第十二讲（2023.4.19）

Thm（积分的绝对连续性）

$f \in L^1 \Rightarrow \forall \varepsilon > 0, \exists \delta > 0$, s.t.

$$\int_E |f|\, dm < \varepsilon, \quad \forall E \text{ with } m(E) < \delta.$$

Pf $\forall k$，令

$$E_k \overset{\text{def}}{=} \{ |f| \le k \}$$

$$g_k \overset{\text{def}}{=} |f| \cdot \chi_{E_k} \qquad \text{（截断）}$$

$\Rightarrow \quad g_k \nearrow |f|$

$\overset{\text{MCT}}{\Longrightarrow} \quad \lim\limits_{k\to\infty} \int g_k\, dm = \int |f|\, dm$

$\Rightarrow \quad \forall \varepsilon > 0, \exists N$ s.t.

$$0 \le \int |f|\, dm - \int g_N\, dm < \varepsilon.$$

令 $\delta \overset{\text{def}}{=} \dfrac{\varepsilon}{2N}$,

$\Rightarrow \quad \forall E$ with $m(E) < \delta$,

$$\int_E |f|\,dm = \int_E (|f| - g_N)\,dm + \int_E g_N\,dm$$
$$< \frac{\varepsilon}{2} + N \cdot m(E)$$
$$< \varepsilon .$$

<u>Thm</u> (Lebesgue控制收敛定理, DCT)

设 $f_k \to f$ a.e. 且 $\underbrace{\exists\, g \in L^1 \text{ s.t. } |f_k| \le g \text{ a.e.}}_{\text{存在可积控制函数.}}$

则

$$\lim_{k\to\infty} \int f_k\,dm = \int f\,dm$$

<u>Pf</u>

$$\left.\begin{array}{l} f_k \to f \text{ a.e.} \\ |f_k| \le g \text{ a.e.} \end{array}\right\} \Rightarrow |f| \le g \text{ a.e.}$$
$$\Rightarrow \int |f|\,dm \le \underbrace{\int g\,dm}_{<+\infty}$$
$$\Rightarrow f \in L^1$$

令 $g_k \overset{\text{def}}{=} |f_k - f|, \quad k = 1, 2, \cdots$

$$\Rightarrow 0 \le g_k \le 2g \text{ a.e.} \quad k = 1, 2, \cdots$$

Fatou

$$\Longrightarrow \int \liminf_{k\to\infty} (2g - g_k)\, dm \le \liminf_{k\to\infty} \int (2g - g_k)\, dm$$

$$\Longrightarrow \cancel{2\int g\, dm} - \int \lim_{k\to\infty} g_k\, dm \le \cancel{2\int g\, dm} - \limsup_{k\to\infty} \int g_k\, dm$$

$$\Longrightarrow \limsup_{k\to\infty} \int g_k\, dm \le \int \lim_{k\to\infty} g_k\, dm = 0.$$

$$\Longrightarrow \lim_{k\to\infty} \int |f_k - f|\, dm = 0$$

$$\Longrightarrow \lim_{k\to\infty} \int f_k\, dm = \int f\, dm$$

<u>Thm</u> (有界收敛定理). 设 f_k, $k=1,2\cdots$ 可测 s.t.

(i) $\exists M$ (const) s.t. $|f_k| \le M$ a.e.

(ii) $\exists E$, $m(E) < \infty$ s.t. $\mathrm{supp}(f_k) \subset E$, $k=1,2\cdots$

(iii) $f_k \to f$ a.e.

则 $$\lim_{k\to\infty} \int f_k\, dm = \int f\, dm$$

Pf $\quad g \overset{def}{=} M\chi_E$

例. 求 $\lim\limits_{k\to\infty} \int_0^\infty \frac{dt}{\left(1+\frac{t}{k}\right)^k t^{\frac{1}{k}}}$

$$\frac{1}{\left(1+\frac{t}{k}\right)^k t^{\frac{1}{k}}} \to e^{-t} \quad \text{as } k\to\infty$$

$$\overset{?}{\Rightarrow} \lim_{k\to\infty} \int_0^\infty \frac{dt}{\left(1+\frac{t}{k}\right)^k t^{\frac{1}{k}}} = \int_0^\infty e^{-t}\,dt = 1.$$

1° 当 $t\in(0,1]$, $k\geq 2$

$$\frac{1}{\left(1+\frac{t}{k}\right)^k t^{\frac{1}{k}}} \leq \frac{1}{\sqrt{t}} \in L^1(0,1]$$

2° 当 $t\in[1,\infty)$, $k\geq 2$

$$\frac{1}{\left(1+\frac{t}{k}\right)^k t^{\frac{1}{k}}} \leq \frac{4}{t^2} \in L^1[1,\infty)$$

$$\left[\ \left(1+\frac{t}{k}\right)^k \geq \binom{k}{2}\cdot\left(\frac{t}{k}\right)^2 \geq \frac{t^2}{4}\ \right]$$

令

$$g(t) \overset{\text{def}}{=} \begin{cases} \dfrac{1}{\sqrt{t}}, & t \in (0,1] \\ \dfrac{4}{t^2}, & t \in (1,\infty) \end{cases}$$

Thm (积分号下求导)

设 $E \subset \mathbb{R}^n$ 可测，函数 $f: E \times (a,b) \to \mathbb{R}$ s.t.

(i) $\forall y \in (a,b)$, $x \mapsto f(x,y)$ 在 E 上可积,

(ii) $\forall x \in E$, $y \mapsto f(x,y)$ 在 (a,b) 上可微.

