具有 C 对称性.

$$C = \sigma_x K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K$$

$$CH = -HC \Rightarrow \sigma_x H^* = -H\sigma_x$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h^T & \Delta^* \\ \Delta^T & -h \end{pmatrix}$$

$$= -\begin{pmatrix} h & \Delta \\ \Delta^\dagger & h^T \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow \underline{\Delta^T = -\Delta}$，对于 BdG Equation 有 particle-hole 对称性.

D类. ① Why we call it D class

② $H \to$ Lie Algerbric

$$\left.\begin{aligned} &X = iH \quad X^\dagger = -X = -\sigma_x X^T \sigma_x \\ &Y^\dagger = -Y = -\sigma_x Y^T \sigma_x \end{aligned}\right\} [X, Y] = Z$$

$Td(X)$ 文小刚 $Hd(M, X)$

Review

$$\begin{cases} \text{幺正} & A = U \Rightarrow \langle\psi|\varphi\rangle = \langle A\psi|A\varphi\rangle \\ \text{反幺正} & A = UK \Rightarrow (\langle\psi|\varphi\rangle)^* = \langle A\psi|A\varphi\rangle \end{cases}$$

Time-reversal

$$T^2 = \begin{cases} +1 \\ -1 \end{cases}$$

$+1$: $T|\psi\rangle = |\psi\rangle \Rightarrow$ A Real wavefunction

-1: Kramers degeneracy

$\boxed{\text{Ref. Sakara, P276}}$

如果 $\begin{cases} THT^{-1} = H \\ T^2 \equiv 1 \end{cases}$

$$\boxed{\begin{aligned} & u^\dagger H u = \lambda \quad \equiv \{1 \\ & \text{or } H = u\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}u^\dagger \quad \equiv\{-1 \\ & \Rightarrow \quad M = \frac{U(M+N)}{U(M)\times U(N)} \end{aligned}}$$

$$\begin{cases} u^* = u \\ u^\dagger u = 1 \end{cases} \Rightarrow u \in O(N) \Rightarrow M = \frac{O(M+N)}{O(M)\times O(N)}$$

Lie 群与 Lie 代数.

群: ① 封闭 ② 单位元 ③ 逆元 ④ 结合律

有限群: $G = \{g_1, g_2, g_3, \cdots, g_N\}$

$|G| = k$.

元素个数无限.

e.g $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$

define:

$\check{R} = U^{\dagger} R(\theta) U$ $R = \begin{pmatrix} R(\theta) & \\ & I \end{pmatrix}$ ⇐ 可约表示

$\check{R}(\theta_1)\check{R}(\theta_2) = \check{R}(\theta_1+\theta_2)$

等价的酉表示。

特殊的群 ① $U(N) = \{u \mid u^{\dagger}u = 1\}$

$u = e^{iH}$, $H = H^{\dagger}$, $uu^{\dagger} = 1$

$H \in U(N)$'s Lie algebra

② $SU(N) = \{u \in U(N) \mid \det(u) = 1\}$

$u = e^{iH}$,

$H \in SU(N)$'s Lie Algebra

③ $O(N) = SO(N) \otimes Z_2$

$O(N) = \{O \mid O^T O = 1\}$

$SO(N) = \{O \in O(N) \mid \det(O) = 1\}$

$R\vec{x} = \vec{x}'$, 旋转加反射

另外一种表示

$O^T O = 1$ or $O^T M O = M$, $M = M^{-1}$ $M^2 = 1$

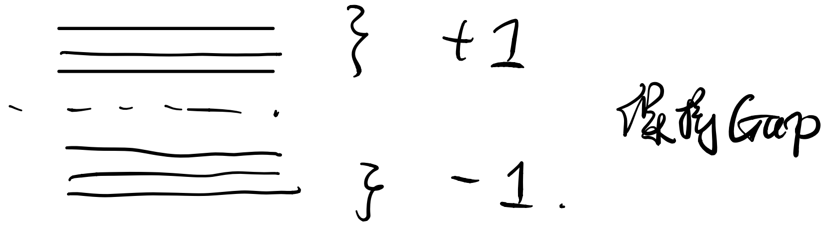
$$M = \begin{cases} 1 \\ \sigma_x \\ \sigma_z \end{cases}$$

Symplectic Group 辛群 Sp(2N)

