2021.6.10.

$\pi_n(M)$. $\begin{cases} \text{不稳定} & \cancel{\$} \dim |M| \text{ 小} \\ \text{稳定} & \dim |M| >> \text{ ?} \end{cases}$

$\uparrow$

Topo Bands.

$\pi_n(G/H)$ 意义/应用

$S^n \to G/H = M$

映射的不等价.

The geometry of ~~sta~~ state space.

ref: Adelman et al Foundations of phys ~~1997~~ 1993.

☆ 文小刚 $H \in$ ~~Multinear~~ Multinear space. / Tensor Hilbert.

$\{H, \int\int\} \to h_1 \otimes h_2 \cdots \otimes h_m$.

Tensor表示.

单比特的态空间.

~~下~~ $f: X \to M$.

$$\rho = \begin{pmatrix} a \\ b \end{pmatrix} (a^* \ b^*) = \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix}$$

更一般 $\rho = \frac{1}{2}(a_0 I + \vec{a}\cdot\vec{\sigma}) = \frac{1}{2}(1 + \vec{a}\cdot\vec{\sigma})$.

$Tr(\rho) = 1 \Leftrightarrow a_0 = 1$, 纯态.

$\vec{x} \to \rho(\vec{x})$.

~~正~~ P_1

态空间.

$\mathrm{Tr}(\rho)=1 \Rightarrow a_0=1$, $\vec{a}$ 呢?

条件. $\rho|\psi\rangle=\lambda|\psi\rangle$, $\lambda\geq 0$, $\Rightarrow \vec{a}$ 有限制

$\mathrm{Tr}(\rho^2)\leq 1$. $\mathrm{Tr}(\rho)=1 \Rightarrow \lambda, 1-\lambda$.

$\mathrm{Tr}(\rho^2)=\frac{1}{4}\mathrm{Tr}[(1+\vec{a}\cdot\vec{\sigma})^2]$.

$=\frac{1}{4}\mathrm{Tr}[1+2\vec{a}\cdot\vec{\sigma}+(\vec{a}\cdot\vec{\sigma})^2]$.

$\mathrm{Tr}(\vec{\sigma})=0$, $(\vec{a}\cdot\vec{\sigma})^2=\vec{a}^2$.

$\Leftrightarrow |\vec{a}|^2\leq 1$.

纯态 $\rho|\psi\rangle=\lambda|\psi\rangle$, $\lambda=0,1$

$\rho=|\psi\rangle\langle\psi|$.

$\Rightarrow \rho^2=|\psi\rangle\langle\psi|\psi\rangle\langle\psi|=\rho$.

$\Leftrightarrow |\vec{a}|=1$, $\Leftrightarrow S^2$ 球面. OR Bloch sphere.

$\rho=U\lambda U^\dagger$.

$\overset{=}{\leftarrow} U\begin{pmatrix}\lambda & 0\\ 0 & 1-\lambda\end{pmatrix}U^\dagger$ $\quad U^\dagger A U=\lambda$. $\quad U\in U(2)$.

1). 图

U自由度. $M=[0,1]_\lambda \otimes U(2)/[U(1)\times U(1)]$

$M=G/H$

$U\to Ue^{i\theta}$

混 $|\vec{a}|^2\le 1 \Rightarrow$ 球 ~~体~~ D^3.

纯 $|a|^2=1 \Rightarrow$ 球 S^2

$$\frac{[0,1]_\lambda \otimes U(2) / U(1)\times U(1)}{I\times S^2}$$

$\underset{4}{U(2)}/U(1)\times U(1). \Rightarrow 4-2=2.$

$\simeq S^2.$

Bott 周期. (N是够大. OR. $N\gg k$.)

1). $\pi_n(S^n)=\mathbb{Z}$.

$\pi_k(S^n)=0$ if $k<n \Rightarrow \pi_k(S^n)\ne 0$, if $k>n$.

2) $\pi_k(U(N))=\pi_k(U(1)\times SU(N))=\begin{cases}0\ ; & k=\text{even}=2n\\ \mathbb{Z}\ , & k=\text{odd}=2n+1.\end{cases}$

3) $\pi_k(O(N))=\pi_k(SO(N))=\begin{cases}0\ , & k=2,4,5,6,\ \text{mod}(8)\\ \mathbb{Z}_2 & k=0,1\ ,\ \text{mod}(8)\\ \mathbb{Z} & k=3,7.\ \text{mod}(8).\end{cases}$

4). $\pi_k(Sp(N))=\begin{cases}0\ , & k=0,1,2,6\\ \mathbb{Z}_2\ , & k=4,5\\ \mathbb{Z}\ , & k=3,7.\end{cases}$ mod(8)

P3.

~~Bott 周~~期: $\pi_k(Sp/U)=\pi_{k+1}(Sp)$

$\cancel{\pi_k(U/O)}=$

Bott 证明

$\pi_k(Sp/U) = \pi_{k+1}(Sp)$

$\pi_k(U/O) = \pi_{k+2}(Sp)$

$\pi_k(O/U) = \pi_{k+1}(O)$

$\pi_k(U/Sp) = \pi_{k+2}(O)$.

