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The Geometry of State Space

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The geometry of the state space of a finite-dimensional quantum mechanical system, with particular reference to four dimensions, is studied. Many novel features, not evident in the two-dimensional space of a single spin, are found. Although the state space is a convex set, it is not a ball, and its boundary contains mixed states in addition to the pure states, which form a low-dimensional submanifold. The appropriate language to describe the role of the observer is that of flag manifolds.

The geometry of the state space of a single spin, $E(\mathbb{C}^2)$, is so well known as to be a standard textbook topic, but there is surprisingly little known about the geometry of the state space of larger systems, apart from general statements such as it is a convex region which is generated by the pure states which are extremal points. Even for $E(\mathbb{C}^4)$, which is the state space for a pair of spins and hence the manifold appropriate to the Einstein–Podolsky–Rosen paradox, little appears to be known.

It is the purpose of this article to describe the structure of $E(\mathbb{C}^4)$, and, as it turns out, many major features of $E(\mathbb{C}^n)$ for arbitrary finite n. The principal motive for this inquiry arose out of the work of the first named authors[(1)] on intuitionist quantum mechanics. In this formulation a special neighborhood basis was introduced, and it was thought desirable to test the properties of this basis on some special examples. The case of $E(\mathbb{C}^2)$ is too special to be of much use, and so the case of $E(\mathbb{C}^4)$ was selected. It turns out that this is sufficient to illustrate the general situation.

The structure of this paper is as follows. In Section 1, a brief review of $E(\mathbb{C}^2)$ is given, following the very nice article of Urbantke[(2)] drawing atten-

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tion to the reasons why this is not a suitable starting point for general study. In Sec. 2, $E(\mathbb{C}^4)$ is studied in detail. It is shown that the natural step of using a Clifford algebra representation of the density matrices is not very helpful. Instead, the manifold of density matrices is described in terms of the diagonalizing unitary matrices $\mathscr{U}(4)$, and the appropriate language is found to be that of flag manifolds. In Section 3 the special case where $E(\mathbb{C}^4)$ reduces to a product $E(\mathbb{C}^2)\times E(\mathbb{C}^2)$ is briefly discussed. In Section 4 the extension to $E(\mathbb{C}^n)$ is outlined, and in Section 5 some reference is made to the topology of intuitionist quantum mechanics.

1. SINGLE SPIN—$E(\mathbb{C}^2)$

The state of a single spin is given by a 2×2 positive-definite hermitian matrix ρ of trace one. If we introduce the Pauli matrices σ_1, σ_2, σ_3, we can write

$$\rho = (a_0 1 + \mathbf{a}\cdot\sigma)/2 \tag{1}$$

Then

$$1 = \operatorname{Tr}\rho = a_0 \tag{2}$$

The condition that ρ be positive definite is fulfilled if both the eigenvalues of ρ are positive, and this will be so if

$$\mathbf{a}^2 \leqslant 1 \tag{3}$$

When $\mathbf{a}^2 = 1$, $\rho^2 = \rho$—pure states. So we see that $E(\mathbb{C}^2)$ is the closed unit ball $\mathbb{B}^3$, with the boundary the sphere $\mathscr{S}^2$ consisting entirely of pure states. All interior points of $\mathbb{B}^3$ are convex combinations of the surface points, which are extremal points.

As already mentioned, this approach does not generalize, and so we must proceed in a different way. As ρ is hermitian, for each such ρ there exists a unitary matrix $U\in\mathscr{U}(2)$ such that

$$\rho = U\Lambda U^* \tag{4}$$

where

$$\Lambda = \begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}, \qquad \lambda_1+\lambda_2 = 1, \qquad \lambda_i \geqslant 0 \tag{5}$$

If

$$U' = \begin{bmatrix} e^{i\theta_1} & \\ & e^{i\theta_2}\end{bmatrix} \in \mathscr{U}(1)\times\mathscr{U}(1) \tag{6}$$

then $U'\Lambda U'^* = \Delta$, and so for a given Λ the corresponding set of density matrices can be partitioned into equivalence classes which are homeomorphic to the left cosets $G/H = \mathscr{U}(2)/\mathscr{U}(1) \times \mathscr{U}(1)$. As the group manifold of $\mathscr{U}(2)$ is four-dimensional and that of $\mathscr{U}(1)$ is one-dimensional, G/H is two-dimensional. The set of different Λ, under the constraint (5), is one-dimensional, and so dim $E(\mathbb{C}^2) = 2 + 1 = 3$, as required.

