第三讲

Cantor 三分集

将 $[0,1]$ 三等分，"挖去"居中之开区间

$$I_1 \overset{\text{def}}{=} \left(\frac{1}{3}, \frac{2}{3}\right)$$

令

$$F_{11} \overset{\text{def}}{=} \left[0, \frac{1}{3}\right], \quad F_{12} \overset{\text{def}}{=} \left[\frac{2}{3}, 1\right]$$

$$G_1 \overset{\text{def}}{=} I_1$$

$$C_1 \overset{\text{def}}{=} F_{11} \cup F_{12}$$

将 F_{11}, F_{12} 分别三等分，"挖去"各自居中之开区间

$$I_{21} \overset{\text{def}}{=} \left(\frac{1}{3^2}, \frac{2}{3^2}\right), \quad I_{22} \overset{\text{def}}{=} \left(\frac{7}{3^2}, \frac{8}{3^2}\right)$$

令

$$F_{21} \overset{\text{def}}{=} \left[0, \frac{1}{3^2}\right], \quad F_{22} \overset{\text{def}}{=} \left[\frac{2}{3^2}, \frac{1}{3}\right]$$

$$F_{23} \overset{\text{def}}{=} \left[\frac{2}{3}, \frac{7}{3^2}\right], \quad F_{24} \overset{\text{def}}{=} \left[\frac{8}{3^2}, 1\right]$$

$$G_2 \overset{\text{def}}{=} I_{21} \cup I_{22}$$

$$C_2 \overset{\text{def}}{=} F_{21} \cup \cdots \cup F_{24}$$

$$\vdots$$

第 k 步，挖去 2^{k-1} 个开区间

$$I_{k1} \overset{\text{def}}{=} \left(\frac{1}{3^k}, \frac{2}{3^k}\right), \quad \cdots\cdots, \quad I_{k,2^{k-1}} \overset{\text{def}}{=} \left(\frac{3^k-2}{3^k}, \frac{3^k-1}{3^k}\right)$$

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②

留下 2^k 个闭区间

$F_{k1} \overset{def}{=} [0, \frac{1}{3^k}]$，$F_{k2} \overset{def}{=} [\frac{2}{3^k}, \frac{3}{3^k}]$，… $F_{k,2^k} \overset{def}{=} [\frac{3^k-1}{3^k}, 1]$

令

$$G_k \overset{def}{=} \bigcup_{i=1}^{2^{k-1}} I_{k,i} \quad (开)$$

$$C_k \overset{def}{=} \bigcup_{i=1}^{2^k} F_{k,i} \quad (闭)$$

$$G \overset{def}{=} \bigcup_{k=1}^{\infty} G_k$$ 称为 Cantor 开集

$$C \overset{def}{=} \bigcap_{k=1}^{\infty} C_k$$，称为 Cantor 三分集

1° $C = [0, 1] \setminus G$

2° $C \neq \phi$，C 闭
(by 闭集套定理)

3° C 是完全集 (HW, Ex. 1)

4° C 不含内点 ($\Leftrightarrow$ 不含区间)

5° C 有连续统基数，i.e. 存在 $[0, 1]$ 到 C 的一一映射

6° Cantor 开集 G 中开区间长度之和 $= 1$

$$\sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} |I_{k,i}| = 1$$

$$\left(\sum_{k=1}^{\infty}\frac{2^{k-1}}{3^k}=\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k=\frac{1}{2}\times\frac{\frac{2}{3}}{1-\frac{2}{3}}=1\right)$$

$\Rightarrow$ Cantor 三分集是零测集

<u>类 Cantor 集</u> (HW: Ex. 4)

Cantor - Lebesgue 函数

<u>外测度</u>

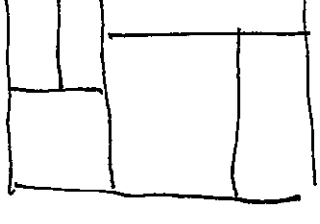
对矩体 $R \overset{\text{def}}{=} [a_1, b_1]\times\cdots\times[a_n, b_n]$

定义 $|R| \overset{\text{def}}{=} (b_1-a_1)\times\cdots\times(b_n-a_n)$

(体积)

<u>Lem 1</u> 如果 $R=\biguplus_{k=1}^{N} R_k$, $R, R_k, k=1,\cdots,N$ 也均为矩体

则 $|R|=\sum_{k=1}^{N}|R_k|$



<u>Lem 2</u> 如果 $R\subset\bigcup_{k=1}^{N} R_k$

则 $|R|\le\sum_{k=1}^{N}|R_k|$ (HW)