定义: Stiefel manifold.

$O^T O = 1,\ \det(O) = \pm 1$

$V_{K,N} = SO(N)/SO(N-k) \simeq O(N)/O(N-k)$

$V^C_{K,N} = U(N)/U(N-k)$.

$V^H_{K,N} = Sp(N)/Sp(N-k)$

可证 $\begin{cases} \pi_i(V_{K,N}) = 0 \\ \pi_i(V^C_{K,N}) = 0 \\ \pi_i(V^H_{K,N}) = 0 \end{cases}$ if $i < N-k$.

求表.

~~H ∈~~

$$H = U\begin{pmatrix}1_N & \\ & -1_M\end{pmatrix}U^+$$

$U \in U(N+M) = G.$

$$U \to U\begin{pmatrix}U_N & 0\\ 0 & U_M\end{pmatrix}$$

$H = U(N)\times U(M).$

$\equiv \}\ +1$

$\updownarrow$ Gap

$\equiv \}\ -1$

$$\mathcal{M} = U(N+M)/\underbrace{U(N)}_{\text{导带}}\times\underbrace{U(M)}_{\text{价带}}$$

求 $\pi_k(\mathcal{M})$.

$$\rho = U\begin{pmatrix}\lambda & 0\\ 0 & 1-\lambda\end{pmatrix}U^+$$

$$\begin{pmatrix}U_N & 0\\ 0 & U_M\end{pmatrix}\begin{pmatrix}1_N & 0\\ 0 & -1_M\end{pmatrix}\begin{pmatrix}U_N^+ & 0\\ 0 & U_M^+\end{pmatrix}.$$

i	$U(M)$ $\pi_i(H)$	$U(N+M)/U(N)$ $\pi_i(G)$	$\pi_i(\mathcal{M})$
	$\mathbb{Z}$	0	0
	0	0	$\mathbb{Z}$
	$\mathbb{Z}$	0	0
	0	0	$\mathbb{Z}$
	$\mathbb{Z}$	0	0
	0	0	

Topo 分类

	T	C	S	已限制 $M = G/H$
A	0	0	0	$U(2n)/U(n)\times U(n)$
AI	1	0	0	$Sp(2n)/Sp(n)\times Sp(n)$
AII	-1	0	0	$O(2n)/O(n)\times O(n)$
AIII	0	0	1	$U(n)$
BDI	1	1	1	$U(2n)/Sp(2n)$
CII	-1	-1	1	$U(2n)/O(2n)$
D	0	1	0	$O(2n)/U(n)$
C	0	-1	0	$Sp(2n)/U(n)$
DIII	-1	-1	0	$O(2n)$

$\pi_d(n)$: 1 | 2 | 3 | 4 | … | 8.

	1	2	3	4	5	6	7	8
A	0	Z	0	Z	0	Z	0	Z
AI	0	0	0	Z	0	Z_2	Z_2	Z
AII	0	Z_2	Z_2	Z	0	0	0	Z
AIII	Z	0	Z	0	Z	0	Z	0
BDI	Z	0	0	0	Z	0	Z	0
CII	Z	0	Z_2	Z_2	Z	Z_2	Z_2	0
D	Z_2	Z	0	0	0	Z	0	Z_2
C	0	Z	0	Z_2	Z_2	Z	0	0
DIII	Z_2	0	Z	0	0	0	Z	Z_2

$\pi_k(U(N)) = \begin{cases} 0, & k = \text{even} \\ Z; & k = \text{odd} \end{cases}$

$\pi_k(O(N)) = \begin{cases} 0, & 2, 4, 5, 6 \\ Z_2, & 8, 1 \\ Z, & 3, 7 \end{cases}$

Bott 周期 可以 找规律

P6.

$$\pi_K(U/Sp) = \pi_{K+2}(O)$$

$$\pi_1(U/Sp) = \pi_3(O)$$

$$\pi_K(O/U) = \pi_{K+1}(O)$$

$$\pi_K(U/O) = \pi_{K+2}(Sp)$$

$K+2 = 0, 1, 2, 6.$

$$= \begin{cases} 0 & \\ \mathbb{Z}_2 & , K+2 = 4, 5 \\ \mathbb{Z} & , K+2 = 3, 7 \end{cases}$$

AZ 分类 (CPT ~~对称~~ sym)

Ref. PRB. 55. 1142 (199??)

$\in$ TRS、PHS、CS

- Wigner … $\longrightarrow$ symmetry
- weyl $\longrightarrow$ gauge

$UU^* \psi = e^{i\theta} \psi$

$T\psi = \hat{\psi}$

$\cancel{P e^{i\lambda}\, e^{-i\lambda}}\ UU^* = e$

$\parallel$

$U^* U = e^{-i\theta}$ P_7

$e^{-i\theta} = e^{i\theta}$

$= e^{i2\theta} = 1$

$\theta = \pi n$