However, this describes the general case where $\lambda_i > 0$, $\lambda_1 \neq \lambda_2$. If $\lambda_2 = 0$, say, then we have a pure state, and then dim $E(\mathbb{C}^2) = 2$, corresponding to the surface $\mathscr{S}^2$. If $\lambda_1 = \lambda_2 = 1/2$, the appropriate left coset is now $\mathscr{U}(2)/\mathscr{U}(2)$, which is trivial, and the state space reduces to a single point.

The manifold of G/H is $\mathscr{S}^2$, which may be interpreted as the Riemann sphere, and hence as the projective space $\mathbb{CP}^1$. The space Δ of the Λ's is the interval [0, 1] of the real line and $E(\mathbb{C}^2)$ is the ball $\mathbb{B}^3$.

2. TWO SPINS—$E(\mathbb{C}^4)$

The only attempt that we are aware of to describe $E(\mathbb{C}^4)$ is that given by Majorana; see also Penrose.[3] They considered the general case of spin j. A pure state vector can be written

$$\psi = \sum_{k=-j}^{j} c_k \psi_k \tag{7}$$

They introduce $2j$ complex numbers $\zeta_1, \ldots, \zeta_{2j}$ which are the roots of the equation

$$a_0 \zeta^{2j} + a_1 \zeta^{2j-r} + \cdots + a_{2j} = 0 \tag{8}$$

where

$$a_r = (-1)^r \frac{c_{j-r}}{\sqrt{(2j-r)!r!}} \tag{9}$$

On writing $\zeta_s = \tan \frac{1}{2}\vartheta_s e^{i\varphi_s}$, each ζ_s can be assigned a point on the unit sphere with angles (ϑ_s, φ_s). So the state ψ can be represented by a set of $2j$ unordered points on $\mathscr{S}^2$, or alternatively as a point in the space $(\mathscr{S}^2 \times \mathscr{S}^2 \times \cdots \times \mathscr{S}^2)/S_{2j} \sim \mathbb{CP}^{2j}$. We shall return to this in Section 4. However, this representation refers only to pure states.

The density matrices ρ are 4×4 positive-definite matrices of trace 1, and we could represent them by

$$\rho = \sum_{i=0}^{3} \sum_{j=0}^{3} x_{ij} \sigma_i \otimes \sigma_j \tag{10}$$

where σ_i are the Pauli matrices, $\sigma_0 = 1, \sigma$. From the trace condition we have

$$x_{00} = \operatorname{Tr} \rho/4 = 1/4 \tag{11}$$

A necessary and sufficient condition for $\rho \geqslant 0$ is to write it as $\rho = \tau^2$, where

$$\tau^* = \tau = \sum_{i=0}^{3} \sum_{j=0}^{3} y_{ij} \sigma_i \otimes \sigma_j \tag{12}$$

and then (11) becomes

$$\sum_{i=0}^{3} \sum_{j=0}^{3} y_{ij}^2 = 1/4 \tag{13}$$

This is the sphere $\mathscr{S}^{15}(1/4)$, and so we have $E(\mathbb{C}^4) \subset \mathscr{S}^{15}(1/4)$. However, the inclusion is proper because the mapping $\tau \to \rho$ is many-one, and it is not clear how to choose a unique representative of each equivalence class. Choosing $\tau > 0$, for example, merely refers the problem on.

The set of all 4×4 hermitian matrices is 16-dimensional, and the condition $\operatorname{Tr} \rho = 1$ restricts the manifold to lie in a plane $P \subset \mathbb{R}^{16}$. The condition $\rho \geqslant 0$ restricts the manifold further so that it is a convex 15-dimensional body.

