第十九讲 (2023.5.12)

Thm (LDT)

$$f \in L^1_{loc} \Rightarrow \lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} f \, dm = f(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Thm (H-L)

$\exists C > 0$, s.t.

$$m(\{Mf > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1, \quad \forall \alpha > 0, \ \forall f \in L^1.$$

where

$$Mf(x) \overset{def}{=} \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f| \, dm$$

Thm (Vitali 覆盖引理 1)

$\forall \mathcal{B} \overset{def}{=} \{B_1, \cdots, B_N\}$, $\exists B_{k_1}, \cdots, B_{k_p} \in \mathcal{B}$ 互不相交, s.t.

$$\sum_{j=1}^{p} m(B_{k_j}) \geq \frac{1}{3^n} m\left(\bigcup_{k=1}^{N} B_k\right)$$

Pf of H-L

令 $E_\alpha \overset{def}{=} \{Mf > \alpha\}$.

$\forall x \in E_\alpha$, $\exists\, r_x$ s.t.

$$\frac{1}{m(B_{r_x}(x))} \int_{B_{r_x}(x)} |f|\,dm > \alpha .$$

$$\Longrightarrow \quad m(B_{r_x}(x)) < \frac{1}{\alpha} \int_{B_{r_x}(x)} |f|\,dm .$$

$\forall K \subset\subset E_\alpha \subset \bigcup_{x \in E_\alpha} B_{r_x}(x)$, $\exists\, B_1, \cdots, B_N \in \{B_{r_x}(x)\}_{x \in E_\alpha}$

s.t.

$$K \subset \bigcup_{k=1}^{N} B_k .$$

Vitali

$\Longrightarrow$ $\exists\, B_{k_1}, \cdots, B_{k_p} \in \{B_1, \cdots, B_N\}$, 互不相交 s.t.

$$m\Big(\bigcup_{k=1}^{N} B_k\Big) \leq 3^n \sum_{j=1}^{p} m(B_{k_j})$$

$$\Longrightarrow \quad m(K) \leq 3^n \sum_{j=1}^{p} m(B_{k_j})$$

$$\leq 3^n \sum_{j=1}^{p} \frac{1}{\alpha} \int_{B_{k_j}} |f|\,dm$$

$$= \frac{3^n}{\alpha} \int_{\bigcup_{j=1}^{p} B_{k_j}} |f|\,dm \leq \frac{3^n}{\alpha} \|f\|_1$$

$$\Longrightarrow \quad m(E_\alpha) = \sup \{ m(K) : K \subset\subset E_\alpha \}$$

$$\leq \frac{3^n}{\alpha} \|f\|_1$$

Pf of LDT

Step 1. 先假设 $f \in C(\mathbb{R}^n)$

$\forall x \in \mathbb{R}^n, \ \forall \varepsilon > 0, \ \exists \delta > 0$, s.t.

$$|f(y) - f(x)| < \varepsilon, \quad \forall y \in B_\delta(x).$$

$\Rightarrow \ \forall r < \delta$,

$$\left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dy - f(x) \right|$$

$$\leq \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)|\,dy < \varepsilon$$

Step 2 设 $f \in L^1_{loc}$.

不妨设 $f \in L^1$ (否则代以 $f\chi_B$)

令

$$E \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f\,dm - f(x) \right| > 0 \right\}$$

只需证 $m(E) = 0$

令

$$E_\alpha \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f\,dm - f(x) \right| > 2\alpha \right\}$$

$$\Rightarrow E = \bigcup_{k=1}^{\infty} E_{\frac{1}{k}}.$$

约证为

Claim $\forall \alpha > 0$, $m(E_\alpha) = 0$.

$\forall \varepsilon > 0$, $\exists\, g \in C_c(\mathbb{R}^n)$, s.t.

$$\|f - g\|_1 < \varepsilon \qquad \left(\because\ C_c(\mathbb{R}^n) \overset{\text{dense}}{\subset} L^1\right)$$

$$\Rightarrow \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dy - f(x) \right|$$

$$\leq \underbrace{\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - g(y)|\,dy}_{\leq M(f-g)(x)}$$

$$+ \underbrace{\left| \frac{1}{m(B_r(x))} \int_{B_r(x)} g(y)\,dy - g(x) \right|}_{\to 0 \text{ as } r \to 0^+ \text{ (by step 1)}} + |g(x) - f(x)|$$

$$\Rightarrow \limsup_{r \to 0^+} \left| \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dy - f(x) \right|$$

$$\leq M(f-g)(x) + |f(x) - g(x)|$$

令 $F_\alpha \overset{\text{def}}{=} \{ M(f-g) > \alpha \}$

$G_\alpha \overset{\text{def}}{=} \{ |f - g| > \alpha \}$

$\Rightarrow E_\alpha \subset F_\alpha \cup G_\alpha$

$$\text{H-L} \Rightarrow m(F_\alpha) \leq \frac{3^n}{\alpha}\|f-g\|_1 < \frac{3^n}{\alpha}\varepsilon$$

$$\text{Tchebyshev} \Rightarrow m(G_\alpha) \leq \frac{1}{\alpha}\|f-g\|_1 < \frac{1}{\alpha}\varepsilon$$

$$\Rightarrow m(E_\alpha) < \frac{3^n+1}{\alpha}\varepsilon.$$

Def 设 $E \in \mathcal{L}$. 如果

$$\lim_{r\to 0^+} \frac{m(E\cap B_r(x))}{m(B_r(x))} = 1,$$

称 x 是 E 的一个 Lebesgue 密度点.

