第十七讲 (2023.5.6)

<u>Cor</u> 设 $E \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$.

(i) 对 a.e. $y \in \mathbb{R}^{n_2}$, $E^y \in \mathcal{L}_{\mathbb{R}^{n_1}}$,

对 a.e. $x \in \mathbb{R}^{n_1}$, $E_x \in \mathcal{L}_{\mathbb{R}^{n_2}}$.

(ii) $y \longmapsto m_{n_1}(E^y)$ 在 $\mathbb{R}^{n_2}$ 上可测

$x \longmapsto m_{n_2}(E_x)$ 在 $\mathbb{R}^{n_1}$ - - -

(iii)
$$m(E) = \int_{\mathbb{R}^{n_2}} m(E^y)\,dy = \int_{\mathbb{R}^{n_1}} m(E_x)\,dx.$$

Q: $\forall y \in \mathbb{R}^{n_2}$, $E^y \in \mathcal{L}_{\mathbb{R}^{n_1}}$ $\overset{?}{\not\Rightarrow}$ $E \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

反例: $E \overset{\text{def}}{=} [0,1] \times A$ with $A \subset \mathbb{R}^1$ 不可测

$$E^y = \begin{cases} [0,1], & \text{if } y \in A \\ \phi, & \text{if } y \in \mathbb{R}^1 \setminus A \end{cases}$$

但 $E \notin \mathcal{L}_{\mathbb{R}^2}$

否则 对 a.e. $x \in [0,1]$, $E_x \in \mathcal{L}_{\mathbb{R}^1}$

而当 $x \in [0,1]$ 时 $E_x = A$.

Def 对 $E_1 \subset \mathbb{R}^{n_1}$ 和 $E_2 \subset \mathbb{R}^{n_2}$

$$E_1 \times E_2 \overset{def}{=} \{(x, y) \in \mathbb{R}^{n_1+n_2} : x \in E_1, y \in E_2\}$$

称为 E_1 与 E_2 之乘积集合.

Prop 1 $\left.\begin{array}{l} E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}} \\ m_{n_2}^*(E_2) > 0 \end{array}\right\} \Longrightarrow E_1 \in \mathcal{L}_{\mathbb{R}^{n_1}}$

Pf $E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

$\Longleftrightarrow \chi_{E_1 \times E_2} \in L^+(\mathbb{R}^{n_1+n_2})$

$\overset{\text{Tonelli}}{\Longrightarrow} (\chi_{E_1 \times E_2})^y \in L^+(\mathbb{R}^{n_1})$ 对 a.e. $y \in \mathbb{R}^{n_2}$

而 $(\chi_{E_1 \times E_2})^y(x) = \chi_{E_1 \times E_2}(x, y) = \chi_{E_1}(x)\chi_{E_2}(y)$

只需证明: $\exists\, y \in E_2$ s.t. $(\chi_{E_1 \times E_2})^y \in L^+(\mathbb{R}^{n_1})$

令 $F \overset{def}{=} \{y \in \mathbb{R}^{n_2} : (E_1 \times E_2)^y \in \mathcal{L}_{\mathbb{R}^{n_1}}\}$

$\overset{\text{Cor}}{\Longrightarrow} m_{n_2}(F^c) = 0$

而 $E_2 = (E_2 \cap F) \cup (E_2 \cap F^c)$

$$\Rightarrow \quad 0 < m^*_{n_2}(E_2) \leq m^*_{n_2}(E_2 \cap F) + \underbrace{m^*_{n_2}(E_2 \cap F^c)}_{=0}$$

$$\Rightarrow \quad m^*_{n_2}(E_2 \cap F) > 0$$

$$\Rightarrow \quad E_2 \cap F \neq \phi$$

Prop 2

$$\left.\begin{array}{l} E_1 \in \mathcal{L}_{\mathbb{R}^{n_1}} \\ E_2 \in \mathcal{L}_{\mathbb{R}^{n_2}} \end{array}\right\} \Rightarrow \begin{array}{l} E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}} \\ m_{n_1+n_2}(E_1 \times E_2) = m_{n_1}(E_1) m_{n_2}(E_2) \end{array}$$

