$$\underset{0}{\pi_2(\mathbb{Z}_2)} \to \underset{0}{\pi_2(G)} \to \underset{0}{\pi_2(G/\mathbb{Z}_2)} \to$$

$$\underset{0}{\pi_1(\mathbb{Z}_2)} \to \underset{\mathbb{Z}_2\otimes\mathbb{Z}}{\pi_1(G)} \to \underline{\pi_1(G/\mathbb{Z}_2)} \to \quad T^* = \mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2$$

$$\to \underset{\mathbb{Z}_2}{\pi_0(\mathbb{Z}_2)} \to \underset{0}{\underline{\pi_0(G)}} \to \pi_0(G/\mathbb{Z})$$

$\pi_1(SO(3)\times U(1)) = \mathbb{Z}_2\times\mathbb{Z}$

$\pi_2(SO(3))$

Review

同伦群

$\pi_k(M) = \pi_k(G/H)$ 计算.

i) 物理意义及应用

ii) 方法,

$\pi_0(M) = m$. $\quad m$ 不连通的分支个数.

$\pi_1(S^1) = \mathbb{Z} \qquad \bigcirc \xrightarrow{f} \bigcirc$

$\pi_1(S^2) = 0$

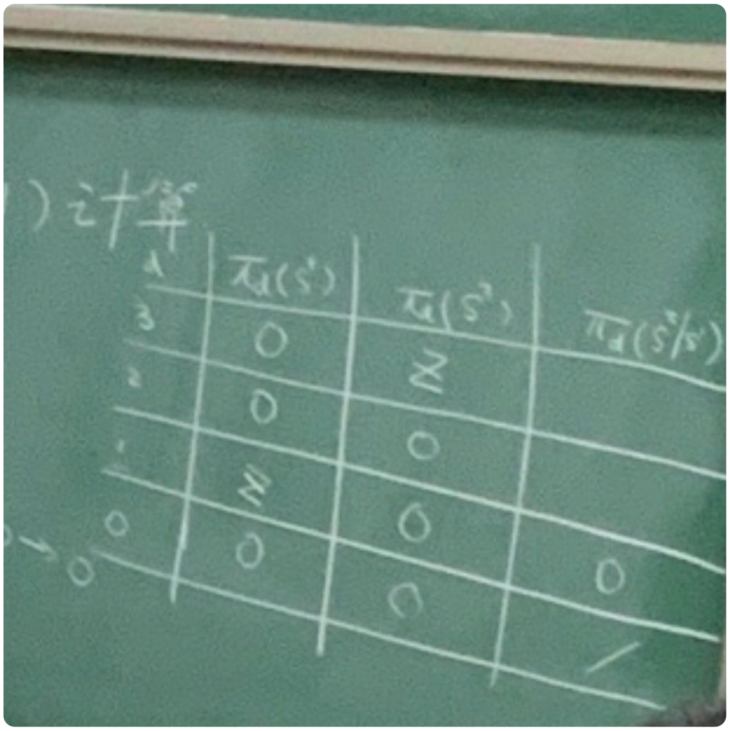
$\pi_2(S^2) = \mathbb{Z}$, $\pi_n(S^n) = \mathbb{Z}$

$\pi_n(S^m) = 0$. $\quad n < m$.

$\pi_k(S^n) \neq 0$. $\quad$ if $k > n$, eg. Hopf map.

$S^3 \to S^2$

$\pi_3(S^2) = \mathbb{Z}$



正合序列 $0 \to A \to B \to 0 \Rightarrow A \simeq B$

$0 \to A \to B \to C \to 0 \Rightarrow B/A \simeq C$

应用. $\left\{ \begin{array}{l} \text{Unstable Homotopy Group} \\ \pi_3(U(N)) \quad N \text{小} \end{array} \right\}$ Topo defect

stable. $\lim_{N\to\infty} U(N) = U$ } Topo band
k-space.

回顾 $M = G/H$, G保持自由能不变的群.
H保持基态不变的群.

1) spin 1 BEC. 1, 0, -1

$$G = U(1) \times SO(3)$$

$$\psi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$H = U(1)$$

$$\Rightarrow M = \frac{U(1) \times SO(3)}{U(1)} = SO(3)$$

$SO(3)$, $SU(2)$, S^3, $RP^3 = S^3/\mathbb{Z}_2$. Lie Group

2). fermion / Boson / Anyon

3) defect in Liquid crystal

} RP^d 有关.

$\vec{x}$, $-\vec{x}$ 等价

2). 自旋-统计关系 Pauli

$$\pi_1(RP^d) = \begin{cases} 0, & d = 0 \\ \mathbb{Z} & d = 1 \\ \mathbb{Z}_2 & d \geq 2. \end{cases}$$

$\psi(\vec{r}_1, \vec{r}_2)$ 质心坐标 $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2} \in R^d$.