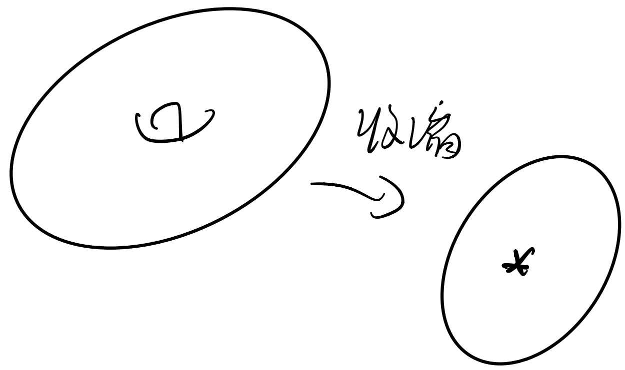
$$\begin{cases} u^\dagger u = 1 \\ u^T \varepsilon u = \varepsilon \end{cases} \qquad \varepsilon = i\sigma_y,\ \sigma_y^T = -\sigma_y \qquad \varepsilon = -\varepsilon^T \quad \varepsilon^2 = -1$$

Topo 分类

H → 平带



$$H = u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u^\dagger$$



$H^\dagger H = 1$ Lie Group

$M = G/H$

目的：1) 确定 H 的 Lie 群 ⇒ Hamiltonian Space

2) $\underbrace{H \xrightarrow{\text{平带}} \tilde{H}}_{\text{Topo 性质}} \Rightarrow \tilde{H} \in$ Lie Group classification space.

or $M = G/H$

找 Lie Group 不一定是 $U(N)$, $SO(N)$ $Sp(2N)$.

可能具有更复杂的形式.

↙

依据 symmetry

D类：(为什么叫D类)

$$H = \begin{pmatrix} h & \Delta \\ \Delta^+ & -h^T \end{pmatrix} \quad H^+ = H \Leftrightarrow \begin{cases} h = h^+ \\ \Delta^T = -\Delta \end{cases}$$

$$C = \Sigma_x K, \quad \Sigma_x = \begin{pmatrix} 0 & I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix} = \sigma_x \otimes I_{n\times n}$$

关于 H ∈ C(?)

$$X = iH, \quad X^+ = -iH^+ = -iH = -X = \Sigma_x X^T \Sigma_x$$

$$\boxed{\begin{aligned} CHC^{-1} &= -H \\ \Sigma_x K H K \Sigma_x &= -H \\ \Sigma_x H^* \Sigma_x &= -H \\ \Sigma_x H^T \Sigma_x &= -H \end{aligned}}$$

$$\begin{aligned} & C(iH)C^{-1} \\ &= \Sigma_x K(iH)K\Sigma_x \\ &= -i\Sigma_x K H K \Sigma_x \\ &= (iH) \end{aligned}$$

$$CXC^{-1} = X.$$

$\Downarrow$

类似于时间反演对称性.

下面证明 X 与 Y 满足上述关系，则 Z = [X, Y] 也一样.

$$\begin{cases} X^+ = -X = \Sigma_x X^T \Sigma_x \\ Y^+ = -Y = \Sigma_x Y^T \Sigma_x \\ Z = [X, Y] \Rightarrow Z^+ = (XY - YX)^+ \end{cases}$$

$$= -Z$$

$$= Y^+ X^+ - X^+ Y^+$$

$$= \Sigma_x Y^T \Sigma_x \Sigma_x X^T \Sigma_x$$

$$-\Sigma_x X^T \Sigma_x \Sigma_x Y^T \Sigma_x$$

$$= \Sigma_x [X, Y]^T \Sigma_x$$

补充李代数

$U \in U(n)$

$$\begin{cases} U = e^{H} \\ U^\dagger = e^{H^\dagger} \end{cases} \quad H^\dagger = -H \quad \begin{cases} e^{H_1} e^{H_2} \text{ 都是幺成} \\ e^{[H_1, H_2]} \text{ 也是幺成} \end{cases}$$

$$\tilde{X} = U_0 X U_0^{-1}, \quad U_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \otimes I_{n\times n}$$

$$X = iH = \begin{pmatrix} ih & i\Delta \\ i\Delta^\dagger & -ih^T \end{pmatrix}$$

$$\tilde{X} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} ih & i\Delta \\ i\Delta^\dagger & -ih^T \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$= \frac{1}{2} \left(\begin{array}{c|c} ih + i\Delta + i\Delta^\dagger - ih^T & h - \Delta + \Delta^\dagger + h^T \\ \hline -h - \Delta + \Delta^\dagger - h^T & i(h - \Delta - \Delta^\dagger - h^T) \end{array} \right)$$