Lem 1 $\forall E_1 \subset \mathbb{R}^{n_1}, \ \forall E_2 \subset \mathbb{R}^{n_2}$

$$m^*_{n_1+n_2}(E_1 \times E_2) \leq m^*_{n_1}(E_1)\, m^*_{n_2}(E_2)$$

Pf. Case 1 $m^*_{n_1}(E_1) = +\infty, \ m^*_{n_2}(E_2) \neq 0$

或 $m^*_{n_1}(E_1) \neq 0, \ m^*_{n_2}(E_2) = +\infty$

平凡

Case 2 $m^*_{n_1}(E_1), m^*_{n_2}(E_2) < \infty$

$\forall \varepsilon > 0$, 存在 E_1 之方体覆盖 $\{Q^{(1)}_j\}_{j=1}^{\infty}$

E_2 - - - - $\{Q^{(2)}_k\}_{k=1}^{\infty}$

s. t.

$$\sum_{j=1}^{\infty} |Q_j^{(1)}| < m_{n_1}^*(E_1) + \varepsilon$$

$$\sum_{k=1}^{\infty} |Q_k^{(2)}| < m_{n_2}^*(E_2) + \varepsilon$$

$\Longrightarrow$ $\{Q_j^{(1)} \times Q_k^{(2)}\}_{j,k}$ 是 $E_1 \times E_2$ 的矩体覆盖.

(Note: Ex. 15 chpt 1)

$$\Longrightarrow m_{n_1+n_2}^*(E_1 \times E_2) \le \sum_{j,k} |Q_j^{(1)} \times Q_k^{(2)}|$$

$$= \left(\sum_{j=1}^{\infty} |Q_j^{(1)}|\right)\left(\sum_{k=1}^{\infty} |Q_k^{(2)}|\right)$$

$$< \left(m_{n_1}^*(E_1) + \varepsilon\right)\left(m_{n_2}^*(E_2) + \varepsilon\right)$$

$\varepsilon \to 0^+$

$$\Longrightarrow m_{n_1+n_2}^*(E_1 + E_2) \le m_{n_1}^*(E_1)\, m_{n_2}^*(E_2)$$

<u>Case 3</u> $m_{n_1}^*(E_1) = +\infty$, $m_{n_2}^*(E_2) = 0$

或 $m_{n_1}^*(E_2) = 0$, $m_{n_2}^*(E_2) = +\infty$

只考虑前者

$$E_1 = \bigcup_{k=1}^{\infty} E_1^{(k)} \quad \text{with } E_1^{(k)} \overset{\text{def}}{=} B_k(0) \cap E_1$$

$$\Longrightarrow E_1 \times E_2 = \bigcup_{k=1}^{\infty} \left(E_1^{(k)} \times E_2\right)$$

$$\Longrightarrow m^*_{n_1+n_2}(E_1 \times E_2) \leq \sum_{k=1}^{\infty} m^*_{n_1+n_2}(E_1^{(k)} \times E_2)$$

$$\overset{\text{Case 2}}{\leq} \sum_{k=1}^{\infty} \underbrace{m^*_{n_1}(E_1)\, m^*_{n_2}(E_2)}_{=0}$$

$$= 0 = m^*_{n_1}(E_1)\, m^*_{n_2}(E_2)$$

Pf of Prop 2

只证明 $E_1 \times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$

(然后用 Fubini)

$$E_1 \in \mathcal{L}_{\mathbb{R}^{n_1}} \Longleftrightarrow E_1 = G_1 \setminus Z_1$$

G_1: $\mathbb{R}^{n_1}$ 中 G_δ 集　　Z_1: $\mathbb{R}^{n_1}$ 中零集

$$E_2 \in \mathcal{L}_{\mathbb{R}^{n_2}} \Longleftrightarrow E_2 = G_2 \setminus Z_2$$