相对坐标 $\vec{r} = \left(\frac{\vec{r}_1 - \vec{r}_2}{2}\right) \in \left(R^d - \{0\}\right)$

$\Uparrow$

"硬核条件"

$(\vec{r}_1, \vec{r}_2) \simeq R^d \times (R^d - \{0\})$

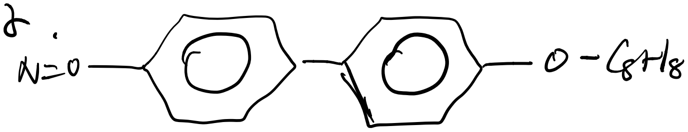
全同性. $\vec{r} \sim -\vec{r}$, 描述同一个系统

$$(\vec{r}_1,\vec{r}_2) \simeq R^d \times [(R^d-\{0\})/(\vec{r} \sim -\vec{r})]$$

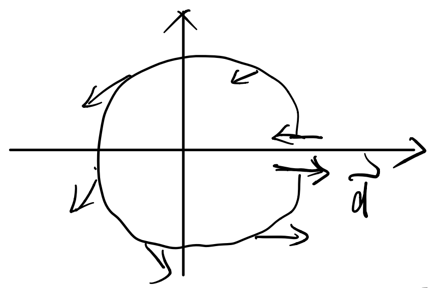
$$(R^d-\{0\})/\mathbb{Z}_2 \simeq R^+ \times S^d/\mathbb{Z}_2 = R^+ \times RP^{d-1},$$

3). 液晶(古老) $\Leftrightarrow$ soft matter physics.

有机分子.



Half-vortex.



1) $\vec{d}$, x, y平面

2) $\vec{d}$ 在 x, y, z 平面.

$\vec{d}$ 和 $-\vec{d}$ 等价.

假设 $|\vec{d}|=1$.

$\vec{d} = r|\vec{d}|$, 其中 $r \in R^+$, $\vec{d} \in S^d$

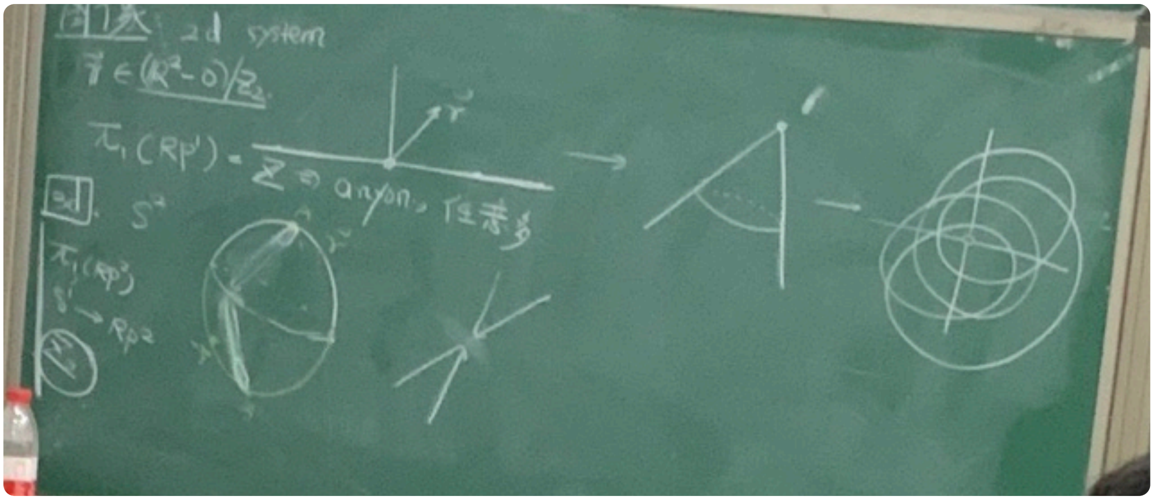
$\pi_1(R^+ \times S^d/\mathbb{Z}_2)$

$\vec{r} \sim -\vec{r}$

$$\pi_1(R^+ \times S^d/\mathbb{Z}_2) = \pi_1(S^d/\mathbb{Z}_2) = \pi_1(RP^d) = \begin{cases} \mathbb{Z}, & \text{if } d=1 \\ \mathbb{Z}_2, & \text{if } d=2 \end{cases}$$

固体中的 defect/disorder, little 书

图像: Rui Yu, et.al PRB



4). 单极子.

i) Dirac monopole. → ii) Wu-Yang monopole.

iii). Shanker monopole.

1977 Application of homotopy group

iv) t' Hopf monopole.

