset{\text{Thm 1}}{\implies} F - G = \text{const}$

令 $C \overset{\text{def}}{=} F(x) - \int_a^x F'(t)\,dt$

$\overset{\text{取 } x=a}{\implies} C = F(a)$

$\implies \int_a^x F'(t)\,dt = F(x) - F(a)$.

(ii) Prop 4 + LDT

HW: Ex. 20, 32.