HW:

18. Prove the following assertion: Every measurable function is the limit a.e. of a sequence of continuous functions.

$f \in \mathcal{L}(E)$, $f_n(x) = \begin{cases} f(x), & \text{if } f(x) \in [-n, n], \\ n, & \text{if } f(x) > n \\ -n, & \text{if } f(x) < -n \end{cases}$ then $f_n(x) \to f(x)$, $\forall x$.

Lusin $\Rightarrow \exists F_n$ closed, $m(E \setminus F_n) < \frac{1}{2^n}$, s.t. $f_n(x) \in C(F_n)$

$G_n := \bigcap_{k \geq n} F_k$, then $m(E \setminus G_n) \leq \frac{1}{2^{n-1}}$, $G_n \subset G_{n+1} \subset \cdots$

Tietz $\Rightarrow f_n(x) \in C(G_n) \rightsquigarrow g_n(x) \in C(E)$. $g_n|_{G_n} = f_n$

If $\exists n$, $x \in G_n$, then $g_k(x) = f_k(x)$ for all $k \geq n$. therefore $g_k(x) \to f(x)$.

$m\{x \mid g_k(x) \not\to f(x)\} \leq m(E \setminus G_n) \leq \frac{1}{2^{n-1}}$, $\forall n \Rightarrow g_k \to f$ a.e.

22. Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \quad \text{almost everywhere.}$$

Suppose there exists such a f. $\exists x_1, f(x_1) = 0$, $x_2, f(x_2) = 1$

$\Rightarrow f^{-1}((0,1))$ not null. It is open $\Rightarrow m(f^{-1}((0,1))) > 0$

But $m(\chi_{[0,1]}^{-1}((0,1))) = 0$

Measurable function is the map between σ-algebra.

X_i: set, $\mathcal{M}_i = \{\text{measurable sets}\}$ σ-algebra.

$f: (X_1, \mathcal{M}_1) \to (X_2, \mathcal{M}_2)$ measurable iff $\forall U \in \mathcal{M}_2$. $f^{-1}(U) \in \mathcal{M}_1$

$\iff \exists S$ generating $\mathcal{M}_2$, $f^{-1}(S) \subset \mathcal{M}_1$

Similar definition: continuous: $f^{-1}(\mathcal{T}_2) \subset \mathcal{T}_1$

Eg. $f: (\mathbb{R}^n, \underset{\substack{\uparrow \\ \text{Lebesgue measurable set}}}{\mathcal{M}}) \to (\mathbb{R}^n, \underset{\substack{\uparrow \\ \text{Borel set}}}{\mathcal{B}})$ Lebesgue measurable function

Eg. Right continuous function $\left(\lim_{x \downarrow x_0} f(x) = f(x_0)\right)$ is measurable.

Indeed, $f : (\mathbb{R}, \mathcal{T}_{\text{Sorgenfrey}}) \to (\mathbb{R}, \mathcal{T}_{\text{Euclidean}})$

$\uparrow$ union of $[a,b)$-interval $\qquad \mathcal{T}_{\text{Euclidean}} \subset \mathcal{B}$

Eg. $f : \mathbb{R} \to \mathbb{R}$ continuous $\Rightarrow$ $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ measurable.

Eg. $f : \mathbb{R} \to \mathbb{R}$ cts, $g : \mathbb{R} \to \mathbb{R}$ measurable

$\Rightarrow$ $f \circ g : \mathbb{R} \to \mathbb{R}$ measurable, $g \circ f$ may not.

$$\mathcal{M} \xrightarrow{g} \mathcal{B} \xrightarrow{f} \mathcal{B} \qquad g^{-1} \circ f^{-1}(\mathcal{B}) \subset g^{-1}(\mathcal{B}) \subset \mathcal{M}$$

Tietz Extension Theorem.

$E \subset \mathbb{R}^n$ measurable, $A \subset E$ closed, $\forall f: A \to [0,1]$ continuous, $\exists g \in C(E)$, $g|_A = f$.

Pf:

A technique:

$A_1 = \{x \in A \mid f(x) \geq \frac{1}{3}\}$, $B_1 = \{x \in A \mid f(x) \leq -\frac{1}{3}\}$ are disjoint closed subset of E, $g(x) := \frac{1}{3} \cdot \frac{d(x, B_1) - d(x, A_1)}{d(x, B_1) + d(x, A_1)}$ continuous, $\operatorname{Im} g \subset [-\frac{1}{3}, \frac{1}{3}]$

$$g(A_1) = \frac{1}{3},\ g(B_1) = -\frac{1}{3},\ |f - g|_A| \leq \frac{2}{3}$$

$f_1 = f: A \to [-1,1] \Rightarrow \exists g_1 \in C(E)$, $|f_1 - g_1|_A| \leq \frac{2}{3}$

$f_2 = f_1 - g_1|_A : A \to [-\frac{2}{3}, \frac{2}{3}] \Rightarrow \exists g_2 \in C(E)$, $|f_2 - g_2|_A| \leq (\frac{2}{3})^2$

$f_n = f_{n-1} - g_{n-1}|_A : A \to [-(\frac{2}{3})^{n-1}, (\frac{2}{3})^{n-1}] \Rightarrow \exists g_n \in C(E)$, $|f_n - g_n|_A| \leq (\frac{2}{3})^n$

$g = g_1 + \cdots + g_n + \cdots$ is continuous (converges uniformly) and coincides with f on A.

Construct a monotone function f on $\mathbb{R}$ such that f do NOT continuous on any interval.

$\mathbb{Q} = \{r_n\}$, $f(x) = \sum \chi_{[r_n, \infty)}(x) \cdot \frac{1}{2^n}$, $\forall (a,b)$, $\exists r_k \in \mathbb{Q} \cap (a,b)$ and $f(x)$ do not continuous at $\{r_k\}$.