$\tilde{X} = \tilde{X}^* = -\tilde{X}^T \Leftrightarrow$ 实/反对称

$\Rightarrow \tilde{X} \in SO(2N) \xrightarrow{\text{记做}} D_N \Rightarrow$ D类

和C代数

辛和C类 $\Rightarrow CHC^{-1}=H$

不同 D: $\begin{cases} C=\Sigma_x K \Rightarrow C^2=1 \\ C=\Sigma_y K \Rightarrow C^2=-1 \end{cases}$ $\begin{cases} X^{\dagger}=-X=\Sigma_x X^T \Sigma_x \\ X^T=-X=\Sigma_y X^T \Sigma_y \end{cases}$

$$\begin{cases} \Sigma_x=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\otimes I \\ \Sigma_y=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\otimes I \end{cases}$$

可以证明 $X \in Sp(2N)$. C类中.

$Sp(2N) \hat{=} C_N$, $\Rightarrow$ C类

证明: $U \in Sp(2N)$

$$\begin{cases} U^{\dagger}U=1 \\ U^T\Sigma_y U=\Sigma_y \end{cases} \Leftrightarrow U=e^X \text{, with } X^{\dagger}=-X \quad \text{辛群.}$$

$$U^T\Sigma_y U=\Sigma_y$$

要求 $e^{X^T}\Sigma_y e^X=\Sigma_y$

$\Leftrightarrow$
$$(1+X^T)\Sigma_y(1+X)$$
$$=(1+X^T)(\Sigma_y+\Sigma_y X)$$

$\Leftrightarrow$ $= X^T\Sigma_y+\Sigma_y X+\Sigma_y=\Sigma_y$

$\Leftrightarrow$ $X^T\Sigma_y+\Sigma_y X=0$

$\Leftrightarrow$ $\Sigma_y X^T\Sigma_y=-X$ 辛代数.

$\{$ Hamiltonian space $\quad H$

形变

| classification Space $\qquad \tilde{\mathcal{H}} = \boxed{M}$

D类 $CHC^{-1} = -H$

$$H|\phi\rangle = \varepsilon|\phi\rangle \Leftrightarrow C|\phi\rangle \Leftrightarrow HC|\phi\rangle = -\varepsilon C|\phi\rangle$$

$\varepsilon \to +1 \qquad \varepsilon \to -1 \qquad U(N)$

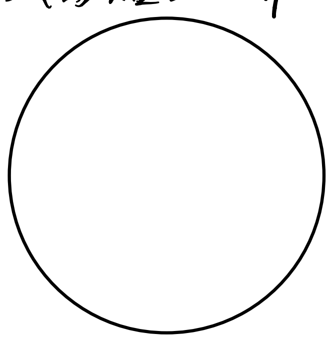
	T	C	S	H	$M = G/\alpha$ (分类空间)
A	0	0	0	$U(N)$	$U(N+M)/U(N)\times U(M)$
D	0	+1	0	$SO(2N)$	$O(2N)/U(N)$
C	0	-1	0	$Sp(2N)$	$Sp(2N)/U(N)$
				$SO(2N)/O(N)$	

DIII, CI,

Topo index.

Winding number ⟺ 高维立体角

$|R_N| = 1$



$$\int dxdy = \frac{1}{2}\int d\omega$$

$$\omega = xdy - ydx$$

① 1D: $\frac{1}{2\pi i}\oint \frac{dz}{z}$

② 2D: $C_2\int \varepsilon^{ij} x_i dx_j$

③ 3D: $C_3\int \varepsilon^{ijk} x_i dx_j \wedge dx_k$.