Consider $\operatorname{Tr} \rho^2 \leqslant \operatorname{Tr} \rho = 1$. If we rotate the axes so that they lie in the plane $\sum_{i=1}^{4} \rho_{ii} = 1$, with the 4-axis perpendicular to that plane, and shift the origin to the point (1/4, 1/4, 1/4, 1/4), we have

$$\operatorname{Tr} \rho^2 = \sum_{i=1}^{3} \rho_{ii}'^2 + \sum_{i<j} (x_{ij}^2 + y_{ij}^2) + \tfrac{1}{4} \leqslant 1 \tag{14}$$

where $\rho_{ij} = (x_{ij} + i y_{ij})/\sqrt{2} = \bar{\rho}_{ji}$, $i \neq j$. This region is the ball $\mathbb{B}^{15}(\sqrt{3}/2)$, and the only points of $E(\mathbb{C}^4)$ which lie on its boundary are the pure states. There are other boundary points of $E(\mathbb{C}^4)$, e.g., $\rho_{11} = \rho_{22} = 1/2$, $\rho_{ij} = 0$ otherwise, which lie inside $\mathbb{B}$, and so $E(\mathbb{C}^4)$ is a proper convex subset of $\mathbb{B}$, in contrast to the case of $E(\mathbb{C}^2)$. This follows from the following lemma, which summarizes some well-known properties of convex sets. For brevity we shall, in future, denote $E(\mathbb{C}^4)$ by $\mathscr{E}$.

Lemma 1. If $\rho, \rho' \in \operatorname{Int} \mathscr{E}$, and $\rho_t = (1-t)\rho + t\rho'$, then $\rho_t \in \operatorname{Int} \mathscr{E}$ for $0 \leqslant t \leqslant 1$. If $\exists t_0 > 1$ such that $\rho_{t_0} \in \partial \mathscr{E}$, then t_0 is unique and $\rho_t \notin \mathscr{E}$ for $t > t_0$ and $\rho_t \in \operatorname{Int} \mathscr{E}$ for $0 \leqslant t < t_0$. When $t = t_0$, $\det \rho_{t_0} = 0$.

Proof. The first two statements follow from the properties of convex sets.[4] When $t = t_0$, there exists a vector $\xi^0 \colon (\xi_1^0, \ldots, \xi_4^0)$ such that

$(\rho_{t_0})_{ij}\, \xi_i^0 \xi_j^0 = 0$, and if we choose $U \in \mathscr{U}(4)$, so that ρ_{t_0} is diagonal with eigenvalues $\lambda_1^0,..., \lambda_4^0$, at least one of λ_i must be zero. Hence det $\rho_{t_0} = 0$.

We have seen from the example given just before the lemma that boundary points can be mixed states, and we now show that the boundary is made up of 3-simplices.

Lemma 2. The states ρ form a convex set with the pure states as extremal points. The boundary is composed of convex combinations of any three independent pure states, i.e., it is made up of 3-simplices.

Proof. If P_1 and P_2 are pure states, then the convex combination $\lambda_1 P_1 + \lambda_2 P_2$, $\lambda_1 + \lambda_2 = 1$, λ_1, $\lambda_2 \geqslant 0$ is a state because it is a positive hermitian matrix of trace 1. Conversely, $\rho = U \Lambda U^*$ can be written in terms of a standard pure state P_1 as

$$\rho = \lambda_1 U_1 P_1 U_1^* + \lambda_2 U_2 P_1 U_2^* + \lambda_3 U_3 P_1 U_3^* + \lambda_4 U_4 P_1 U_4^*$$

where $U_j = U S_{(1j)}$, $S_{(1j)}$ being the permutation interchanging the positions 1 and j, and, as $\sum \lambda_j = 1$, this is a convex combination of pure states.

On the boundary ρ has rank $\leqslant 3$, which means that at least one of λ_i is zero, and so ρ is a convex combination of at most three pure states. Conversely, if $\rho = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$, the rank of ρ is $\leqslant 3$. If we start with a given 3-simplex, the whole boundary will be generated from it by application of the group $\mathscr{U}(4)$.