$\{f_{m,n}(x)\}$ measurable, $\forall m,n$, and

CanNOT be used without proof!

(i) $\lim_{n\to\infty} f_{m,n}(x) = g_m(x)$ a.e.

(ii) $\lim_{m\to\infty} f_{m,n}(x) = h(x)$ a.e.

then $\exists \{f_{m_k, n_k}(x)\}$, s.t. $\lim_{k\to\infty} f_{m_k,n_k}(x) = h(x)$ a.e.

Rmk: 1. This is a useful lemma, since almost everywhere convergence is NOT topological, which means:

$\mathcal{U} = \{f_\alpha\}_\alpha$, $\overline{\mathcal{U}} = \{h \mid \exists \alpha_n, f_{\alpha_n} \to h \text{ a.e.}\}$, $\overline{\overline{\mathcal{U}}} = \{g(x) \mid \exists h_n \in \overline{\mathcal{U}}, h_n \to g \text{ a.e.}\}$

$\overline{u} \neq \overline{\overline{u}}$ generally. The definition of closure failed.

But in fact, $\overline{u} = \overline{\overline{u}}$, because this lemma.

2. In HW 1B, one can use simple functions to approximate f, and use continuous functions to approximate simple function.

Another proof of HW1B:

$f = f_+ - f_-$, $f_+, f_- \in \mathcal{L}^+(E)$, it suffices to prove:

$E_n^k = \{x \in E \mid \frac{k}{2^n} < f(x) \leq \frac{k+1}{2^n}\}$, $n \in \mathbb{N}$, $k = -n\cdot 2^n, -n\cdot 2^n+1 \cdots, n\cdot 2^n - 1$.

$E_n^{-n\cdot 2^n - 1} = \{x \in E \mid f(x) \leq -n\}$, $E_n^{n\cdot 2^n} = \{x \in E \mid f(x) > n\}$

$$g_n(x) = \sum_{k=-n\cdot 2^n - 1}^{n\cdot 2^n} \frac{k+1}{2^n} \chi_{E_n^k} \to f(x), \forall x$$

$\forall$ E measurable, $\exists$ U_n open, $U_n \supset E$, $m^*(U_n \setminus E) < \frac{1}{2^n}$

$\exists$ F_n closed, $F_n \subset E$, $m^*(E \setminus F_n) < \frac{1}{2^n}$.

$$h_n(x) = \frac{d(x, F_n)}{d(x, F_n) + d(x, U_n^c)}$$

$h_n|_{F_n} = 1$, $h_n|_{U_n^c} = 0$ $\Rightarrow h_n \to \chi_E$ a.e.

Proof of the prop:

Egorov does NOT require $|f| < \infty$ a.e.

Case 1: $m(E) < \infty$.

$\forall k$, $\exists E_k$, $m(E \setminus E_k) < \frac{1}{2^k}$, $g_m \rightrightarrows h$ on E_k

$\exists F_m$, $m(E \setminus F_m) < \frac{1}{2^m}$, $f_{n,m} \rightrightarrows g_m$ on F_m

$m(E \setminus E_k \cap \bigcap_{m=k}^{\infty} F_k^m) \leq m(E \setminus \bigcap_{m=k}^{\infty} F_k^m) < \frac{1}{2^{k-1}}$

$\Rightarrow$ on $E_k \cap \bigcap_{m=k}^{\infty} F_k^m$, $\forall l \in \mathbb{N}$, $\exists M_{k,l}$, $\forall m \geq M$, $\exists N_{m,l}$, $\forall n \geq N$

$$|f_{m,n} - h| < \frac{1}{l}$$

We can choose $M_{k,l+1}$ be a subsequence of $M_{k,l}$

$m_k = M_{k,k}$, $\forall n_k > N_{m_k,k}$, $|f_{m_k,n_k} - h| < \frac{1}{k}$ on $E_k \cap \bigcap_{m=k}^{\infty} F_k^m$

$\Rightarrow f_{m_k,n_k} \to h$ a.e. on E.

Case 2 : $m(E) = \infty$

On $E \cap B(0, l)$, $\exists\, m^l_{n_k}$, s.t. $\forall\, n^l_{n^l_k} > N_{m^l_{n^l_k}, k}$, $f_{m^l_{n^l_k}, n^l_{n^l_k}} \to h$ a.e. as $k \to \infty$

We can choose $\{m^{l+1}_{n^{l+1}_k}\}_k$ as a subsequence of $\{m^l_{n^l_k}\}_k$

with $n^{l+1}_1 > n^l_l$, and

have $\{n^{l+1}_{n^{l+1}_k}\}_k$ w.r.t. $m^{l+1}_{n^{l+1}_k}$. Since $n^{l+1}_{n^{l+1}_k}$ may larger than $n^l_{n^{l+1}_k}$,

so we change the value of $n^l_{n^{l+1}_k}$, s.t. $n^l_{n^{l+1}_k} = n^{l+1}_{n^{l+1}_k}$.

$n_l = n^l_l$ and we get a sequence $f_{m_{n_l}, n_{n_l}} \to h$ a.e.

Proof of " a.e. convergence is not a topological convergence " :

$\exists\, f_k \to f$ in measure but not a.e. $\Rightarrow \exists\, U \ni f$, $|\{f_k\} \cap U^c| = \infty$

$\{f_{n_k}\} = \{f_k\} \cap U^c$, $\exists\, \{n'_k\} \subset n_k$, $f_{n'_k} \to f$ a.e.

$f_{n,k} = \chi_{[\frac{k}{n}, \frac{k+1}{n}]}$ $\quad f_{1,0}, f_{2,0}, f_{2,1}, f_{3,0}, f_{3,1}, f_{3,2} \cdots$