Def 设 $E \subset \mathbb{R}^n$，令

$$m_*(E) \overset{\text{def}}{=} \inf\left\{ \sum_{k=1}^{\infty} |Q_k| ,\quad E \subset \bigcup_{k=1}^{\infty} Q_k \right\}$$

称为 E 的外测度

$$m_* : 2^{\mathbb{R}^n} \longrightarrow [0, +\infty]$$

例： $m_*(\phi) = 0$

$m_*(\{a\}) = 0$

$\# E < \infty \implies m_*(E) = 0$

C —— Cantor 三分集

$\implies m_*(C) = 0$

$$C \overset{\text{def}}{=} \bigcap_{k=1}^{\infty} C_k , \qquad C_k \overset{\text{def}}{=} \bigcup_{i=1}^{2^k} F_{k,i}$$

$$m_*(C) \le \sum_{i=1}^{2^k} |F_{k,i}| = \left(\frac{2}{3}\right)^k \longrightarrow 0$$

例 $m_*(Q) = |Q|$

Pf $m_*(Q) \le |Q|$ 平凡

下证 $|Q| \le m_*(Q)$

$\forall \varepsilon > 0$, $\exists Q_k$, $k=1,2,\cdots$ s.t.

$$Q \subset \bigcup_{k=1}^{\infty} Q_k$$

且

$$\sum_{k=1}^{\infty} |Q_k| < m_*(Q) + \varepsilon$$

对每个 Q_k, $\exists P_k$ (开方体) s.t.

$$Q_k \subset P_k \quad 且 \quad |P_k| < (1+\varepsilon)|Q_k|$$

Q 紧 $\Longrightarrow$ $\exists P_1, \cdots, P_N$ s.t.

$$Q \subset \bigcup_{k=1}^{N} P_k$$

$$\overset{\text{Lem 2}}{\Longrightarrow} \quad |Q| \leq \sum_{k=1}^{N} |P_k|$$

$$\leq (1+\varepsilon) \sum_{k=1}^{N} |Q_k|$$

$$< (1+\varepsilon)(m_*(Q) + \varepsilon)$$

$$\overset{\varepsilon \to 0^+}{\Longrightarrow} \quad |Q| \leq m_*(Q)$$

例. R 矩体 $\Rightarrow$ $m_*(R) = |R|$

$|R| \leq m_*(R)$ 同前例

下证 $m_*(R) \leq |R|$

令

$\Gamma_k \overset{\text{def}}{=} \{$边长为 2^{-k} 的二进方体$\}$

$\mathcal{F}_k \overset{\text{def}}{=} \{Q \in \Gamma_k : Q \cap R \neq \phi\}$

$\mathcal{F}_k' \overset{\text{def}}{=} \{Q \in \mathcal{F}_k : Q \subset R\}$

$\mathcal{F}_k'' \overset{\text{def}}{=} \{Q \in \mathcal{F}_k : Q \not\subset R\}$

$\Rightarrow \mathcal{F}_k = \mathcal{F}_k' \cup \mathcal{F}_k''$

claim $\#\mathcal{F}_k'' = O(2^{k(n-1)})$

事实上. $\mathcal{F}_k'' \subset \{Q \in \Gamma_k : Q \cap \partial R \neq \phi\}$

而 $\#\{Q \in \Gamma_k : Q \cap \partial R \neq \phi\}$

$$\lesssim \frac{\text{Area}(\partial R) \times 2^{-k} \times 2}{2^{-kn}}$$

$$= O(2^{k(n-1)})$$

现在,

$$\underbrace{\sum_{Q \in \mathcal{F}_k'}}_{\text{内部互不相交}} |Q| \leq |R|$$

[illegible]

$$\sum_{Q \in \mathcal{F}_k''} |Q| \leq C \cdot 2^{k(n-1)} \cdot 2^{-kn} = C \cdot 2^{-k}$$

$$\Longrightarrow m_*(R) \leq \sum_{Q \in \mathcal{F}_k} |Q| \leq |R| + C \cdot 2^{-k}$$

$$\xRightarrow{k \to \infty} m_*(R) \leq |R|$$

<u>Prop</u> $E_1 \subset E_2 \Longrightarrow m_*(E_1) \leq m_*(E_2)$

（単調性）

<u>Prop</u> $m_*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m_*(E_k)$

（劣可加性）