For further elucidation of the structure of $\mathscr{E}$, we shall study the properties of the diagonalization matrix given by (4). There are a large number of special cases which arise when two or more of the eigenvalues are equal. For definiteness, we order the eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$. The various possibilities are set out in Table I. For brevity we put $G = \mathscr{U}(4)$.

If $\lambda_4 = 0$, we have the same classification except that the last line is omitted as it no longer corresponds to a state. The dimensions of the associated flag manifolds are unchanged, but the dimension of the eigenvalue space, and hence the submanifold dimension, decreases by one. The manifold of pure states is $\mathbb{F}_1$, which is six-dimensional. As these are the extremal points of $\mathscr{E}$, they generate the whole of $\mathscr{E}$ by taking convex combinations. The pure states form a *very small manifold, and yet they are able to generate all the states.*

There are a number of undefined concepts which enter into this table, and they will now be explained.

Stability Groups. If U' commutes with Λ, then $U' \Lambda U'^* = \Lambda$, and the set of all such elements will form a subgroup $H_\Lambda \subset \mathscr{U}(4)$ which is the

Table I.

Eigenvalues	Stability group H	Flag manifold G/H	Dimension of flag manifold	Dimension of eigenvalue space	Dimension of sub-manifold
$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$	$(\mathscr{U}(1))^4$	$\mathbb{F}_{123}$	12	3	15
$\lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > 0$	$\mathscr{U}(2) \times (\mathscr{U}(1))^2$	$\mathbb{F}_{23}$	10	2	12
$\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4 > 0$	$\mathscr{U}(1) \times \mathscr{U}(2) \times \mathscr{U}(1)$	$\mathbb{F}_{13}$	10	2	12
$\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 > 0$	$(\mathscr{U}(1))^2 \times \mathscr{U}(2)$	$\mathbb{F}_{12}$	10	2	12
$\lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 > 0$	$\mathscr{U}(1) \times \mathscr{U}(3)$	$\mathbb{F}_1$	6	1	7
$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > 0$	$\mathscr{U}(2) \times \mathscr{U}(2)$	$\mathbb{F}_2$	8	1	9
$\lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 > 0$	$\mathscr{U}(3) \times \mathscr{U}(1)$	$\mathbb{F}_3$	6	1	7
$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 > 0$	$\mathscr{U}(4)$	$\mathbb{F}_0$	0	0	0

stability group of Λ. For example, for Λ: $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, H_Λ will be the set of diagonal matrices:

$$\begin{bmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & e^{i\theta_3} & \\ & & & e^{i\theta_4} \end{bmatrix}, \quad \text{which is } (\mathscr{U}(1))^4$$

If Λ: $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$, H_Λ is the group of matrices

$$\begin{bmatrix} \mathscr{U}(2) & \\ & \mathscr{U}(2) \end{bmatrix}, \quad \text{or } \mathscr{U}(2) \times \mathscr{U}(2)$$

The others are similarly derived.

So the distinct ρ_Λ corresponding to a given Λ are given by $\rho = U\Lambda U^*$, where $U \in G/H_\Lambda$ and the set $\{\rho_\Lambda\}$ is homeomorphic to the homogeneous space G/H_Λ. As all the H_Λ are block diagonal, they will leave invariant a nested sequence of complex subspaces $V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_{d_\Lambda}} \subset V \equiv \mathbb{C}^4$. The values of the dimensions $k_1, k_2, \ldots$ are listed in Table II.

Table II.

