第八讲

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①

Thm 1 设 $f \geq 0$ 可测，则 $\exists \varphi_k \geq 0$, simple, $k=1,2,\cdots$ s.t. $\varphi_k \nearrow f$.

如果 f 有界，则 $\varphi_k \rightrightarrows f$.

Pf 对 $k=1,2,\cdots$

$j=0,1,2,\cdots,2^{2k}-1$

令

$$E_{k,j} \overset{def}{=} \left\{ \frac{j}{2^k} \leq f < \frac{j+1}{2^k} \right\}$$

$$F_k \overset{def}{=} \left\{ f \geq 2^k \right\}$$

（对每个 k 这些集合互不相交）

$$\varphi_k \overset{def}{=} \sum_{j=0}^{2^{2k}-1} \frac{j}{2^k} \chi_{E_{k,j}} + 2^k \chi_{F_k}$$

即

$$\varphi_k(x) = \begin{cases} \frac{j}{2^k} & \text{if } x \in E_{k,j} \\ 2^k & \text{if } x \in F_k \end{cases}$$

$$\Rightarrow \quad 0 \leq \varphi_k \leq f$$

1° $\varphi_k \leq \varphi_{k+1}, \quad \forall k,$

(1) 如果 $x \in F_k$

<u>Case 1</u> $x \in F_{k+1}$

$$\varphi_{k+1}(x) = 2^{k+1} > 2^k = \varphi_k(x)$$

<u>Case 2</u> $x \in F_k \setminus F_{k+1}$

$$F_k \setminus F_{k+1} = \{2^k \leq f < 2^{k+1}\}$$

$$= \bigcup_{j=2^{2k+1}}^{2^{2k+2}-1} E_{k+1,j}$$

$$\Rightarrow \varphi_{k+1}(x) \geq \underbrace{\frac{2^{2k+1}}{2^{k+1}}}_{\text{在 } E_{k+1,j}\ (2^{k+1} \leq j < 2^{2k+2}-1) \text{ 中最小可能取值}} = 2^k = \varphi_k(x)$$

(2) 如果 $\underbrace{x \notin F_k}$

$\Downarrow$

$x \in E_{k,j}$ for some $j \in \{0, 1, \cdots, 2^{2k}-1\}$

而 $E_{k,j} = E_{k+1,2j} \cup E_{k+1,2j+1}$

$$\Longrightarrow \quad \varphi_{k+1}(x) \geq \frac{2j}{2^{k+1}} = \frac{j}{2^k} = \varphi_k(x)$$

2° $\forall x \in \mathbb{R}^n$, $\varphi_k(x) \to f(x)$ as $k \to \infty$

<u>Case 1</u> $\quad \underbrace{f(x) = +\infty}$

$$\Downarrow$$

$$x \in \bigcap_{k=1}^{\infty} F_k \Longrightarrow \varphi_k(x) = 2^k, \ k = 1, 2, \cdots$$

$$\Longrightarrow \varphi_k(x) \to +\infty$$

<u>Case 2</u> $\quad \underbrace{f(x) < +\infty}$

$$\Downarrow$$

$$\exists\, k_0 \ \text{s.t.} \ f(x) < 2^{k_0}$$

$$\Longrightarrow \forall k > k_0, \ \exists\, j \ \text{s.t.} \ x \in E_{k,j}$$

$$(\because x \notin F_k)$$

by defn of $E_{k,j}$

$$\Longrightarrow \quad 0 \leq f(x) - \varphi_k(x) \leq \frac{1}{2^k}$$

$$\Longrightarrow \varphi_k(x) \to f(x) \ \text{as} \ k \to \infty$$

若 f 有界，同上可得

$$0 \leq f(x) - \varphi_k(x) \leq \frac{1}{2^k}, \quad \forall x \in \mathbb{R}^n$$

$$\Longrightarrow \varphi_k \rightrightarrows f$$

Def $\operatorname{supp}(f) \overset{\text{def}}{=} \overline{\{f \neq 0\}}$ 称为 f 的支集

如果 $\operatorname{supp}(f)$ 是紧集，称 f 有紧支集

Cor $f \geq 0$ 可测 $\Rightarrow \exists\, \varphi_k \geq 0$ simple, 紧支, s.t. $\varphi_k \nearrow f$

Pf $f \geq 0$ 可测 $\Rightarrow \exists\, \tilde{\varphi}_k \geq 0$ simple s.t. $\tilde{\varphi}_k \nearrow f$

令 $\varphi_k \overset{\text{def}}{=} \tilde{\varphi}_k \cdot \chi_{B_k(0)}$

$\Rightarrow \varphi_k$ simple, $\operatorname{supp}(\varphi_k) \subset \overline{B_k(0)}$

$\forall\, x \in \mathbb{R}^n$, $\exists\, k_0$ s.t. $x \in B_{k_0}(0)$

$\Rightarrow \forall\, k \geq k_0$, $x \in B_k(0)$

$\varphi_k(x) = \tilde{\varphi}_k(x)$

Thm 2 f 可测 $\Rightarrow \exists\, \varphi_k$, $k=1,2,\cdots$ simple s.t. $\forall\, x \in \mathbb{R}^n$

$$0 \leq |\varphi_1(x)| \leq |\varphi_2(x)| \leq \cdots \leq |f(x)|$$

且 $\lim\limits_{k\to\infty} \varphi_k(x) = f(x)$.

