第十讲 (2023.4.12)

Def 设 $\varphi \overset{\text{def}}{=} \sum_{k=1}^{N} a_k \chi_{E_k} \geq 0$, $\biguplus_{k=1}^{N} E_k = \mathbb{R}^n$,

令

$$\int_{\mathbb{R}^n} \varphi \, dm = \int_{\mathbb{R}^n} \varphi(x) dx \overset{\text{def}}{=} \sum_{k=1}^{N} a_k m(E_k)$$

(可能为 $+\infty$)

称为 φ 在 $\mathbb{R}^n$ 上的(Lebesgue)积分.

对 $E \subset \mathbb{R}^n$ 可测,

$$\int_E \varphi \, dm \overset{\text{def}}{=} \int_{\mathbb{R}^n} \underbrace{\varphi \cdot \chi_E}_{\text{simple}} \, dm = \sum_{k=1}^{N} a_k m(E \cap E_k)$$

Rmk 以上定义 well-defined. 即: 如果

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} = \sum_{j=1}^{M} b_j \chi_{F_j}$$

$$\biguplus_{k=1}^{N} E_k = \mathbb{R}^n = \biguplus_{j=1}^{M} F_j$$

则

$$\sum_{k=1}^{N} a_k m(E_k) = \sum_{j=1}^{M} b_j m(F_j)$$

Indeed, $E_k = \biguplus_{j=1}^{M} (E_k \cap F_j)$

$$F_j = \biguplus_{k=1}^{N} (E_k \cap F_j)$$

$$\Rightarrow \quad m(E_k) = \sum_{j=1}^{M} m(E_k \cap F_j)$$

$$m(F_j) = \sum_{k=1}^{N} m(E_k \cap F_j)$$

而则 $\quad E_k \cap F_j \neq \phi \Rightarrow a_k = b_j$

例: Dirichlet 函数 $D = \chi_{\mathbb{Q}}$

$$\int_{\mathbb{R}} D\, dm = 1 \cdot \underbrace{m(\mathbb{Q})}_{=0} + 0 \cdot \underbrace{m(\mathbb{R} \setminus \mathbb{Q})}_{=+\infty} = 0.$$

例:

$$\int_{\mathbb{R}^n} \chi_E \, dm = m(E)$$

$$\int_E \chi_F \, dm = m(E \cap F)$$

<u>Prop</u> $\forall\, \varphi, \psi \geq 0$ simple, $\forall\, \alpha, \beta \geq 0$,

(正线性) $\quad \int (\alpha\varphi + \beta\psi)\, dm = \alpha \int \varphi\, dm + \beta \int \psi\, dm$

<u>Pf</u> $\quad \int \alpha\varphi\, dm = \alpha \int \varphi\, dm$ 平凡

下证 $\int (\varphi+\psi)\,dm = \int \varphi\,dm + \int \psi\,dm$

设
$$\left.\begin{aligned} \varphi &= \sum_{k=1}^{N} a_k \chi_{E_k} \\ \psi &= \sum_{j=1}^{M} b_j \chi_{F_j} \end{aligned}\right\} \text{标准化表示}$$

$$\Rightarrow \quad E_k = \biguplus_{j=1}^{M} (E_k \cap F_j), \quad k = 1, 2, \cdots, N,$$
$$F_j = \biguplus_{k=1}^{M} (E_k \cap F_j), \quad j = 1, 2, \cdots, M.$$

$$\Rightarrow \quad \chi_{E_k} = \sum_{j=1}^{M} \chi_{E_k \cap F_j}$$
$$\chi_{F_j} = \sum_{k=1}^{N} \chi_{E_k \cap F_j}$$

$$\Rightarrow \quad \varphi + \psi = \sum_{k=1}^{N} \sum_{j=1}^{M} (a_k + b_j) \chi_{E_k \cap F_j}$$

$$\Rightarrow \quad \int (\varphi + \psi) = \sum_{k=1}^{N} \sum_{j=1}^{M} (a_k + b_j)\, m(E_k \cap F_j)$$
$$= \sum_{k=1}^{N} a_k \sum_{j=1}^{M} m(E_k \cap F_j)$$
$$+ \sum_{j=1}^{M} b_j \sum_{k=1}^{N} m(E_k \cap F_j)$$

$$\stackrel{\text{可加性}}{=} \sum_{k=1}^{N} a_k m(E_k) + \sum_{j=1}^{M} b_j m(F_j)$$

$$= \int \varphi \, dm + \int \psi \, dm$$

Prop (可加性)

设 $\varphi \geq 0$ simple, E_1, E_2 可测, $E_1 \cap E_2 = \phi$.

$$\int_{E_1 \cup E_2} \varphi \, dm = \int_{E_1} \varphi \, dm + \int_{E_2} \varphi \, dm.$$