G_2: $\mathbb{R}^{n_2}$ 中 G_δ 集　　Z_2: $\mathbb{R}^{n_2}$ 中零集

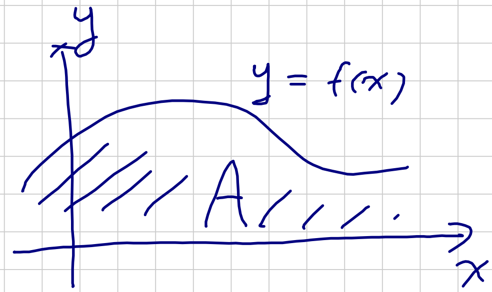
$G_1 \times G_2$ 是 $\mathbb{R}^{n_1+n_2}$ 中 G_δ 集

$$(G_1 \times G_2) \setminus (E_1 \times E_2)$$
$$\subset \Big[\underbrace{(G_1 \setminus E_1) \times G_2}_{\text{零集 (by Lem 1)}}\Big] \cup \Big[G_1 \times \underbrace{(G_2 \setminus E_2)}_{\text{零集}}\Big]$$

$$\Longrightarrow \quad m^*_{n_1+n_2}\Big(\big((G_1\times G_2)\setminus(E_1\times E_2)\big)\Big)=0$$

$$\Longrightarrow \quad E_1\times E_2 \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$$

积分的几何意义



Prop 3 设 f 是 $\mathbb{R}^n$ 上非负函数

$$A \overset{\text{def}}{=} \{(x,y)\in\mathbb{R}^{n+1} : 0\le y\le f(x)\}$$

(i) $f\in L^+(\mathbb{R}^n) \iff A\in\mathcal{L}_{\mathbb{R}^{n+1}}$

(ii) $f\in L^+(\mathbb{R}^n) \Longrightarrow \int_{\mathbb{R}^n} f(x)\,dx = m_{n+1}(A)$

Lem 2 设 f 是 $\mathbb{R}^{n_1}$ 上可测函数，令

$$\tilde f(x,y) \overset{\text{def}}{=} f(x), \quad (x,y)\in\mathbb{R}^{n_1+n_2}$$

$\Longrightarrow$ $\tilde f$ 是 $\mathbb{R}^{n_1+n_2}$ 上可测函数.

Pf $\forall a\in\mathbb{R}$,

$$E(a) \overset{\text{def}}{=} \{x\in\mathbb{R}^{n_1} : f(x)<a\} \in \mathcal{L}_{\mathbb{R}^{n_1}}$$

$$\Rightarrow \{(x, y) \in \mathbb{R}^{n_1+n_2} : \tilde{f}(x, y) < a\}$$

$$= E(a) \times \mathbb{R}^{n_2} \in \mathcal{L}_{\mathbb{R}^{n_1+n_2}}$$

$\uparrow$

(by prop 2)

<u>Pf of Prop 3</u>

(i) " $\Longrightarrow$ "

$$f \in L^+(\mathbb{R}^{n_1}) \xrightarrow{\text{Lem 2}} F(x, y) \overset{\text{def}}{=} y - f(x)$$

在 $\mathbb{R}^{n+1}$ 上可测

$$\Rightarrow A = \{y \geq 0\} \cap \{F \leq 0\} \in \mathcal{L}_{\mathbb{R}^{n+1}}$$

" $\Longleftarrow$ "

$\forall x \in \mathbb{R}^n$, $\quad A_x = [0, f(x)] \in \mathcal{L}_{\mathbb{R}^1}$

且 $\quad x \longmapsto m_1(A_x) \in L^+(\mathbb{R}^n)$ (by Cor)

而 $\quad f(x) = m_1(A_x)$

$$\Rightarrow f \in L^+(\mathbb{R}^n)$$

(ii)

$$m_{n+1}(A) \underset{\text{by Cor}}{=} \int_{\mathbb{R}^n} m_1(A_x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx$$

<u>Prop 4</u> $\quad f$ 在 $\mathbb{R}^n$ 上可测

$$\Longrightarrow (x, y) \longmapsto f(x-y) \text{ 在 } \mathbb{R}^{2n} \text{ 上可测}$$