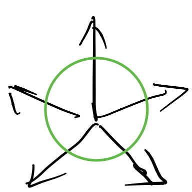
了解 texture 概念.

i) Dirac Monopole

$$\oint \vec{B}\cdot d\vec{s} = g \qquad \vec{B} = \nabla\times\vec{A}$$

$$\vec{B} = g\frac{\vec{r}}{4\pi|\vec{r}|^3} \qquad \int\left(\frac{\partial f}{\partial x}\right)dx$$

$r=1$

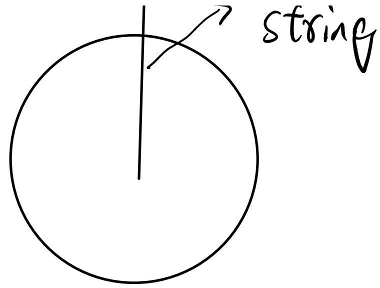


$|\vec{B}| = 1$

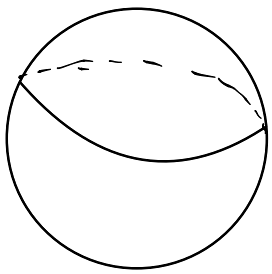
$d=1 \quad d\geq 2 \quad \cdots$

$SO(3)$, $\pi_d(SO(3)) = (Z_2, 0, Z, Z_2, Z_2 \cdots)$

$\oint \vec{B}\cdot d\vec{s} = g$, $\nabla\times\vec{A} = \vec{B}$



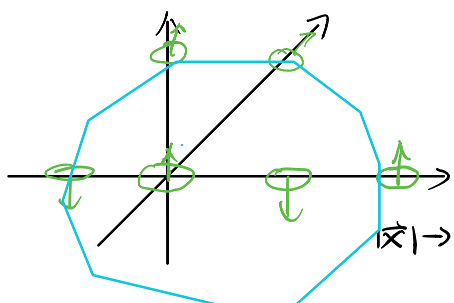
Yang-Wu solution.



$= \oint_{\Sigma_1} + \oint_{\Sigma_2}$

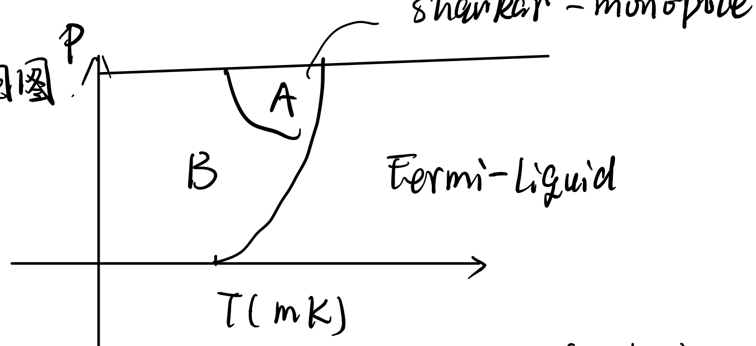
shankar monopole (^{3}He superfluid A phase)

texture $\vec{\Omega} = \frac{\vec{r}}{|\vec{r}|} f(\vec{r})$, $f(\vec{r}) = \begin{cases} 2\pi & r=0 \\ 0 & r\to\infty \end{cases}$



$|\vec{x}|\to\infty$ ，指向一致.

^{3}He 相图



$\pi_d(SO(3))$
$\pi_d(S^d)$ 与 Monopole 有关系

BKT 相变和 skymion

BKT. $H = J \vec{S}_i \cdot \vec{S}_j$

$\vec{S}_i = (\cos\theta_i, \sin\theta_i) \in S^1$，平面内的向量.

$\pi_1(S^1) = \mathbb{Z}$.

skymion.

$d=3$. $|\vec{S}|=1$. or $\vec{S} \in S^2$，球面.

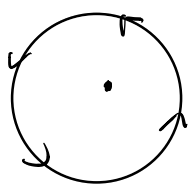
$\pi_2(S^2) = \mathbb{Z}$.

$d=2$ skymion.

$R^2 + \{\infty\} \simeq S^2$ 前提 $r \to \infty$

$\vec{S}(\vec{r}) = \vec{S}'$

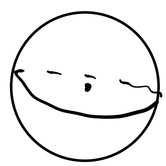
7), defects 分类.



S^1

$d=2$

$\frac{1}{2\pi i} \oint \frac{dz}{z}$



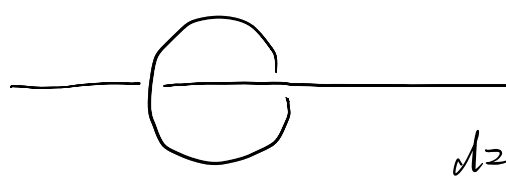
S^2

$d=3$.

$\oint \vec{E} \cdot d\vec{S}$

point defects

line defects



$d=3$

	$d=1$	$d=2$	$d=3$	$d=4$
π_0	point	line	surface	- - -
π_1	texture	p	L	S
π_2		t	p	

TI_3

paper:

Y. X. zhan. Z. D. zhang

PRL (2012)

如何描述 量子霍尔效应的体系

特点: 1) 能级多.

2) 费米面可调.

$n \left\{ \equiv \right. \rightarrow \text{——} +1$

$m \left\{ \equiv \right. \rightarrow \text{——} -1$

$$H = U^{+} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} U \in U(N+M)$$

$$M \in U(N+M) / U(N) \times U(M)$$

$$\pi_d(M)$$