Case	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
Dimensions	(1, 2, 3)	(2, 3)	(1, 3)	(1, 2)	1	2	3	0

The set of subspaces $V_{k_1} \subset V_{k_2} \ldots$ with fixed $(k_1, k_2, \ldots, k_{d_A})$ is called a *flag manifold* $\mathbb{F}_{k_1 \ldots k_{d_A}}$, and they have complex dimension

$$\dim_{\mathbb{C}} \mathbb{F} = \sum_{j=1}^{d_A} k_j(k_{j+1} - k_j), \ (k_{d_A+1} = n, \dim_{\mathbb{R}} \mathbb{F} = 2 \dim_{\mathbb{C}} \mathbb{F})$$

This explains the notation in the third column of Table I, and how the numbers in the fourth are calculated. The special case $d_A = 1$ is called a *Grassmann manifold*, the space of linear subspaces of dimension k_1. It is a complex projective space which can be canonically embedded into the complex projective space $\mathbb{CP}^N$, where $N = \binom{n}{k_1} - 1$. For $n = 4$, the only additional case (because $k_1' = n - k_1$ is the same manifold as k_1) is $\mathbb{F}_2$, when $k_1 = 2$, $N = 5$ and the embedding is proper. This is the well-known case first analyzed by Grassmann, and is fundamental in twistor theory. All flag manifolds can be realised as projective spaces, and, as is obvious in this context, are all compact. The case where $k_{j+1} - k_j = 1 \forall j$ is called *complete, full, or standard.*

The eigenvalues λ_i are restricted by $\sum_{i=1}^4 \lambda_i = 1$, $\lambda_i \geqslant 0$ and span a 4-simplex Δ_4' which is generated by the extremal points, $\mathbf{e}_1' = (1, 0, 0, 0)$, $\mathbf{e}_2' = (0, 1, 0, 0)$, $\mathbf{e}_3' = (0, 0, 1, 0)$, $\mathbf{e}_4' = (0, 0, 0, 1)$, so that $\Delta_4' = \sum_{i=1}^4 \lambda_i \mathbf{e}_i'$. Because the symmetric group $S_4 \subset \mathscr{U}(4)$, there will be double counting in the sets $\{\rho_A\}$ unless $\{\lambda_i\}$ are ordered as in Table I. So we should restrict ourselves to the 4-simplex Δ_4 generated by the extremal points $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (1/2, 1/2, 0, 0)$, $\mathbf{e}_3 = (1/3, 1/3, 1/3, 0)$, $\mathbf{e}_4 = (1/4, 1/4, 1/4, 1/4)$, i.e., $\Delta_4 = \sum_{i=1}^4 \alpha_i \mathbf{e}_i$ with $\sum_{i=1}^4 \alpha_i = 1$, $\alpha_i \geqslant 0$. Then we have

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1/2 & 1/2 & & \\ 1/3 & 1/3 & 1/3 & \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \tag{15}$$

The boundaries $\partial\Delta_4$ of Δ_4 are obtained by putting up to three of the α_i equal to zero, thereby giving the eight regions of Table I, together with the seven boundary regions obtained by putting $\lambda_4 = 0$. We can summarize the foregoing discussion in the statement

Theorem 1. The state space $\mathscr{E}$ can be written as a disjoint union

$$\mathscr{E} = (\Delta_4 \times \mathbb{F}_{123}) \cup (\partial\Delta_4 \times \mathbb{F}_{\partial})$$

where the $\partial\Delta_4$ are obtained by putting one or more $\alpha_i = 0$ and $\mathbb{F}_{\partial}$ are the corresponding flag manifolds listed in Table I.

From the polar decomposition for any matrix $G \in GL(4, \mathbb{C}) \equiv G_c$, i.e., $G = AU$ with $U \in \mathscr{U}(4)$, A positive definite, we can infer that the set of all positive-definite matrices is given by

$$\{A\} = G_c/G \tag{16}$$

However, the condition Tr $A = 1$ has still to be imposed.

In terms of physics, the regions Δ_4 and $\partial\Delta_4$ describe the various experimentally possible mixtures of spin states, whereas the flag manifolds describe the different observer situations.