(iii) $\exists g \in L^1(E)$ s.t.

$$\left| \frac{\partial f}{\partial y}(x,y) \right| \le g(x), \quad \forall (x,y) \in E \times (a,b)$$

则

$$\frac{\partial}{\partial y} \int_E f(x,y)\,dx = \int_E \frac{\partial f}{\partial y}(x,y)\,dx$$

Pf 对 (x,y), $\forall t_k \to 0$, with $y + t_k \in (a,b)$

$$f_k(x) \overset{\text{def}}{=} \frac{f(x, y+t_k) - f(x,y)}{t_k}$$

$$\Rightarrow \quad f_k(x) \to \frac{\partial f}{\partial y}(x,y)$$

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$$|f_k(x)| \overset{\text{中值定理}}{\le} \sup_{y\in(a,b)} \left|\frac{\partial f}{\partial y}(x,y)\right| \overset{\text{(iii)}}{\le} g(x)$$

$$\overset{\text{DCT}}{\Longrightarrow} \int_E \frac{\partial f}{\partial y}(x,y)\,dx = \lim_{k\to\infty} \int_E f_k(x)\,dx$$

$$= \lim_{k\to\infty} \frac{\int_E f(x,y+t_k)\,dx - \int_E f(x,y)\,dx}{t_k}$$

$$= \frac{\partial}{\partial y}\int_E f(x,y)\,dx$$

复值函数之积分

<u>Def</u> 对函数 $f: E\to\mathbb{C}$，如果 $\mathrm{Re}f$，$\mathrm{Im}f$ 均可测，

则称 f 可测.

如果 $\int_E |f|\,dm < \infty$，则称 f 可积，并

令

$$\int_E f\,dm \overset{\text{def}}{=} \int_E \mathrm{Re}f\,dm + i\int_E \mathrm{Im}f\,dm$$

Lebesgue 积分与 Riemann 积分

Thm 对 $[a,b]$ 上实值函数 f,

Riemann 可积 $\Longrightarrow$ Lebesgue 可积

且

$$\int_{[a,b]} f\,dm = \int_a^b f(x)\,dx$$

Pf 对 $[a,b]$ 上划分

$$P: \ a = x_0 < x_1 < \cdots < x_n = b$$

令

$$S(f,P) \overset{def}{=} \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \qquad \left(\begin{matrix}\text{Darboux}\\ \text{上和}\end{matrix}\right)$$

$$s(f,P) \overset{def}{=} \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \qquad (\text{下和})$$

$$M_i \overset{def}{=} \sup_{x\in[x_{i-1},x_i]} f(x), \qquad m_i \overset{def}{=} \inf_{x\in[x_{i-1},x_i]} f(x)$$

$$\overline{\int_a^b} f \overset{def}{=} \inf_P S(f,P) \qquad (\text{上积分})$$

$$\underline{\int_a^b} f \overset{def}{=} \sup_P s(f,P) \qquad (\text{下积分})$$

f Riemann 可积 $\iff \overline{\int_a^b} f = \underline{\int_a^b} f$

存在单调划分序列 $\{P_k\}_{k=1}^{\infty}$ s.t.

$$S(f, P_k) \searrow \overline{\int_a^b} f$$

$$s(f, P_k) \nearrow \underline{\int_a^b} f$$

设 $P_k : a = x_0^{(k)} < x_1^{(k)} < \cdots < x_{n_k}^{(k)} = b$,

令

$$\varphi_k \stackrel{\text{def}}{=} \sum_{i=1}^{n_k} M_i \chi_{(x_{i-1}, x_i]}$$

$$\psi_k \stackrel{\text{def}}{=} \sum_{i=1}^{n_k} m_i \chi_{(x_{i-1}, x_i]}$$

$$\Longrightarrow \varphi_k \searrow, \quad \psi_k \nearrow \quad (\text{w.r.t. } k)$$

且

$$\psi_k \le f \le \varphi_k$$

令

$$g \stackrel{\text{def}}{=} \lim_{k\to\infty} \psi_k$$

$$h \stackrel{\text{def}}{=} \lim_{k\to\infty} \varphi_k$$

$$\Longrightarrow g \le f \le h$$

设 $|f| \leq M$ （Riemann 可积 $\Rightarrow$ 有界）

$$\Rightarrow \quad |\varphi_k| \leq M, \quad |\psi_k| \leq M.$$

$$\overset{DCT}{\Longrightarrow} \int_{[a,b]} |g| \, dm = \lim_{k \to \infty} \int_{[a,b]} |\psi_k| \, dm \leq M(b-a)$$

$$\Rightarrow \quad g \in L^1[a,b] \quad \underline{\text{且}}$$

$$\int_{[a,b]} g \, dm \overset{DCT}{=} \lim_{k \to \infty} \int_{[a,b]} \psi_k \, dm$$

$$= \lim_{k \to \infty} s(f, P_k) = \underline{\int_a^b} f$$

同理

$$\int_{[a,b]} h \, dm = \overline{\int_a^b} f$$

$$f \text{ Riemann 可积} \iff \overline{\int_a^b} f = \underline{\int_a^b} f$$

$$\iff \int_{[a,b]} h \, dm = \int_{[a,b]} g \, dm$$

$$\Rightarrow \int_{[a,b]} (\underbrace{h-g}_{\geq 0}) \, dm = 0$$

$$\Rightarrow \quad g = h \quad a.e.$$

$$\Longrightarrow \quad f = g \quad a.e.$$

$g \in L^1[a,b]$

$$\Longrightarrow \quad f \in L^1[a,b]$$

(1)

$$\int_{[a,b]} f \, dm = \int_{[a,b]} g \, dm = \int_a^b f = \int_a^b f(x)dx$$

Rmk: Lebesgue积分没有覆盖条件收敛的广义积分

例: $f(x) = \dfrac{\sin x}{x}$

$f \notin L^1(\mathbb{R})$，但作为广义积分，$\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$

HW: 11. 12. 15