回顾基本的 Brown Motion

$$dx = \mu dt + \sigma dw \Leftrightarrow P(x,t) \Leftrightarrow \underbrace{\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}}_{\text{Brown motion}}$$

微观过程. 宏观分布.

Fokker-plank equation $\begin{cases} \frac{\partial P}{\partial t} + \partial_x J = 0 \\ J = \mu p - \frac{1}{2}\partial_x \sigma^2 P \end{cases} \Rightarrow \int dx\, p(x) = 1$

$P(x,t) \geq 0$.

如果我们已知 $p(x,t)$. 问其对应的随机过程 (逆问) 是否存在.

$dx = (\ )dt + (\ )dw$?

上一节课 ① Random matrix $\Rightarrow$ 联合概率分布 (本征值).

$$P(\lambda_1, \lambda_2, \cdots \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{\beta}{2}\sum_i \lambda_i^2}$$

② Disordered many-body model

$H = H_{man} + H_{disorder}$.

$$\partial_t u = A\nabla^2 u + Bu^3 + \eta.$$

今天. 已知 $P(\lambda_1, \lambda_2, \cdots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{1}{2}\sum_i \lambda_i^2}$

问 λ_i 满足什么样的随机过程.

① λ_i 本征值 $\Rightarrow$ 理解为第 i 个粒子的位置

② 目的. P是数学结论. 还是有不一样的物理意义?

$$\begin{cases} d\lambda_i = \mu_i dt + \sigma_i dW_i \\ \text{找出 } \mu_i \text{ 和 } \sigma_i . \end{cases}$$

potential.

$\mu_i = ?$ drift term $\quad \frac{d\vec{v}}{dt} = \vec{f} \Leftrightarrow d\vec{v} = \vec{f}\, dt.$

Ref: ① Mehta 随机矩阵 第九章

② Dyson 原始论文, 1962. "A Brownian Motion Model for the eigenvalues of a Random matrix"

$$F = c e^{-\beta W} = c \prod_{i<j} |\lambda_i - \lambda_j|^{\beta} e^{-\frac{\beta}{2a^2}\sum_i \lambda_i^2}.$$

$$W = -\sum_{i<j} \ln|\lambda_i - \lambda_j| + \underbrace{\sum_i \frac{1}{2a^2} \lambda_i^2}_{\text{谐振势能}} \quad \text{Potential.}$$

Coulomb gas model. $\quad E(x_i) = -\frac{\partial W}{\partial x_i}$

$$= \sum_{i<j} \frac{1}{x_i - x_j} - \frac{x_i}{2a^2}$$

$$\frac{d^2\lambda_i^2}{dt^2} = -f\frac{d\lambda_i}{dt} + \underset{\text{外力}}{E(x_i)} + \underset{\text{随机力}}{A_i(t)}$$

$$\beta = \frac{1}{K_B T} \qquad \langle A_i(t) A_j(t') \rangle = \frac{2k_B T}{f}\delta(t - t')$$

代入 Fokker-Plank equation

$$f\frac{\partial P}{\partial t}=\sum_j k_BT\frac{\partial^2 P}{\partial x_j^2}-\frac{\partial}{\partial x_j}(E(x_j)P)$$

稳定解 $\frac{\partial P}{\partial t}=0$.

$$\Rightarrow \sum_j \frac{\partial}{\partial x_j}\left[k_BT\frac{\partial}{\partial x_j}P-E(x_j)P\right]=0.$$

$$\boxed{k_BT\frac{\partial}{\partial x_j}P=E(x_j)P}$$

$$\Rightarrow E(x_j)=-\frac{\partial W}{\partial x_j}$$

$$d\lambda_i=\sqrt{\frac{2}{\beta N}}dw_i+\frac{1}{N}\sum_{j=0}\frac{dt}{(x_i-x_j)}$$

Dyson $\Rightarrow$
$$\begin{cases} f\langle \delta x_i\rangle = E(x_i)\delta t \\ f\langle \delta x_i^2\rangle = 2k_BT\delta t\end{cases}$$

下面讨论一个有趣的图像.

$$\begin{cases} \text{矩阵}: M=(M_{ij})_{N\times N} \Rightarrow \text{元素 } M_\mu,\ \mu=1,\cdots,L. \\ P(M_1,\cdots,M_L)dM_1dM_2\cdots dM_L \text{ 为 } [M_i, M_i+dM_i] \text{ 之间几率} \end{cases}$$

$$\begin{cases} f\langle \delta M_\mu \rangle = -M_\mu / a^2 \, \delta t \\ f\langle \delta M_\mu^2 \rangle = g_\mu k_B T \, \delta t \end{cases} \qquad g_\mu = \text{const}$$

$$f \frac{\partial P}{\partial t} = \sum_\mu \left[\frac{1}{2} k_B T g_\mu \frac{\partial^2 P}{\partial M_\mu^2} + \frac{1}{a^2} \frac{\partial}{\partial M_\mu} (M_\mu P) \right]$$

假设 M 对角化 $\Rightarrow t+\delta t$, $M \to M + \delta M$

$M_{ii} = \lambda_i$ 是在 t 时刻的本征值.

$t \to t+\delta t \quad \lambda_i' = \lambda_i + \delta\lambda_i$

利用二阶 perturbation

$$\delta\lambda_i = \delta M_{ii} + \sum_j \left(\frac{|\delta M_{ij}|^2}{\lambda_i - \lambda_j} \right)$$

$$\delta\lambda_i = (\quad) dw_i + \sum_j \left(\frac{\beta \delta t}{\lambda_i - \lambda_j} \right)$$

从二阶微扰角度解释

$|\lambda_i - \lambda_j|^\beta$ 因子.