π_3

paper:

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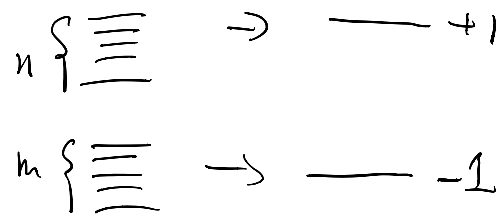
PRL (2012)

如何描述 量子霍尔效应的体系

特点： 1) 能级多.

2) 费米面可调.



$$H = U^{+}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} U \in U(N+M)$$

$$M \in U(N+M) / U(N)\times U(M)$$

$$\pi_d(M)$$

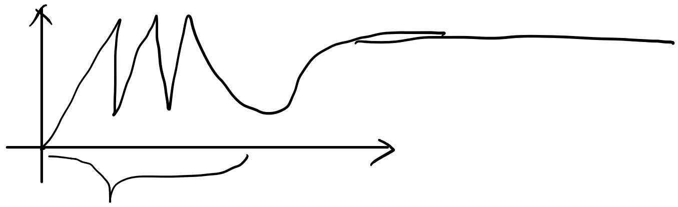
Review

$\pi_n(G/H)$ 意义，应用

$S^n \to G/H = M$ 有多少不等价的映射

1) M

2) $\pi_n(M)$ $\begin{cases} \text{不稳定} & \dim M \text{ 小} \\ \text{稳定} & \dim M \text{ 大} \end{cases}$



The Geometry of state space Ref:

Adelman et al

"Foundation of Phys" 1993

☆ 文小刚

Multilinear Hilbert space

$\mathcal{H} \in$ Linear Hilbert Space

一个系统 $\{H, SS\}$ $SS \in h_1 \otimes h_2 \otimes h_3 \otimes h_4 \cdots \otimes h_n$

单比特的态空间: M

关注于 $f: X \to M$

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \to \rho = \begin{pmatrix} a \\ b \end{pmatrix}(a^* \ b^*) = \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix}$$

更一般地 $\rho = \frac{1}{2}(I + \vec{a}\cdot\vec{\sigma})$

$$\boxed{\begin{array}{l} \psi \sim c\psi \\ CP^d \text{ space} \end{array}}$$

态空间:

$$\begin{cases} Tr(\rho) = 1 \\ \rho = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i| , \text{ with } \lambda_i > 0 \end{cases}$$

$\vec{a}$ 有限制

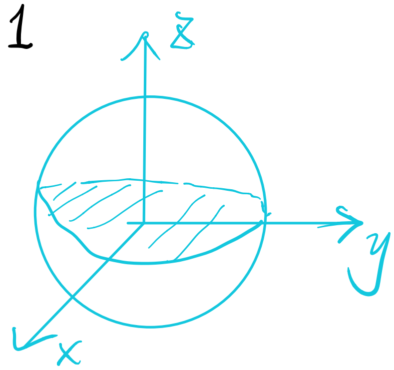
$\Downarrow$

$Tr[\rho^2] = Tr[\frac{1}{4}(1+\vec{a}\cdot\vec{\sigma})^2] \Leftarrow Tr(\rho^2) \le 1$

$= \frac{1}{4}\mathrm{Tr}[1+\vec{a}^2] \Rightarrow |\vec{a}|^2 \leq 1$

$\Downarrow$

态空间是一个由Bloch球包含的点，



若ρ是一个纯态，则态空间是一个Bloch球。

$$\rho = u \lambda u^{+}$$

$$= u \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} u^{+}, \quad u \in U(2) \text{ Group}$$

U矩阵具有自由度。

$$\underset{\text{等价于}}{U \simeq} U \underbrace{\begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}}_{U(1)\times U(1) \text{ Group}}$$

order-parameter space

$$M = [0,1]_\lambda \otimes U(2)/[U(1)\otimes U(1)] \quad \text{混态} \qquad I \times S^2$$

$$M = \{0,1\}_\lambda \otimes \underbrace{U(2)}_{\dim(4)}/\underbrace{[U(1)\otimes U(1)]}_{\dim(2)} \quad \text{纯态} \qquad \simeq S^2$$

M 限制 $\begin{cases} \text{纯/混} \\ \text{TR} \\ \text{PH} \\ \text{symmetry} \end{cases}$

Bott 周期　　N 很大

1) $\pi_n(S^n) = \mathbb{Z}$

$\pi_k(S^n) = 0$ if $k < n$，$k > n$ 时 $\pi_k(S^n)$ 有可能不等于 0

2) $\pi_k(U(N)) = \pi_k(U(1) \times SU(N)) = \begin{cases} 0, & k = \text{even} = 2n \\ \mathbb{Z} & k = \text{odd} = 2n+1 \end{cases}$