The mathematics of flag manifolds contains a very beautiful structure theory which will now be outlined, although at present it is not clear to us what the physical significance of this decomposition is. In order to do this, it is necessary to work with G_c rather than G although it is the latter which is physically relevant. Instead of the diagonal and block-diagonal subgroups H, the appropriate complexifications H_c are the Borel and parabolic subgroups. The *standard Borel subgroup* B is the subgroup of upper triangular matrices, general Borel subgroups being obtained from B by conjugation within G_c, and it is clear that B also leaves $\mathbb{F}_{123}$ invariant, so that

$$\mathbb{F}_{123} \cong G/(\mathscr{U}(4))^4$$

Accordingly, in studying the properties of $\mathbb{F}_{123}$, one can just as well work with the complex groups. Parabolic subgroups P are subgroups for which $B \subset P \subset G_c$, and to each P there corresponds a flag manifold from among the list given. For example, if P is of the form

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

then it leaves $\mathbb{F}_{23}$ invariant, so that now

$$\mathbb{F}_{23} \cong G_c/P \cong G/\mathscr{U}(2) \times (\mathscr{U}(1))^2$$

The key step in the structure analysis of $\mathbb{F}_{123}$ (or of any complete flag manifold) is the construction of the *Bruhat decomposition*. This asserts that

$$G_c = \bigcup_{w \in W} BwB \tag{17}$$

where W is the Weyl group of the associated Lie algebra. For $GL(4, \mathbb{C})$ it is the symmetric group S_4. Equation (17) is called a double coset decomposition of G_c and describes a representation of an arbitrary nonsingular matrix as a product of two upper triangular matrices with a permutation matrix sandwiched between. This decomposition is a disjunct decomposition and is nonunique to the extent that if $S_w \subset B$ is the subgroup for which $S_w w \subset B$, then $BS_w wB = BwB$. So, in order to have a unique decomposition, we choose a representative U_w from the left cosets B/S_w, and then

$$G_c = \bigcup_{w \in W} U_w wB \tag{18}$$

The flag manifold—homogeneous space—has therefore the decomposition

$$G_c/B = \bigcup_{w \in W} B_w, \qquad B_w = U_w w$$

The sets B_w, which are called *Bruhat cells*, give a partitioning of $\mathbb{F}_{123}$, each cell being labelled by $w \in W$. The elements of $W \cong S_4$ can be ordered by a length function $l(w)$, which is the smallest integer such that w can be written as a product of the generating reflections (12) and (23). In S_4, $l(w)$ ranges from 0, the identity, to 6, the element (14)(23). Then it can be shown that $B_w \cong \mathbb{C}^{l(w)}$, i.e., complex dim $B_w = l(w)$. *Schubert cells* are the closure of Bruhat cells and include all the lower order ones:

$$\bar{B}_w = \bigcup_{w' \prec w} B'_w \tag{19}$$

This sequence of Bruhat cells generalizes the decomposition of the projective space $\mathbb{CP}^n$:

$$\mathbb{CP}^n = \bigcup_{k=0}^{n} \mathbb{C}^k$$

with the important distinction that while the latter is a chain decomposition, (19) is a lattice decomposition, there being in general several cells for each $l(w)$.

The smaller flag manifolds have a similar decomposition. To each parabolic subgroup P_θ there corresponds a subgroup $W_\theta \subset W$ of the Weyl group such that

$$P_\theta = BW_\theta B \tag{20}$$

and, as a consequence, (18) is replaced by

$$G_c = \bigcup_{w \in W^\theta} U_w wP_\theta \tag{21}$$

where $W^{\theta} = W_{\theta} \backslash W$ is the right coset with respect to W_{θ}. The ensuing Bruhat decomposition is obtained by collapsing together Bruhat cells which lie in the same right W_{θ} coset.

The Schubert cells therefore correspond physically to an increasing range of observer possibilities, with the highest cell having the same dimension as the corresponding flag manifold. However, because this decomposition is made using G_c, rather than G, a simple interpretation has not yet been found.

3. PRODUCT SPACE—$E(\mathbb{C}^2) \times E(\mathbb{C}^2)$

If the system is prepared so that there is no correlation between the two spins, the density matrix ρ can be written as a direct product $\rho = \rho^{\mathrm{I}} \times \rho^{\mathrm{II}}$. This means that (10) is replaced by

$$\rho = (1^{\mathrm{I}} + \mathbf{b}^{\mathrm{I}} \cdot \sigma^{\mathrm{I}})/2 \times (1^{\mathrm{II}} + \mathbf{b}^{\mathrm{II}} \cdot \sigma^{\mathrm{II}})/2 \tag{22}$$

The manifold of pure states is then the product $\mathcal{S}^2 \times \mathcal{S}^2$ and has dimensions 4. The state space $E(\mathbb{C}^2) \times E(\mathbb{C}^2)$ is the convex hull.

