# 第二十讲（2023.5.17）

Thm (LDT)

$$f \in L^1_{loc} \Longrightarrow \lim_{r\to 0^+} \frac{1}{m(B_r(x))}\int_{B_r(x)} f\,dm = f(x) \text{ for a.e. } x$$

令 $\varphi \overset{def}{=} \frac{1}{v_n}\chi_{B_1(0)}$, $\quad v_n \overset{def}{=} m(B_1(0)) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$

$$\Longrightarrow \int \varphi\,dm = 1.$$

$$\varphi_t(x) \overset{def}{=} t^{-n}\varphi(t^{-1}x)$$

$$\Longrightarrow \quad (f*\varphi_t)(x) = \frac{1}{v_n t^n}\int f(x-y)\chi_{B_1(0)}(t^{-1}y)\,dy$$

$$= \frac{1}{v_n t^n}\int_{B_t(0)} f(x-y)\,dy$$

$$= \frac{1}{m(B_r(x))}\int_{B_r(x)} f(y)\,dy$$

LDT $\Longleftrightarrow$ $\forall f \in L^1_{loc}$, $\quad f*\varphi_t \to f$ a.e. as $t\to 0^+$.

恒等之逼近 (Approximations to the Identity)

**Def** 设 $\{K_t\}_{t>0}$ 是一族可测函数. s.t.

(A1) $\int K_t \, dm = 1$

(A2) $\exists C_1 > 0$ s.t.

$$|K_t(x)| \le \frac{C_1}{t^n}, \quad \forall t \in (0,1)$$

(A3) $\exists C_2 > 0$, s.t.

$$|K_t(x)| \le \frac{C_2 t}{|x|^{n+1}}, \quad \forall t > 0, \ \forall x \in \mathbb{R}^n \setminus \{0\}$$

则称 $\{K_t\}_{t>0}$ 是 A.I.

**Thm** 设 $\{K_t\}_{t>0}$ 是 A.I.

$\forall f \in L^1$, $f * K_t \to f$ a.e. as $t \to 0^+$

**Lem** 设 $f \in L^1$, $x \in L_f$

$$g(r) \overset{\text{def}}{=} \frac{1}{r^n} \int_{|y| \le r} |f(x-y) - f(x)| \, dy$$

则

(i) $g$ 在 $(0, \infty)$ 上连续, 且 $\lim_{r \to 0^+} g(r) = 0$.

(ii) $g$ 有界

Pf (i) $\forall r \in (0, \infty)$,

$$g(r+h) - g(r)$$

$$= \underbrace{\frac{1}{(r+h)^n} \int_{B_{r+h}(0) \setminus B_r(0)} |f(x-y) - f(x)|\, dy}_{\longrightarrow 0 \text{ as } h \to 0} \quad (\text{積分之絕對連續性})$$

$$+ \underbrace{\left[ \frac{1}{(r+h)^n} - \frac{1}{r^n} \right]}_{\longrightarrow 0 \text{ as } h \to 0} \int_{B_r(0)} |f(x-y) - f(x)|\, dy$$

$$\lim_{r \to 0^+} g(r) = v_n \lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\, dy$$

$$= 0 \quad (\because\ x \in L_f)$$

(ii) $\left.\begin{array}{l} g \in C(0, 1] \\ g(0+) \text{ 存在} \end{array}\right\} \Rightarrow g$ 在 $(0, 1]$ 上有界

若 $r > 1$,

$$g(r) \le \frac{1}{r^n} \int_{|y| \le r} |f(x-y)|\, dy + v_n |f(x)|$$

$$\le \|f\|_1 + v_n |f(x)| .$$

Pf of Thm

$$|(f*K_t)(x)| \leq \int |f(x-y)-f(x)|\,|K_t(y)|\,dy$$

$$= \int_{|y|\leq t} + \sum_{k=0}^{\infty} \int_{2^k t<|y|\leq 2^{k+1}t}$$

$$\int_{|y|\leq t} |f(x-y)-f(x)|\,|K_t(y)|\,dy$$

$$\overset{(A2)}{\leq} \frac{C_1}{t^n} \int_{|y|\leq t} |f(x-y)-f(x)|\,dy$$

$$= C_1\, g(t)$$

$$\int_{2^k t<|y|\leq 2^{k+1}t} \overset{(A3)}{\leq} \frac{C_2 t}{(2^k t)^{n+1}} \int_{|y|\leq 2^{k+1}t} |f(x-y)-f(x)|\,dy$$

$$= \frac{C_2 \cdot 2^n}{2^k} \frac{1}{(2^{k+1}t)^n} \int_{|y|\leq 2^{k+1}t} |f(x-y)-f(x)|\,dy$$

$$= \frac{C_3}{2^k}\, g(2^{k+1}t)$$

$$\Longrightarrow \quad |(f*K_t)(x) - f(x)| \leq C\left[g(t) + \sum_{k=0}^{\infty} \frac{1}{2^k} g(2^{k+1}t)\right]$$

令 $M \stackrel{\text{def}}{=} \sup\limits_{t\in(0,\infty)} g(t)$

$\forall \varepsilon > 0$, $\exists N$, s.t.

$$\sum_{k=N}^{\infty} \frac{1}{2^k} < \varepsilon$$

则当 $t$ 充分小时

$$g(t) < \varepsilon, \quad g(2^{k+1}t) < \frac{\varepsilon}{N}, \quad k = 0, 1, 2, \dots, N-1$$

$$\left(\because \lim_{t\to 0^+} g(t) = 0\right)$$

$$\Longrightarrow \quad |(f*K_t)(x) - f(x)| \leq C\left[2\varepsilon + M\varepsilon\right]$$

(当 $t$ 充分小)

<u>Thm</u> 设 $\{K_t\}_{t>0}$ 是 A.I.