Pf

$$\text{LHS} = \int \varphi \chi_{E_1 \cup E_2} \, dm$$

$$= \int \varphi (\chi_{E_1} + \chi_{E_2}) \, dm$$

$$= \int \varphi \chi_{E_1} \, dm + \int \varphi \chi_{E_2} \, dm$$

$$= \text{RHS}$$

Prop (单调性) 设 $\varphi, \psi \geq 0$ simple

$$\varphi \leq \psi \Rightarrow \int \varphi \, dm \leq \int \psi \, dm$$

Pf 设 $\varphi = \sum a_k \chi_{E_k}$, $\psi = \sum b_j \chi_{F_j}$ (标准表示)

$$\varphi \le \psi \implies a_k \le b_j \quad \text{if } E_k \cap F_j \ne \emptyset$$

$$\implies \int \varphi \, dm = \sum_{k,j} a_k \, m(E_k \cap F_j)$$

$$\le \sum_{k,j} b_j \, m(E_k \cap F_j) = \int \psi \, dm$$

<u>Def</u> 设 f 在 $\mathbb{R}^n$ 上非负可测，令

$$\int_{\mathbb{R}^n} f \, dm = \int_{\mathbb{R}^n} f(x) \, dx$$

$$\stackrel{\text{def}}{=} \sup\left\{ \int_{\mathbb{R}^n} \varphi \, dm : \varphi \text{ simple}, \ 0 \le \varphi \le f \right\}$$

称为 f 在 $\mathbb{R}^n$ 上的积分.

如果 $\int f \, dm < +\infty$，则称 f (Lebesgue) 可积，记为 $f \in L^1$.

如果 f 在 E 上非负可测，令

$$\int_E f \, dm \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f \cdot \chi_E \, dm$$

如果 $\int_E f \, dm < \infty$，称 f 在 E 上可积

记为 $f \in L^1(E)$.

Prop (单调性) 设 $f, g \geq 0$ 可测

$$f \leq g \implies \int f\,dm \leq \int g\,dm$$

Pf $\forall \varphi$ simple, with $0 \leq \varphi \leq f$,

$$\overset{f \leq g}{\implies} \varphi \leq g$$

$$\implies \int \varphi\,dm \leq \int g\,dm$$

$$\implies \int f\,dm \leq \int g\,dm$$

Prop 设 $f \geq 0$ on E, 可测

$$\int_E f\,dm = 0 \iff f = 0 \text{ a.e. on } E$$

Pf: "$\Leftarrow$" 平凡

"$\Rightarrow$" $\forall k$, 令

$$E_k \overset{\text{def}}{=} \left\{ f > \frac{1}{k} \right\}.$$

$$\Rightarrow \quad \frac{1}{k} m(E_k) = \int_{E_k} \frac{1}{k}\, dm$$

$$\leq \int_{E_k} f\, dm \leq \int_E f\, dm = 0$$

$$\Rightarrow \quad m(E_k) = 0, \quad \forall k.$$

$\{f>0\} = \bigcup_k E_k$

$$\Longrightarrow \quad m(\{f>0\}) = 0$$

<u>Cor</u> 改变 f 在一个零集上的取值，不改变积分值.

<u>Prop</u> 设 $f \geq 0$ 且 $f \in L^1(E)$，则 f a.e. 有限.

<u>Pf</u> $\forall k$，令

$$E_k \overset{def}{=} \{f > k\}$$

$$\Rightarrow \quad \{f = +\infty\} = \bigcap_{k=1}^{\infty} E_k$$

$$k\, m(E_k) = \int_{E_k} k\, dm \leq \int_E f\, dm < \infty$$

$$\Rightarrow \quad m(E_k) \leq \frac{1}{k} \int_E f\, dm$$

$\Rightarrow \lim_{k\to\infty} m(E_k) = 0.$

而 $m(E_1) \leq \int_E f\,dm < \infty$

且 $E_k \searrow \{f = +\infty\}$

测度连续性 $\Longrightarrow$ $m(\{f=+\infty\}) = \lim_{k\to\infty} m(E_k) = 0$

Thm (Levi，单调收敛定理，MCT)

设 $f_k \geq 0$ on E，可测，$k=1,2,\cdots$

$f_k \nearrow f$ a.e. on E，则

$$\lim_{k\to\infty} \int_E f_k\,dm = \int_E f\,dm$$

Pf 不妨设 $f_k \nearrow f$ pointwise

（可修改 f_k 在零集上取值）

积分单调性 $\Longrightarrow$ $\underbrace{\int_E f_k\,dm}_{\text{is 广义实数列}} \nearrow$

$$\Longrightarrow \lim_{k\to\infty} \int_E f_k \, dm$$ 存在（可能为 $+\infty$）

<u>Case 1</u> $\lim_{k\to\infty} \int_E f_k \, dm = +\infty$

$$f_k \leqslant f \Longrightarrow \int_E f \, dm \geqslant \int_E f_k \, dm \to +\infty$$

$$\Longrightarrow \int_E f \, dm = +\infty$$

<u>Case 2</u> $\lim_{k\to\infty} \int_E f_k \, dm < \infty$

同上 $\lim_{k\to\infty} \int_E f_k \, dm \leqslant \int_E f \, dm$

<u>Claim</u> $\lim_{k\to\infty} \int_E f_k \, dm \geqslant \int_E f \, dm$

$\forall \varphi$ simple, $0 \leqslant \varphi \leqslant f$

$\forall \alpha$, $0 < \alpha < 1$

令 $E_k \overset{\text{def}}{=} \{ f_k \geqslant \alpha \varphi \}$, $k = 1, 2 \cdots$

$$\Longrightarrow E_k \nearrow E$$