It might be thought, intuitively, that this state space is of lower dimension than $E(\mathbb{C}^4)$, but surprisingly it is not. This is because a convex combination of two product states is not a product state except when there is a common factor. Although it is difficult to determine the shape of $E(\mathbb{C}^2) \times E(\mathbb{C}^2)$, its dimensionality can be determined from the following argument. Consider the set of all vectors whose endpoints lie in the set of pure states $\mathscr{P}(\mathbb{C}^2 \times \mathbb{C}^2)$. These will span a linear subspace P of $\mathbb{R}^{16}$, and $\dim \overline{\mathrm{conv}}\, \mathscr{P}(\mathbb{C}^2 \times \mathbb{C}^2) = \dim P$. A simple calculation shows that $\dim P = 15$, which is the same as $\dim E(\mathbb{C}^4)$. So $E(\mathbb{C}^2) \times E(\mathbb{C}^2)$ will be a proper subset of $E(\mathbb{C}^4)$, of nonzero measure. It is certainly proper because it does not contain any pure states which are not product states, but it is a nontrivial subset.

4. EXTENSION TO $E(\mathbb{C}^n)$

All that has been said about $E(\mathbb{C}^4)$ extends with little difficulty to $E(\mathbb{C}^n)$. It is a convex subset of the $(n^2 - 1)$-dimensional ball $\mathbb{B}(\sqrt{(n-1)/n})$ touching at the pure states, but also having boundary points which are strictly inside $\mathbb{B}$. The pure states are described by the manifold

$$\mathbb{F}_1 = \mathscr{U}(n)/\mathscr{U}(1) \times \mathscr{U}(n-1) \cong \mathbb{CP}^{n-1} \tag{23}$$

This is the manifold of Majorana and Penrose described in Section 2, but it is now part of a more general theory. $\mathbb{F}_1$ has complex dimension $(n-1)$, and so has real dimension $2(n-1)$. All of its points are extremal points whose convex hull is $E(\mathbb{C}^n)$.

Table I can be appropriately extended to include all the flag manifolds $\mathbb{F}_{k_1 k_2 \ldots k_{d_\Lambda}}$, describing the observers, and the n-simplices Δ_n, together with their boundaries, represent the physical state of the system. Each of the flag manifolds can be decomposed into Bruhat and Schubert cells following the Bruhat ordering $l(w)$ over the Weyl group.

It is interesting to note that the type of analysis of the structure of ρ using the diagonalization equation $\rho = U\Lambda U^*$ was used by von Neumann and Wigner[5] in their analysis of noncrossing levels in molecular physics, with the difference that there ρ was replaced by the Hamiltonian H and so conditions of positive definiteness and unit trace were not imposed.

5. TOPOLOGY OF INTUITIONIST QUANTUM MECHANICS

In Ref. 1 it was shown that the set of neighborhoods

$$\mathcal{N}(\rho, P, \varepsilon) = \{\sigma \mid (\mathrm{Tr}(\sigma - \rho) P) < \varepsilon\} \tag{24}$$

where P is an arbitrary projection operator, provides a basis of neighborhoods for the weak-*topology on the state space $\mathscr{E}$.

A sub-basis of neighborhoods is the set $\mathcal{O}(P, \varepsilon)$:

$$\mathcal{O}(P, \varepsilon) = \{\rho \in \mathscr{E} \mid \mathrm{Tr}(\rho P) > 1 - \varepsilon\} \tag{25}$$

From (1) we can write

$$P = (1 + \mathbf{x} \cdot \sigma)/2$$

with $\mathbf{x}^2 = 1$. Then

$$1 - \varepsilon < \mathrm{Tr}\, \rho P \Leftrightarrow \mathbf{a} \cdot \mathbf{x} > 1 - 2\varepsilon \tag{26}$$

So for given P, $\rho \in \mathcal{O}(P, \varepsilon)$ iff $\mathbf{a}$ lies in the polar cap of the sphere.