Cor. 设 $E \in \mathcal{L}$.

1° a.e. $x \in E$ 都是 E 的密度点

2° a.e. $x \in E^c$ 都不是 ...

Pf

$$\frac{m(E\cap B_r(x))}{m(B_r(x))} = \frac{1}{m(B_r(x))}\int_{B_r(x)} \chi_E \, dm$$

$$\to \chi_E(x) \quad \text{as } r \to 0^+$$

for a.e. $x \in \mathbb{R}^n$.

Thm

$$f \in L^1_{loc} \Rightarrow \underbrace{\lim_{r\to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| = 0}_{\text{这样的 } x \text{ 称为 } f \text{ 的 Lebesgue 点.}} \quad \text{for a.e. } x$$

Thm' $\quad f \in L^1_{loc} \Rightarrow$ a.e. $x \in \mathbb{R}^n$ 都是 f 的 Lebesgue 点

Pf 令

$$L_f \overset{def}{=} \{ f \text{ 的 Lebesgue 点} \}$$

要证明 $m(\mathbb{R}^n \setminus L_f) = 0$.

$\forall q \in \mathbb{Q}$, $\exists E_q \subset \mathbb{R}^n$ with $m(E_q) = 0$, s.t.

$$\lim_{r\to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - q| \, dy = |f(x) - q|,$$

$$\forall x \in \mathbb{R}^n \setminus E_q$$

令 $\quad E \overset{def}{=} \bigcup_{q\in\mathbb{Q}} E_q$

$\Rightarrow m(E) = 0$.

Claim $\quad \mathbb{R}^n \setminus E \subset L_f$

$\forall x \in \mathbb{R}^n \setminus E = \bigcap_{q \in \mathbb{Q}} (\mathbb{R}^n \setminus E_q)$,

不妨设 $f(x)$ 有限 ($\because$ f a.e. 有限)

$\forall \varepsilon > 0$, $\exists q \in \mathbb{Q}$, s.t.

$$|f(x) - q| < \frac{\varepsilon}{2}$$

$$\Rightarrow \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy$$

$$\leq \underbrace{\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - q| \, dy}_{\to |f(x) - q| \text{ as } r \to 0^+} + |q - f(x)|$$

$$\Rightarrow \limsup_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy$$

$$= 2|f(x) - q| < \varepsilon$$

$$\Rightarrow \lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0.$$

Def 设 $x \in \mathbb{R}^n$，如果 $\mathcal{F}_x \subset \mathcal{L}$ 满足

(i) $\forall \varepsilon > 0$，$\exists E \in \mathcal{F}_x$ s.t. $\operatorname{diam} E < \varepsilon$

(ii) $\exists c > 0$, s.t.

$$m(E) > c\, m(B^E(x)), \quad \forall E \in \mathcal{F}_x$$

这里 $B^E(x)$ 是以 x 为心，包含 E 之最小开球.

则称 $\mathcal{F}_x$ 正则收缩于 x.

例：

$\{B : B \ni x\}$

$\{Q : Q \ni x\}$

$\{B_{2r}(x) \setminus B_r(x)\}_{r>0}$

正则收缩于 x

$\{R : R$ 矩体, $R \ni x\}$ 不正则收缩于 x

$\{R : R$ 为 $\mathbb{R}^2$ 中矩体, 长.宽比固定, $R \ni x\}$ 正则收缩于 x

Cor 设 $f \in L^1_{loc}$, $x \in L_f$, $\mathcal{F}_x$ 正则收缩于 x

则

$$\lim_{\substack{\operatorname{diam}(E) \to 0 \\ E \in \mathcal{F}_x}} \frac{1}{m(E)} \int_E f\, dm = f(x).$$

Pf $\forall E \in \mathcal{F}_x$

$$\frac{1}{m(E)} \int_E |f(y) - f(x)|\, dy$$

$$\le \frac{1}{c} \frac{1}{m(B^E(x))} \int_{B^E(x)} |f(y) - f(x)|\, dy$$

$\to 0$ as $\operatorname{diam} B^E(x) \to 0$

<u>Cor</u> $f \in L^1_{loc} \implies \lim_{\substack{m(B)\to 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f\, dm = f(x)$

for a.e. $x \in \mathbb{R}^n$

<u>Cor</u> $f \in L^1_{loc}(\mathbb{R}^1) \implies \forall$ a.e. $x \in \mathbb{R}^1$,

$$\lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(y)\, dy = f(x)$$

$\implies F' = f$ a.e.

where $F(x) = \int_a^x f(t)\, dt$

<u>HW</u>: Ex 4. (P. 146)