<u>Pf</u> $\quad \forall a \in \mathbb{R}$

$$E(a) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : f(x) > a \} \in \mathcal{L}_{\mathbb{R}^n}$$

须证

$$\tilde{E}(a) \overset{\text{def}}{=} \{ (x, y) \in \mathbb{R}^{2n} : f(x-y) > a \} \in \mathcal{L}_{\mathbb{R}^{2n}}$$

令

$$\Phi : \mathbb{R}^{2n} \to \mathbb{R}^n$$

$$(x, y) \mapsto x - y$$

$$\Longrightarrow \tilde{E}(a) = \Phi^{-1}(E(a))$$

<u>只须证</u>

<u>claim</u> $\quad \Phi^{-1}(E) \in \mathcal{L}_{\mathbb{R}^{2n}}, \quad \forall E \in \mathcal{L}_{\mathbb{R}^n}$

(1) 证 Φ 连续

Step 1 $\quad \forall\, G \overset{G_\delta}{\subset} \mathbb{R}^n, \quad \Phi^{-1}(G) \overset{G_\delta}{\subset} \mathbb{R}^{2n}$

$$G = \bigcap_{k=1}^{\infty} \underbrace{G_k}_{\text{开集}}$$

$$\Rightarrow \Phi^{-1}(G) = \bigcap_{k=1}^{\infty} \underbrace{\Phi^{-1}(G_k)}_{\mathbb{R}^{2n}\text{ 中开集}}$$

$\Rightarrow \Phi^{-1}(G)$ 是 $\mathbb{R}^{2n}$ 中 G_δ 集

Step 2 $\quad \forall\, Z \overset{null}{\subset} \mathbb{R}^n, \quad \Phi^{-1}(Z) \overset{null}{\subset} \mathbb{R}^{2n}$

$m_n(Z) = 0$

$\Rightarrow \exists\, G \subset \mathbb{R}^n$, G_δ 集 s.t. $Z \subset G$

且 $m_n(G) = 0$

令 $\quad \tilde{G} \overset{def}{=} \Phi^{-1}(G)$

$\overset{\text{Step 1}}{\Longrightarrow} \quad \tilde{G}$ 是 $\mathbb{R}^{2n}$ 中 G_δ 集

v)

$$m_{2n}(\tilde{G}) = \int_{\mathbb{R}^{2n}} \chi_{\tilde{G}} \, dm_{2n}$$

$$\overset{\text{Fubini}}{=\!=} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} (\chi_{\tilde{G}})^y \, dx \right] dy$$

$\Rightarrow$ |i-ξ

$$(\chi_{\tilde{G}})^y(x) = 1 \iff (x, y) \in \Phi^{-1}(G)$$

$$\iff x - y \in G$$

$$\iff x \in G + y$$

$$\iff \chi_{G+y}(x) = 1$$

Th ξ $\quad m_{2n}(\tilde{G}) = \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \chi_{G+y} \, dx \right] dy$

$$= \int_{\mathbb{R}^n} m_n(G+y) \, dy$$

$$= \int_{\mathbb{R}^n} m_n(G) \, dy = 0$$

$\Phi^{-1}(Z) \subset \tilde{G}$

$$\Longrightarrow \quad m_{2n}(\Phi^{-1}(Z)) = 0$$

Step 3 $\quad \forall E \in \mathcal{L}_{\mathbb{R}^n}$

$$E = \underset{G_\delta\text{集}}{G} \setminus \underset{\text{零集}}{Z}$$

$$\Longrightarrow \Phi^{-1}(E) = \Phi^{-1}(G) \setminus \Phi^{-1}(Z)$$

$$\overset{\text{Step 1,2}}{\Longrightarrow} \Phi^{-1}(E) \in \mathcal{L}_{\mathbb{R}^{2n}}$$

Def：设 f, g 在 $\mathbb{R}^n$ 上可测，如果对 a.e. $x \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \quad \text{存在},$$

则称之为 f 与 g 之卷积，记为

$$(f * g)(x) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

Thm $\quad f, g \in L^1 \Longrightarrow f * g$ a.e. 存在 且

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1$$

(HW : Ex. 21)