Not surprisingly, when we look at $E(\mathbb{C}^4)$ it is not possible to say as much. However, a number of points can be made. The first is to note that from

$$\mathrm{Tr}\, \rho P = \mathrm{Tr}(U\Lambda U^* P)$$

the same Λ is obtained if U belongs to the double coset $\mathscr{U}' \backslash \mathscr{U}(4)/\mathscr{U}''$, where $U \in \mathscr{U}'$ if $U^*PU = P$, and $U \in \mathscr{U}''$ if $U\Lambda U^* = \Lambda$. So, for example, if P is the pure state P_1, $\mathscr{U}' = \mathscr{U}(1) \times \mathscr{U}(3)$, and $\mathscr{U}''$ belongs to the stability group appropriate to Λ. Hence, if $\rho \in \mathcal{O}$, then so do all the points in the orbit of ρ under $\mathscr{U}'$.

Next we show that the sets $\mathcal{O}(P, \varepsilon)$ cover $\mathscr{E}$ when $\{P\}$ are any set of four orthogonal pure states.

Lemma 3. $\forall \rho$, $\exists j$ such that $\operatorname{Tr} \rho P_j > 1 - \varepsilon$ if $\varepsilon > 3/4$.

Proof. Suppose $\operatorname{Tr} \rho P_i \leqslant 1 - \varepsilon \forall i$.

Then $1 = \operatorname{Tr} \rho = \operatorname{Tr} \rho \sum_{i=1}^4 P_i = \sum_{i=1}^4 \operatorname{Tr} \rho P_i \leqslant 4(1-\varepsilon) < 1$, a contradiction. These four neighborhoods are not disjoint.

At present we do not have an explicit algebraic characterization of the neighborhoods $\mathcal{O}(P, \varepsilon)$ such as is available for $E(\mathbb{C}^2)$.

6. CONCLUSION

Unfortunately, or fortunately, depending on taste, there still remain a number of open questions.

(1) As mentioned in the text the decomposition into Bruhat and Schubert cells has not been given an interpretation in terms of observers. A possible line of enquiry is to use the Murnaghan[6] factorization of $\mathscr{U}(4)$

$$U = U_{34}(\phi_3, \sigma_6)\, U_{23}(\theta_3, \sigma_5)\, U_{24}(\phi_2, \sigma_4)\, U_{12}(\theta_2, \sigma_3)\, U_{13}(\theta_1, \sigma_2) \times U_{14}(\phi_1, \sigma_1)\, D(\delta_1, \delta_2, \delta_3, \delta_4) \tag{27}$$

where $D(\cdot)$ is the diagonal matrix and U_{pq} are the 4×4 matrices with $(U_{pq})_{ij} = 1$ for $i = j$, $i, j \neq p, q$ and

$$(U_{pq})_{ij} = \begin{bmatrix} \cos\theta & -\sin\theta e^{-i\sigma} \\ \sin\theta e^{i\sigma} & \cos\theta \end{bmatrix} \text{for } i, j = p, q \text{ and } (U_{pq})_{ij} = 0 \text{ otherwise}$$

This is a left coset decomposition with respect to $(\mathscr{U}(1))^4$ and possibly gives an explicit expression for B_w.

(2) The neighborhoods $\mathcal{O}(P, \varepsilon)$ for general projectors need more explicit characterization.

(3) As the equation for the boundary, $\det \rho = 0$, is a quartic polynomial in sixteen variables, constrained by the linear equation $\operatorname{Tr} \rho = 1$, it cannot be directly visualized but two-dimensional sections of it can, and are readily amenable to computer graphics.

It would not be possible to conclude this paper without paying a tribute to the wide and penetrating contributions which Asim Barut, in whose honor this issue is prepared, has made to physics. Not only has he done outstanding work in the mainstream of theoretical ideas, but he has also been prepared to open or reopen less fashionable but just as interesting areas in a definitive way.

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