$\forall f \in L^1$, $\|f*K_t - f\|_1 \to 0$ as $t \to 0^+$

Lem（平均连续性）

设 $1 \le p < \infty$, $f \in L^p$, 则

$$\|\tau_h f - f\|_p \to 0 \quad \text{as} \quad h \to 0$$

其中 $(\tau_h f)(x) \overset{\text{def}}{=} f(x-h)$

Pf Step 1 先假设 $f \in C_c(\mathbb{R}^n)$

当 $|h| < 1$ 时 $\operatorname{supp}(\tau_h f) \subset \underbrace{\operatorname{supp}(f) + \overline{B_1(0)}}_{\text{记为 } K}$

$$\int |(\tau_h f)(x) - f(x)|^p \, dx$$

$$\le \left[\sup_K |(\tau_h f)(x) - f(x)|\right]^p m(K)$$

$\to 0$ as $h \to 0$ (by 一致连续性)

Step 2 一般情形

$\forall f \in L^p$, $\forall \varepsilon > 0$, $\exists g \in C_c(\mathbb{R}^n)$ s.t.

$$\|f - g\|_p < \varepsilon/3$$

$$\Rightarrow \|\tau_h f - f\|_p \le \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p$$

$$= \underbrace{2\|f-g\|_p}_{<\frac{2}{3}\varepsilon} + \underbrace{\|\tau_h g - g\|_p}_{<\frac{\varepsilon}{3},\ \text{当}|h|\text{充分小}}$$

**Pf of Thm**

首先 $\|K_t\|_1 = \displaystyle\int_{|y|\le t} |K_t|\,dy + \int_{|y|\ge t} |K_t|\,dy$

$$\le \underbrace{\int_{|y|\le t} \frac{C_1}{t^n}\,dy}_{\le \frac{C_1}{t^n} v_n t^n} + \underbrace{\int_{|y|\ge t} \frac{C_2\, t}{|y|^{n+1}}\,dy}_{= \int_{|x|\ge 1} \frac{C_2 t\cdot t^n}{t^{n+1}|x|^{n+1}}\,dx}$$

$$\le C$$

$$\|f * K_t - f\|_1$$

$$= \int \left| \int [f(x-y) - f(x)]\, K_t(y)\,dy \right| dx$$

Tonelli:

$$\le \int \left[ \int |f(x-y) - f(x)|\,dx \right] |K_t(y)|\,dy$$

$$= \int \|\tau_y f - f\|_1\, |K_t(y)|\,dy$$

<u>由 Lem</u>

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$\|\tau_y f - f\|_1 < \varepsilon, \quad \forall y \in B_\delta(0)$$

$$\Longrightarrow \|f * k_t - f\|_1$$

$$\leq \underbrace{\int_{|y|<\delta} \|\tau_y f - f\|_1 |k_t(y)| \, dy}_{\leq C\varepsilon}$$

$$+ \underbrace{\int_{|y|\geq\delta} \|\tau_y f - f\|_1 |k_t(y)| \, dy}_{\leq 2\|f\|_1 \underbrace{\int_{|y|\geq\delta} |k_t(y)| \, dy}_{\leq \int_{|y|\geq\delta} \frac{C_2 t}{|y|^{n+1}} dy}}$$

$< \varepsilon$ 当 $t$ 充分小

<u>HW</u>: Ex. 1 (b)

A. I 之例子

上半平面之 poisson 核.

$$K_y(x) \overset{def}{=} \frac{1}{\pi} \frac{y}{x^2+y^2}, \quad x \in \mathbb{R},\ y>0$$

$$K(x) \overset{def}{=} \frac{1}{\pi} \frac{1}{1+x^2}$$

$$K_y(x) = y^{-1} K(y^{-1}x)$$

$$\int_{\mathbb{R}} K(x)\,dx = \frac{1}{\pi} \arctan x \Big|_{-\infty}^{\infty} = 1.$$

$$\Rightarrow \int K_y\,dx = 1.$$

$$|K_y(x)| \leq \frac{1}{\pi} \frac{1}{y}, \quad \forall y>0$$

$$|K_y(x)| \leq \frac{1}{\pi} \cdot \frac{y}{x^2}, \quad \forall x \in \mathbb{R}\setminus\{0\}.\ \forall y>0$$

$\forall f \in L^1$

$$\left.\begin{array}{l} \|f*K_y - f\|_1 \to 0 \\ f*K_y \to f \quad a.e. \end{array}\right\} \text{as } y \to 0^+$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial y^2}\right) K_y(x) = 0$$

$$(Pf)(x,y) \overset{\text{def}}{=} (f * K_y)(x) \text{ 调和 in } \mathbb{R}^2_+ \text{，且}$$

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+ \\ u = f & \text{on } \mathbb{R}\times\{0\}. \end{cases}$$

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