第四讲

Def (外测度) $m_*: 2^{\mathbb{R}^n} \to [0,+\infty]$

$$m_*(E) \overset{def}{=} \inf\left\{ \sum_{k=1}^{\infty} |Q_k| : \{Q_k\}_{k=1}^{\infty} 是 E 的方体覆盖 \right\}$$

Prop 1 (单调性) $E_1 \subset E_2 \Rightarrow m_*(E_1) \leq m_*(E_2)$

Prop 2 (次可加性)

$$m_*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m_*(E_k)$$

Pf 不妨设 $\forall k$, $m_*(E_k) < \infty$

(若 RHS 和式中有一项为 $+\infty$, 则不等式平凡成立

$\forall \varepsilon > 0$, $\forall k$, $\exists Q_j^{(k)}$, $j = 1, 2, \cdots$ s.t.

$$E_k \subset \bigcup_{j=1}^{\infty} Q_j^{(k)}$$

且

$$\sum_{j=1}^{\infty} |Q_j^{(k)}| \leq m_*(E_k) + \frac{\varepsilon}{2^k}$$

$$\Rightarrow m_*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k,j} |Q_j^{(k)}|$$

$$\leq \sum_{k=1}^{\infty} \left(m_*(E_k) + \frac{\varepsilon}{2^k}\right)$$

$$= \sum_{k=1}^{\infty} m_*(E_k) + \varepsilon$$

$$\overset{\varepsilon \to 0+}{\Longrightarrow} m_*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m_*(E_k)$$

Prop 3 (外正则性)

$$m_*(E) = \inf\{ m_*(G) : G \text{ 开}, E \subset G \}$$

Pf $\forall \varepsilon > 0$, $\exists Q_k$, $k=1,2,\cdots$ s.t. $E \subset \bigcup_{k=1}^{\infty} Q_k$

则

$$\sum_{k=1}^{\infty} |Q_k| < m_*(E) + \frac{\varepsilon}{2}$$

$\forall k$, $\exists P_k$ 开方体 s.t.

$$Q_k \subset P_k \quad \text{且} \quad |P_k| < |Q_k| + \frac{\varepsilon}{2^{k+1}}$$

令

$$G \overset{def}{=} \bigcup_{k=1}^{\infty} P_k \quad (\text{开})$$

$\Rightarrow E \subset G$ 且

次可加性

$$\begin{aligned} m_*(E) \le m_*(G) &\le \sum_{k=1}^{\infty} |P_k| \\ &\le \sum_{k=1}^{\infty} \left(|Q_k| + \frac{\varepsilon}{2^{k+1}} \right) \\ &\le m_*(E) + \varepsilon \end{aligned}$$

prop 4

$$\mathrm{dist}(E_1, E_2) > 0 \Rightarrow m_*(E_1 \cup E_2) = m_*(E_1) + m_*(E_2)$$

Pf 由次可加性

$$m_*(E_1\cup E_2)\le m_*(E_1)+m_*(E_2)$$

下证 $m_*(E_1\cup E_2)\ge m_*(E_1)+m_*(E_2)$

$\forall \varepsilon>0$，$\exists\, Q_k$，$k=1,2,\cdots$ s.t.

$$E_1\cup E_2\subset\bigcup_{k=1}^{\infty}Q_k$$

且

$$\sum_k |Q_k| < m_*(E_1\cup E_2)+\varepsilon$$

不妨设 $\forall k$，$\operatorname{diam} Q_k<\frac{1}{2}\operatorname{dist}(E_1,E_2)$

（否则细分 Q_k，得到新方体覆盖 且 $\sum_k|Q_k|$ 不变）

$\Rightarrow$ 每个 Q_k 仅与 E_1 和 E_2 之一相交

令

$$I_1\overset{def}{=}\{k: Q_k\cap E_1\ne\phi\}$$

$$I_2\overset{def}{=}\{k: Q_k\cap E_2\ne\phi\}$$

$$\Rightarrow\quad E_1\subset\bigcup_{k\in I_1}Q_k,\quad E_2\subset\bigcup_{k\in I_2}Q_k$$

$$\begin{aligned}\Rightarrow m_*(E_1)+m_*(E_2)&\le\sum_{k\in I_1}|Q_k|+\sum_{k\in I_2}|Q_k|\\&=\sum_{k=1}^{\infty}|Q_k|\\&<m_*(E_1\cup E_2)+\varepsilon\end{aligned}$$

$$\Rightarrow m_*(E_1) + m_*(E_2) \leq m_*(E_1 \cup E_2)$$

Prop 5 设 Q_k, $k=1,2,\cdots$ 内部互不相交，则

$$m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) = \sum_{k=1}^{\infty} |Q_k|.$$

