3. Bessel functions. Consider *Bessel's differential equation* (of order $n$):

$$x^2y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0, \qquad (\dagger)$$

for integer $n > 0$.

(a) Find the indicial exponents $\alpha_1$, $\alpha_2$ (with $\operatorname{Re}\alpha_1 > \operatorname{Re}\alpha_2$) for the local series expansion of ($\dagger$) about $x = 0$.

(b) Determine the series $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha_1}$ that solves ($\dagger$), giving the coefficients $a_k$ in closed form. Find $a_0$ such that the series is the expansion of the *Bessel functions of first kind*,

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(-x^2/4\right)^k}{k!(k+n)!}. \qquad (\#)$$

(c) Using (#), show that the following recursion relation is true for all integers $n \geq 0$:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

(d) For any integer $n \geq 0$, show that

$$\boxed{\int_0^1 x\left[J_n(\alpha x)\right]^2 \mathrm{d}x = \frac{1}{2}\left[J_n'(\alpha)\right]^2,}$$

where $\alpha$ is a zero of $J_n$. [*Hint: Substitute $z = \alpha x$, integrate by parts, and use the fact that $J_n$ satisfies Bessel's equation.*]

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