Polarization

$\nabla \cdot \vec{D} = -\rho(t)$ $\quad\quad \nabla \cdot (\partial_t \vec{P} - \vec{j}) = 0 \quad\quad (\vec{E} = 0)$

$$\Rightarrow \Delta P_\alpha = \int_0^T dt\, j_\alpha = e\int_0^T \sum_n dt \int_{BZ} \frac{d\vec{q}}{(2\pi)^d} \Omega^n_{q_\alpha, t}$$

$$= e\nu_\alpha$$

（证明方法见沈顺清附录）

提醒 SSH $Q = 0, \frac{1}{2}$

$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

$Q = \int \rho\, dx$

$\dot{Q} = \int \dot{\rho}\, dx = \int \nabla \cdot \vec{J}\, dx = 0$

$G = \frac{ne^2}{h} \quad n \in \mathbb{Z} \quad\quad \int \vec{B} \cdot d\vec{S} \in \mathbb{Z}$

几何相

Zak phase $\quad \gamma_n = \frac{2\pi}{a} X_{nn} \rightarrow$ polarization

$\int d\omega$ 不能整体使用 Stokes 定理

Manifold → 今天 $\xrightarrow[\partial]{\text{边界}}$ 同调群 $H_d(M)$

Manifold ↙ 切空间 (tangent space) T_pM $\left(\frac{\partial}{\partial x^i}\right)$ → Lie algebra，算子

Manifold → 余切空间 T_p^*M (dx^i) $\xrightarrow{d}$ $H^d(M)$ 上同调群

k-形式 $\int d\omega$

tensor 张量

A. Zee

A tensor is sth. transform like a tensor.

Multilinear

为什么？ $\langle v, d\omega \rangle = \langle \partial v, \omega \rangle$

$A \sim A + \nabla\phi \quad\quad \omega \sim \omega + d\eta \quad\quad (d\omega = d\omega + d^2\eta = d\omega)$

群元 = mapping

群 < 群表示论
群同态同构定理 → 同伦群 → Lie群 / 对称性 → gu与路… 无关

Noether　　　　　　Topo 分类

问题：分类是什么
└→ 同伦群 or 其它

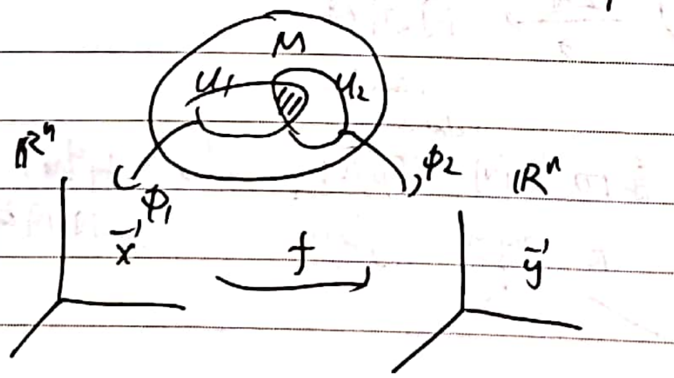
理解：1) Topo defect
2) Topo band
3) BEC

1) Manifold

局部同胚于 $\mathbb{R}^n$ 空间的空间

Homeomorphism

数学语言：M is a manifold / 避免使用坐标　物理规律和坐标无关



存在 $p \in U_1 \cap U_2$

$$\begin{cases} \vec{x} = \phi_1(p) \Rightarrow p = \phi_1^{-1}(x) \\ \vec{y} = \phi_2(p) \Rightarrow y = \phi_2 \circ \phi_1^{-1}(x) = f(x) \end{cases}$$

$$\Rightarrow f = \phi_2 \circ \phi_1^{-1}$$

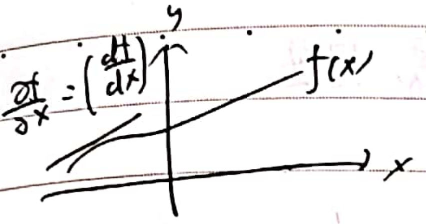
Ref. Nakahara 书 Chap 5

Gauge potential, knots, gravity Chap 3

切空间 / 切向量

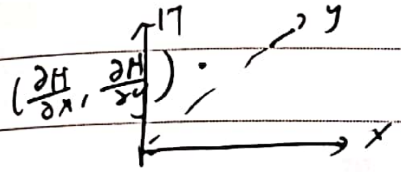
$f: M \to \mathbb{R}$　　$f(\vec{x}) \in \mathbb{R}$　　$f \in C^\infty(M)$