Pf 由次可加性，LHS $\leq$ RHS

下证 LHS $\geq$ RHS

$\forall \varepsilon > 0$, $\forall k$, $\exists \tilde{Q}_k$（收缩 Q_k 得到）s.t.

1° $\tilde{Q}_k \subset Q_k$

2° $|\tilde{Q}_k| > |Q_k| - \frac{\varepsilon}{2^k}$

3° $\text{dist}(\tilde{Q}_k, \tilde{Q}_j) > 0$, $\forall j, k$, $j \neq k$.

$\forall N$,

$$\begin{aligned} m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) &\geq m_*\left(\bigcup_{k=1}^{N} \tilde{Q}_k\right) \\ &\overset{\text{Prop 4}}{=} \sum_{k=1}^{N} |\tilde{Q}_k| \\ &\geq \sum_{k=1}^{N} \left(|Q_k| - \frac{\varepsilon}{2^k}\right) \end{aligned}$$

$$\overset{N\to\infty}{\Longrightarrow} m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \geq \sum_{k=1}^{\infty} |Q_k| - \varepsilon$$

$$\overset{\varepsilon\to 0}{\Longrightarrow} m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \geq \sum_{k=1}^{\infty} |Q_k|$$

Prop 6 (平移不变性)

设 $E \subset \mathbb{R}^n$，则

$$m_*(E+h) = m_*(E), \quad \forall h \in \mathbb{R}^n$$

Pf 略

问：是否存在 $\mu: 2^{\mathbb{R}^n} \to [0, +\infty]$ s.t.

(i) $\mu(\phi) = 0$

(ii) $\mu(R) = |R|$, $\forall R$ 矩体

(iii) 可数可加

(iv) 平移不变？

不存在！

Def 设 $E \subset \mathbb{R}^n$.

1° 如果 $\forall \varepsilon > 0$，$\exists G$ 开 s.t. $E \subset G$ 且

$$m_*(G \setminus E) < \varepsilon,$$

则称 E 是 Lebesgue 可测的（简称可测）.

$$\mathcal{L} \overset{def}{=} \{\mathbb{R}^n \text{中 Lebesgue 可测集}\}$$

2° 如果：$\forall A \subset \mathbb{R}^n$,

$$m_*(A) = m_*(A \cap E) + m_*(A \cap E^c),$$

则称 E 是 Carathéodory 可测的.

HW (选作) (L) 可测 $\Leftrightarrow$ (C) 可测

Prop 7 开集可测

Prop 8 零测集可测.

Pf 由外正则性,

$$m_*(E) = \inf\{m_*(G) : G \text{ 开}, E \subset G\}.$$

$\Rightarrow \forall \varepsilon > 0, \exists G$ 开 s.t. $E \subset G$ 且

$$m_*(G) < \underbrace{m_*(E)}_{=0} + \varepsilon = \varepsilon$$

$$\Rightarrow m_*(G \setminus E) \leq m_*(G) < \varepsilon.$$

例: Cantor 三分集可测

Prop 9 $E_k, k=1,2,\cdots$ 可测 $\Rightarrow \bigcup_{k=1}^{\infty} E_k$ 可测

($\mathcal{L}$ 对可数并运算封闭)

Pf: $\forall \varepsilon > 0, \exists G_k$ 开 s.t.

$$E_k \subset G_k \text{ 且 } m_*(G_k \setminus E_k) < \frac{\varepsilon}{2^k}$$

令 $G \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} G_k$ (开)

$$G \setminus \left(\bigcup_{k=1}^{\infty} E_k \right) \subset \bigcup_{k=1}^{\infty} (G_k \setminus E_k)$$

$$\Rightarrow m_*\left(G \setminus \bigcup_{k=1}^{\infty} E_k \right) \leqq \sum_{k=1}^{\infty} m_*(G_k \setminus E_k) < \varepsilon$$