$$T^2\psi = T\tilde{\psi} = \tilde{\tilde{\psi}} = \lambda\psi \quad , \lambda = e^{i\theta}$$

$$T^2 = e^{i\theta} \Rightarrow UKUK = e^{i\theta} \Rightarrow UU^* = e^{i\theta}$$

Review

拓扑分类—Bott 周期律.

推导：技巧

stiefel manifold

短正合序列 ☆ $A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4$

主题：为什么对称性很重要 $M = G/H$

单粒子

1) H的分类

2) Topo 相变 ⇒ 能带

key: $\boxed{\text{H 属于（ ）Lie群 / Lie代数。}}$

Lie群：A、AI、AII

C D DI

History：Wigner 1951，1958

Dyson 1962，

多体物理 ⇒ H ⇒ 随机矩阵.

矩阵元独立同分布

① $H = H^\dagger$

② 对称性不变

下面：Symmetry，时间反演，粒子空穴对称，手征对称

BdG equation，Bogoliubov - de - Gennes

$H_{BdG} \in$ Lie Algebra g

ref. Atland, Zirnbauer PRB, 55, 1142 (1997)

$|\langle\varphi|\psi\rangle| = |\langle\hat{A}\varphi|\hat{A}\psi\rangle| \Rightarrow A$ is unitary or Anti-unitary

$$A = uK$$

Time-reversal $T = UK$ 其中 $T^2 = \pm 1$

K，复共轭算符.

$T^2 = 1$, eg $T = K$, $\sigma_x K$

$T^2 = -1$, eg. $T = i\sigma_y K$

讨论 $TH = HT$

OR $\begin{cases} T^2 = 1, & \text{有什么后果} \\ T^2 = -1, & \end{cases}$

实表示 $\Rightarrow H = u\lambda u^{\dagger}$

Kramers degeneracy

$$H = u\lambda u^{\dagger} \uparrow \text{U(N) Group} \xrightarrow{T^2=1} \text{O(N) Group}$$

$T^2 = 1$, $TH = HT$

$H|\phi\rangle = \lambda|\phi\rangle$, $\lambda \in \mathbb{R}$

$$\underbrace{TH|\phi\rangle = \lambda T|\phi\rangle = H(T|\phi\rangle) = \lambda(T|\phi\rangle)}$$

H 本征值为 λ 时，本征态是 $|\phi\rangle$ 和 $T|\phi\rangle$

怎么判定 $|\phi\rangle$ 和 $T|\phi\rangle$ 关系。

$$\langle\phi|T\phi\rangle = \begin{cases} 1, & \text{本征值为单重.} \\ 0, & \text{本征值为两重.} \end{cases}$$

$$|\psi\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad |\phi\rangle = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

区分.
$\langle A^{+}\phi|\psi\rangle \quad \langle\phi|A\psi\rangle$

$$\langle\phi|A\psi\rangle = (b_1^*, b_2^*, b_n^*)\, U \begin{pmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_N^* \end{pmatrix} = \sum b_i^* U_{ij} a_j^*$$

$$|A^{+}\phi\rangle = (UK)^{+}|\phi\rangle = K^{*}U^{+}|\phi\rangle = K(U^{+}|\phi\rangle)$$
$$= K\left((U^{+})_{ij}\, b_j\right)$$
$$= K(b_j U_{ji}^*)$$
$$= b_j^* U_{ji}$$

$$\Rightarrow \langle A^{+}\phi|\psi\rangle = b_j U_{ji}^* a_i = (\langle\phi|A\psi\rangle)^*$$

Summary

A = unitary

$\langle\phi|A\psi\rangle = \langle A^{+}\phi|\psi\rangle$

A = Anti unitary

$\langle\phi|A\psi\rangle = (\langle A^{+}\phi|\psi\rangle)^*$

证明: $T^2 = -1 \Rightarrow |\phi\rangle \perp T|\phi\rangle$

$\langle\phi|T\phi\rangle = 0.$

证 $\langle\phi|T\phi\rangle = (\langle T^{+}\phi|\phi\rangle)^*$

$= -(\langle T\phi|\phi\rangle)^*$

$= -\langle\phi|T\phi\rangle$

$T^2 = -1 \Rightarrow \underline{\underline{T^{+} = -T}}$

$\Rightarrow \langle\phi|T\phi\rangle = 0$

$|\phi\rangle$ 和 $|T\phi\rangle$ 正交。

$\{ \; T^2 = -1 \Rightarrow \langle\phi|T\phi\rangle = 0$, degenerate

$$\Big\{\ T^2=1 \Rightarrow T|\phi\rangle = |\phi\rangle$$

e.g. $T=K \Rightarrow K|\phi\rangle = |\phi\rangle \Rightarrow$ wavefunction 可取成实数.