Pf
$$f = f^+ - f^-$$
$$|f| = f^+ + f^-$$

$\exists\ \varphi_k^{(1)} \nearrow f^+,\quad \varphi_k^{(2)} \nearrow f^-$

令 $\varphi_k \overset{\text{def}}{=} \varphi_k^{(1)} - \varphi_k^{(2)}$

$\Rightarrow\ \varphi_k \to f$ pointwise

则 $|\varphi_k| = \varphi_k^{(1)} + \varphi_k^{(2)} \nearrow |f|$

Thm f 可测 $\Rightarrow$ 存在阶梯函数 $\psi_k,\ k=1,2,\cdots$ s.t. $\psi_k \to f$ a.e.

Lem 1
$$\{f_k \to f\} = \bigcap_{l=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} \left\{ |f_k - f| < \frac{1}{l} \right\}$$
(第一次课：HW4)

Lem 2 设 $\{f_k\}_{k=1}^{\infty}$, $\{g_k\}_{k=1}^{\infty}$ 可测

$$\left.\begin{array}{l} f_k \to f \text{ a.e.} \\ \sum_{k=1}^{\infty} m(\{f_k \neq g_k\}) < \infty \end{array}\right\} \Rightarrow g_k \to f \text{ a.e.}$$

Pf 首先，$\forall \varepsilon > 0$

$$\{|g_k - f| \geq \varepsilon\}$$
$$\subset \{|f_k - g_k| \geq \frac{\varepsilon}{2}\} \cup \{|f_k - f| \geq \frac{\varepsilon}{2}\}$$
$$\subset \{f_k \neq g_k\} \cup \{|f_k - f| \geq \frac{\varepsilon}{2}\}$$

另一方面，由 Lem 1，

$$\{g_k \not\to f\} = \bigcup_{l=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \{|g_k - f| \geq \frac{1}{l}\}$$
$$\subset \bigcup_{l=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \left[\{f_k \neq g_k\} \cup \{|f_k - f| \geq \frac{1}{l}\}\right]$$
$$\subset \underbrace{\left(\limsup_{k\to\infty} \{g_k \neq f_k\}\right)}_{\text{上限集}} \cup \{f_k \not\to f\}$$

$$f_k \to f \text{ a.e.} \implies m(\{f_k \not\to f\}) = 0.$$

$\implies$ 只需证明

$$m\left(\limsup_{k\to\infty} \{g_k \neq f_k\}\right) = 0$$

而

$$\sum_{k=1}^{\infty} m(\{g_k \neq f_k\}) < \infty$$

Borel-Cantelli $\underset{\text{(Ex. 16)}}{\Longrightarrow}$ $m\left(\limsup\limits_{k\to\infty}\{g_k \neq f_k\}\right) = 0.$

Pf of Thm 2

Step 1 $f = \chi_E$, $m(E) < \infty$ 之情形

claim $\forall \varepsilon > 0$, 存在阶梯函数 ψ s.t.

$$m(\{\psi \neq \chi_E\}) < \varepsilon$$

$\exists\, Q_1, \cdots, Q_N$ s.t.

$$m\left(E \,\Delta \left(\bigcup_{k=1}^{N} Q_k\right)\right) < \frac{\varepsilon}{2}$$

将 $\bigcup_{k=1}^{N} Q_k$ 划分为有限个内部不交矩体之并, i.e.

$$\bigcup_{k=1}^{N} Q_k = \biguplus_{j=1}^{M} \tilde{R}_j$$

~~再把~~每个 $\tilde{R}_j$ 收缩为 R_j. s.t.

(i) $m\left(E \,\Delta \left(\bigcup_{j=1}^{M} R_j\right)\right) < \varepsilon$

(ii) R_j, $j = 1, 2, \cdots, M$ 互不相交.

$$\Rightarrow \quad \chi_E(x) = \sum_{j=1}^{M} \chi_{R_j}(x), \quad \forall x \in \left(E \triangle \bigcup_{j=1}^{M} R_j\right)^c$$

$$\left[\begin{aligned} &\because \left(E \triangle \bigcup_{j=1}^{M} R_j\right)^c \\ &= \underbrace{\left[E^c \cap \left(\bigcup_{j=1}^{M} R_j\right)^c\right]}_{\text{在这里 } \chi_E = 0 = \sum_{j=1}^{M} R_j} \cup \underbrace{\left[E \cap \left(\bigcup_{j=1}^{M} R_j\right)\right]}_{\text{在这里 } \chi_E = 1 = \sum_{j=1}^{M} R_j} \end{aligned} \right]$$

$$\Rightarrow \quad m\left(\left\{\chi_E \neq \sum_{j=1}^{M} \chi_{R_j}\right\}\right) < \varepsilon$$

<u>Step 2</u> $\forall \varphi$ simple，可测，$\forall \varepsilon > 0$，

$\exists \psi$ 阶梯函数，s.t.

$$m(\{\psi \neq \varphi\}) < \varepsilon.$$

<u>Step 3</u> 对一般可测函数 f，

$\exists \varphi_k$ simple，可测，$k = 1, 2 \cdots$

s.t. $\varphi_k \to f$

$\overset{\text{Step 2}}{\Longrightarrow}$ 对每个 k，$\exists \psi_k$ 阶梯函数，s.t.

$$m(\{\psi_k \neq \varphi_k\}) < \frac{1}{2^k}$$

Lem 2

$\Longrightarrow \quad \psi_k \to f \quad a.e.$