3) $\pi_k(O(N)) = \pi_k(SO(N)) = \begin{cases} 0, & k = 2, 4, 5, 6 \mod(8) \\ \mathbb{Z}_2, & k = 0, 1 \mod(8) \\ \mathbb{Z}, & k = 3, 7 \mod(8) \end{cases}$

4) $\pi_k(SP(N)) = \begin{cases} 0, & k = 0, 1, 2, 6 \\ \mathbb{Z}_2, & k = 4, 5 \\ \mathbb{Z}, & k = 3, 7 \end{cases} \mod(8)$

注 G/H

记 $SP = \lim_{N\to\infty} SP(N)$

$O = \lim_{N\to\infty} O(N)$

Bott 证明

$$\begin{cases} \pi_k(SP/U) = \pi_{k+1}(SP) \\ \pi_k(U/O) = \pi_{k+2}(SP) \\ \pi_k(O/U) = \pi_{k+1}(O) \\ \pi_k(U/SP) = \pi_{k+2}(O) \end{cases}$$

定义 Stiefel manifold

$V_{kN} = SO(N)/SO(N-k) \simeq O(N)/O(N-k)$

$V^C_{kN} = U(N)/U(N-k)$

$V^H_{kN} = SP(N)/SP(N-k)$

可证 $\{\ \pi_i(V_{kN}) \equiv 0$

$$\begin{cases} \pi_i(V^c_{kN}) \equiv 0 \\ \pi_i(V^H_{kN}) \equiv 0 \end{cases}$$

求表

$$H = U \lambda U^{\dagger}$$

$$H = U \begin{pmatrix} 1_N & 0 \\ 0 & 1_M \end{pmatrix} U^{\dagger}$$

U 与 $U \begin{pmatrix} u_N & 0 \\ 0 & u_M \end{pmatrix}$ 等价 $\Rightarrow H = U(N) \times U(M)$

$$\Rightarrow M = U(N+M) / U(N) \times U(M)$$

不合适的构造

$\pi_i(U(N)) = \begin{cases} 0, & \text{even} \\ \mathbb{Z}, & \text{odd} \end{cases}$

i	$U(N)\times U(M)$ $\pi_i(H)$	$U(N+M)$ $\pi_i(G)$	$U(N+M)/U(N)\times U(M)$ $\pi_i(G/H)$	
4	0	0	?	
3	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$	?	
2	0	0	?	
1	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}$	?	
0	0	0	?	

正确做法.

i	$U(M)$ $\pi_i(H)$	$U(N+M)/U(N)$ $\pi_i(G)$	$U(N+M)/U(N)\times U(M)$ $\pi_i(G/H)$	
5	$\mathbb{Z}$	0	0	
4	0	0	$\mathbb{Z}$	
3	$\mathbb{Z}$	0	0	
2	0	0	$\mathbb{Z}$	
1	$\mathbb{Z}$	0	0	
0	0	0		

Topo 分类表

$T^2 = 0, \pm 1$, S

$C^2 = 0, \pm 1$.

	T	C	S	$M = G/H$	$\pi_d(M)$ 1	2	3	4	5	6	7	8
A	0	0	0	$U(2n)/U(n)\times U(n)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$Sp(2n)/Sp(n)\times Sp(n)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
AII	-1	0	0	$O(2n)/O(n)\times O(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AIII	0	0	1	$U(n)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
BDI	1	1	1	$U(2n)/Sp(2n)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
CII	-1	-1	1	$U(2n)/O(2n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

D	0	1	0	$O(2n)/U(n)$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
C	0	-1	0	$Sp(2n)/U(n)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
DIII	-1	+1	0	$O(2n)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
CI				$Sp(2M)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0

Altland and Zirnbauer, PRB, 55, 1142 (1997)

Concept.
1> time-reversed symmetry
2> Particle-Hole symmetry
3> chiral symmetry

基本定理：Wigner 定理.

$$\begin{cases} \text{时间反演.} & T = U_T K \\ \text{粒子空穴对称} & C = U_p K \end{cases}$$

最重要的结果 $T^2 = \pm 1$

特殊的[illegible]

$$\begin{cases} T = K & T^2 = 1 \\ T = \sigma_x K & T^2 = 1 \\ T = \sigma_y K & T^2 = -1 \\ T = \sigma_z K & T^2 = 1 \end{cases}$$

$T\psi = \tilde{\psi}$

$t \rightarrow -t \longrightarrow t$

$\psi \rightarrow \tilde{\psi} \longrightarrow \tilde{\tilde{\psi}}$

$\Downarrow$