$T=\sigma_x K$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} y^* \\ x^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} x = x^* \\ y = y^* \end{cases} \Rightarrow \phi = \begin{pmatrix} x \\ x^* \end{pmatrix}$$

$T=\sigma_z K$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} \cancel{x^*} \\ y^* \end{pmatrix} = \begin{pmatrix} x^* \\ -y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \phi = \begin{pmatrix} x \\ -iy \end{pmatrix}$$

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particle-hole

$C = U_p K$

$$\begin{cases} TH = HT \\ CH = -HC \end{cases}$$

$HC|\phi\rangle = -EC|\phi\rangle$

$$\begin{cases} C^2 = 1 \\ C^2 = -1 \end{cases}$$

e.g. $C = \tau_x K$

$C = \tau_y K$

σ : spin ↑↓

τ : particle-hole 空间

10重分类：为什么是10

$3 \times 3 \times 2 = 18$

Chiral symmetry S 手征 $\Leftrightarrow$ 左手、右手 $\underline{\underline{\gamma_5}}$

以

e.g. $\begin{cases} TH = HT \\ CH = -HC \end{cases}$

$\{\gamma_5, H_D\} = 0$

$$\begin{cases} \gamma_5 H_D = -H_D \gamma_5 \\ \gamma_5^2 = 1 \end{cases}$$

$S = CT = U_C K U_T K$

$= \mathcal{U}_C \mathcal{U}_T^*$

令切对角化

$\mathcal{U}\gamma_5\mathcal{U}^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{U} H_0 \mathcal{U}^+ = -h \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\Downarrow$

$h = \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}$

$$\begin{cases} CTH = CHT = -HCT \\ SH = -HS \end{cases}$$

$\Downarrow$

$$\begin{cases} S^+S = 1 \\ S^2H = HS^2 \end{cases}$$

$H S|\phi\rangle = -E S|\phi\rangle$

结论：① 存在 T 对称，C 对称，一定存在 S 对称性。

② $\begin{cases} \text{存在 } T \text{ 和 } S \Rightarrow \text{一定存在 } C \\ \text{存在 } C \text{ 和 } S \Rightarrow \text{一定存在 } T \end{cases}$

③ 例外 $\{S, H\} = 0$，$T^2 = 0$，$C^2 = 0$

④ $S^2 = 0$，$T^2 = 0$，$C^2 = 0$

10 重，T，C

$$\begin{cases} 1) \ \text{2个都有} & 4 \\ 2) \ \text{只有一种} & 4. \\ 3) \ \text{只有 } S & 1 \\ 4) \ \text{都没有} & 1 \end{cases}$$

BdG Equation 以及 H 的空间

$$H_{BdG} = \sum_{\alpha\beta} h_{\alpha\beta} C_\alpha^+ C_\beta + \frac{1}{2}\Delta_{\alpha\beta} C_\alpha^+ C_\beta^+ + \frac{1}{2}\Delta_{\beta\alpha}^* C_\beta C_\alpha .$$

α, β index $\begin{cases} \text{Real space} \\ k \ \ldots \end{cases}$

1 k space

e.g.

$$H = \varepsilon_1 c_1^\dagger c_1 + \varepsilon_2 c_2^\dagger c_2 + \Delta c_1^\dagger c_2^\dagger + \Delta^* c_2 c_1 + t c_1^\dagger c_2 + t c_2^\dagger c_1$$

$$\to (c_1^\dagger\ c_2^\dagger\ \ c_1\ c_2) \left(\begin{array}{cc|cc} \varepsilon_1/2 & t/2 & 0 & \Delta/2 \\ t^*/2 & \varepsilon_2/2 & -\Delta/2 & 0 \\ \hline 0 & -\Delta^*/2 & -\varepsilon_1/2 & -t^*/2 \\ \Delta^*/2 & 0 & -t/2 & -\varepsilon_2/2 \end{array}\right) \begin{pmatrix} c_1 \\ c_2 \\ c_1^\dagger \\ c_2^\dagger \end{pmatrix}$$

$$+ \varepsilon_1/2 + \varepsilon_2/2$$

$$H_{BdG} = \frac{1}{2}(c^\dagger, c) \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}$$

好处 1)形式唯一，
2). 保证对角化后粒子属性不变.

$$H_{BdG} = \frac{1}{2} \begin{pmatrix} h & \Delta \\ \Delta^\dagger & -h^T \end{pmatrix}$$

具有 C 对称性.

$$C = \sigma_x K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K$$

$$CH = -HC \Rightarrow \sigma_x H^* = -H\sigma_x$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} h^T & \Delta^* \\ \Delta^T & -h \end{pmatrix}$$

$$= -\begin{pmatrix} h & \Delta \\ \Delta^+ & h^+ \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow$ $\Delta^T = -\Delta$ ，对于 BdG Equation 有 particle-hole 对称性.

D类. ① Why we call it D class

② $H \to$ Lie Algerbric

$$X = iH \qquad X^\dagger = -X = -\sigma_x X^T \sigma_x$$

$$Y^\dagger = -Y = -\sigma_x Y^T \sigma_x$$

$$[X, Y] = Z$$