Probability Theory 大数定理，中心极限定理

$$\bar{Y} = \frac{1}{N}(x_1 + x_2 + \cdots + x_N)$$

$$\bar{Y} = N(\mu, \frac{\sigma^2}{N})$$

依概率收敛到1

$$\lim_{N\to\infty} P(|Y-\mu|<\varepsilon) = 1$$

数学： $X_i$ 随机数

iid 独立同分布

大数定理/中心极限定理

布朗运动

$$\langle x^2\rangle - \langle x\rangle^2 \sim 2Dt$$

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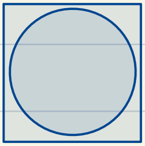
布丰投针实验



$$\frac{\pi}{4} \approx \frac{N_i}{N}$$

$N_i$ 在圆内的针

$N$ 总投针次数

统计力学 ~ 求平均值

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Brown Motion

1905 Einstein $\rightarrow$ $\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2} D$

$$P \sim e^{-\frac{x^2}{4Dt}}$$

$$\langle x^2(t)\rangle - \langle x(t)\rangle^2 = 2Dt$$

1908 Langevin

$$m\ddot{\vec{x}} = -\gamma\dot{\vec{x}} - \vec{\nabla}V(\vec{x}) + \vec{\xi}(t)$$

$\vec{\xi}(t)$，随机力

$\vec{x}$的解是不可导的。

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Wiener 过程

Ito 依腾请

Ito引理

数值求解

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Brown Motion $\to P(x,t)$

$$P(x,t+dt) = \int d\xi\, P(x+\xi,t)w(\xi) = \int d\xi \left(P(x,t) + \frac{\partial P}{\partial x}\xi + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}\xi^2\right) w(\xi)$$

$$\int d\xi\, w(\xi) = 1, \quad \int d\xi\, w(\xi)\xi = 0, \quad \int d\xi\, \xi^2 w(\xi) = \sigma^2$$

$$\Rightarrow P(x,t+dt) = P(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}P(x,t)$$

$$\frac{\partial P(x,t)}{\partial t}dt = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}P(x,t)$$

$$\Downarrow$$

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2}{\partial x^2}P(x,t) \qquad \sigma^2 = 2D\,dt$$

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Ref: Langevin 1908 翻译本，Lemmon，Gythiel. 1997 "On the Brown Motion"

朗之万方程.

$$m\ddot{x} = -\eta\dot{x} + \xi$$

$$m\frac{d^2x}{dt^2} = -\eta\frac{dx}{dt} + \xi$$

$$mx\frac{d^2x}{dt^2} = -\eta x\frac{dx}{dt} + \xi x$$

$$\Rightarrow \frac{m}{2}\frac{d^2}{dt^2}(x^2) - m\left(\frac{dx}{dt}\right)^2 = -\frac{\eta}{2}\frac{d}{dt}x^2 + \xi x.$$

此时的处理有一定的问题，链式法则不再成立（实际上）

定 $Z = \langle \frac{d}{dt}x^2 \rangle_e$ $\langle \cdots \rangle_e$，系统平均.

$$\Rightarrow \frac{m}{2}\frac{dZ}{dt} - \underbrace{\langle mv^2 \rangle}_{=k_BT} = -\frac{\eta}{2}Z + \underbrace{\langle \xi x \rangle}_{=0}$$

$$\Rightarrow \frac{m}{2}\frac{dZ}{dt} - k_BT = -\frac{\eta}{2}Z$$

$$\Rightarrow Z = Ce^{-\frac{\eta t}{m}} + \frac{2k_BT}{\eta}$$

长时间之后 $\Rightarrow Z = \frac{2k_BT}{\eta} \Rightarrow \langle \frac{d}{dt}x^2 \rangle = \frac{2k_BT}{\eta}$

$$\Rightarrow \langle x^2 \rangle = \frac{2k_BT}{\eta}\cdot t$$

$$D = \frac{k_BT}{\eta} \Longleftarrow = 2Dt$$

Plack-sholes Equation

$$dS = (r+\xi)S\,dt \Rightarrow d(\ln S) = (r+\xi)dt$$

$$\Rightarrow \ln S - \ln S_0 = \int_0^t (r+\xi)dt$$

$$\Rightarrow \quad \underline{S = S_0 e^{\gamma t + \int_0^t \xi dt'}}$$

似乎总是挣钱，与生活不符。

对于此类问题的求解需要改变.

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$\dot{x} = a + b\xi(t)$ ， $a$, $b$ 常数. 而 $\langle \xi(t) \rangle = 0$. $\langle \xi(t) \xi(t') \rangle = \delta(t-t')$

$$x = x_0 + at + b\int_0^t \xi(t')dt'$$

定义 $w(t) = \int_0^t \xi(t')dt'$

则 $\langle w(t) w(t') \rangle = \int_0^t dt' \int_0^s ds' \langle \xi(t') \xi(s') \rangle$

$$= \int_0^t dt' \int_0^s ds' \delta(t'-s')$$

$$= \int_0^{\min(t,s)} dt' = \min(t,s)$$

$\langle w^2(t) \rangle = t \quad \Rightarrow \quad \bar{x} = x_0 + at$

$$\langle x^2 \rangle - \langle x \rangle^2 = b^2 t$$

离散化后如何处理.

$$\int_0^t \xi(t')dt' = \sum_n \delta t \, \xi_n$$

其中 $\bar{\xi}_n = 0$. 而 $t = \sum \delta t^2 \langle \xi_n^2 \rangle$

$\Rightarrow \quad \langle \xi_n^2 \rangle = \frac{1}{\delta t}$，上式才能成立.

则 $X = x_0 + at + \sum_{n=0}^{N} \delta t\, \xi_n$. $\xi_n \in N(0, \frac{1}{\delta t})$