$\vec{x} = (x_1, \cdots, x_N)$

数 → 矩阵.　　$dx \to u^{-1}du$

$z \to u(z)$　　$\mathrm{Tr}(\quad)$

$$I = \frac{1}{2\pi i}\oint \mathrm{Tr}(u^{-1}du) \in \mathbb{Z}$$

1) $u = \mathrm{diag}(f_1(z), f_2(z), \dots f_N(z))$ 显然,

2) 非对角

2d

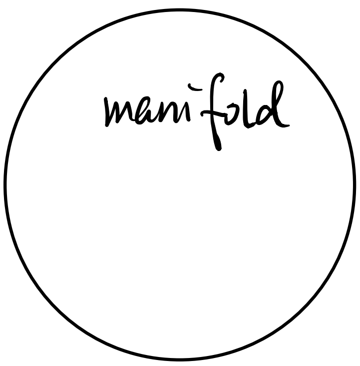
$$C_2 \int \mathrm{Tr}(u^{-1}du \wedge u^{-1}du) \propto \int \mathrm{Tr}[(u^{-1}du)^2] \propto \int \mathrm{Tr}(A^2)$$

问: 为何用 d?　答: 无需坐标, 自动规范不变.

TKNN Number

$$\int \left(\frac{\langle \phi | \frac{\partial H}{\partial k_x} | \phi \rangle \langle \phi | \frac{\partial H}{\partial k_y} | \phi \rangle \cdots}{\quad} \right) dk_x\, dk_y$$

缺点: ① 长 ② 和 k_x 显式相关.



$$\int \langle \phi | \frac{\partial}{\partial \vec{R}} | \phi \rangle \cdot d\vec{R} = \int \langle \phi | d | \phi \rangle$$

$\uparrow$ u^{-1}　$\uparrow$ u　$\Rightarrow \mathrm{Tr}(u^{-1}du)$

class A at $d = 2n+2$

$H(k)|u_\alpha(k)\rangle = E_\alpha(k)|u_\alpha(k)\rangle$

Lie Group

$N^+ \left\{ \begin{array}{l} \text{- - - - -} \\ \text{- - - - -} \\ \text{- - - - -} \end{array} \right.$

$\underline{Q = 1-2P}$

$N^- \left\{ \begin{array}{l} \text{- - - - -} \\ \text{- - - - -} \\ \text{- - - - -} \end{array} \right.$

$\big\} P(k) = \sum_\alpha |u_\alpha^-(k)\rangle\langle u_\alpha^-(k)|$

不是李代数.

$A^{ab}(k) = A_\mu^{ab}(k)dk^\mu = \langle u_a^-(k)|d|u_b^-(k)\rangle$

$F^{ab}(k) = dA^{ab} + (A^2)^{ab}$

OR $F = dA + A^2$, $F' = uFu^{-1}$

意义: $\left\{ \begin{array}{ll} \vec{A}. & \text{规范势} \\ F = dA. & \text{磁场} \end{array} \right.$

$\int \vec{B}\cdot d\vec{S}$ G 定

$= \int \vec{F}\cdot d\vec{S}$

$ch_{n+1}(F) = \frac{1}{(n+1)!}\mathrm{tr}\left[\left(\frac{iF}{2\pi}\right)^{n+1}\right]$

$(2n+1)$ form

$CH_{n+1}(F) = \int ch_{n+1}(F)$ G 定

$CH(F) = \sum_n ch_{n+1}(F)$

$= \sum_n \frac{1}{(n+1)!}\mathrm{tr}\left[\left(\frac{iF}{2\pi}\right)^{n+1}\right]$

$$\sim \mathrm{Tr}\left[e^{\frac{iF}{2\pi}} - 1\right]$$

Next Lecture.

Fu Liang $\mathbb{Z}_2$ Pumping

Review

Topo Insulator

同伦 $\pi_d(M)$

- Topo defect
  - $5d \to X$
- Topo bands (Stable)

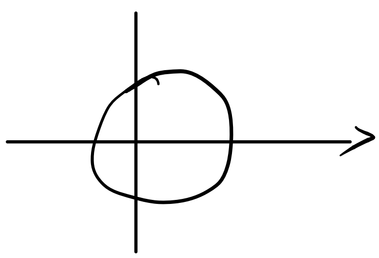
Vortex

liquid crystal

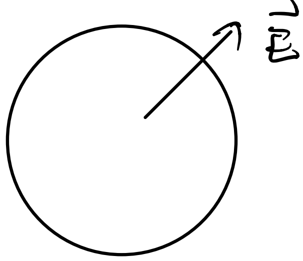
Fermion / Boson / Anyon

Today Topo insulator $\mathbb{Z}_2$ 分类

图像. $\mathbb{Z} \to$ Winding Number



Gauss's Law



$$\oint \vec{E}\cdot d\vec{s} = \mathbb{Z}$$

$\mathbb{Z}_2$ $\quad \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$