$\mathbb{R}^2$ 平面



$\mathbb{R}^3$ 平面

$H(x,y)$



定义 $\hat{V} = \frac{\partial}{\partial x}$ $\quad \hat{V} f = \frac{\partial f}{\partial x}$

$\hat{V}: C^\infty(M) \to \mathbb{R}$

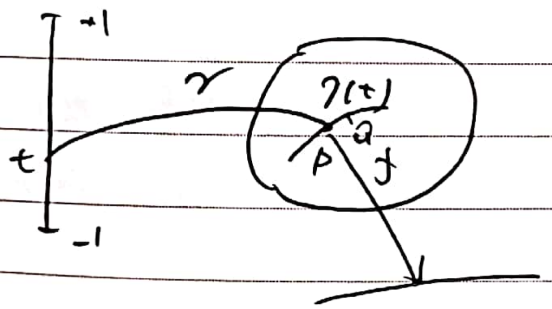
例：$L_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$

$e^{a\frac{\partial}{\partial x}} f(x) \to f(x+a)$

物理意义

$\hat{V} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ $\quad \hat{V}(f) = \left(\frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f\right)$

考虑 M a manifold $\quad f: M \to \mathbb{R}$



$f(p) = f(\gamma(t)) = f \circ \gamma(t)$

$p = \gamma(t)$

$\frac{df}{dt} = \frac{f_Q - f_P}{dt}$ 坐标无关

$= f' \frac{d\gamma}{dt} \qquad p = (x_1, \dots, x_n)$

$= \frac{\partial f}{\partial x^i} \frac{dx^i}{dt}$

$= \left(V^i \frac{\partial}{\partial x^i}\right) f = \hat{V}(f)$

$\hat{V} = \sum V^i \frac{\partial}{\partial x^i} \qquad T_pM$ 切空间

1) $\hat{V}$ 和坐标无关 $\quad \hat{V} = \sum V^i \frac{\partial}{\partial x^i} = \sum V^i(y) \frac{\partial}{\partial y^i}$

但 $V^i(x) \to V^i(y)$ 之间有一个变换

2) $V^i(p) \quad V^i: M \to \mathbb{R}$

3) $\hat{V}$ 是线性空间中的元素，V^i 是系数，$\frac{\partial}{\partial x^i}$ 是基

条件：e_i 线性独立

4) $\frac{\partial}{\partial x^i}$ 是线性独立的

线性 $x^i = A^{ij} y_j$ $\quad \frac{\partial}{\partial x^i} = \sum A^{i\alpha} \frac{\partial}{\partial y^\alpha}$

5) 加法性

6) $\hat{V}, \hat{W} \in T_pM$

$\hat{V} + \hat{W} \in T_pM \quad [\hat{V}, \hat{W}] \in T_pM$

$\hat{V}\hat{M} \notin T_pM$

余切空间 Cotangent space

$df = f_Q - f_P$ 和坐标无关

$= \sum_i \left(\frac{\partial f}{\partial x^i} dx^i\right) \quad \Leftrightarrow \quad$ 1-form $\Rightarrow$ 写成线性空间形式

$= \sum \frac{\partial f}{\partial y^\alpha} \left(\frac{\partial y^\alpha}{\partial x^i}\right)\left(\frac{\partial x^i}{\partial y^\beta}\right) dy^\beta \quad$ Basis dx^i 系数为 $\left(\frac{\partial f}{\partial x^i}\right)$

$= \sum \frac{\partial f}{\partial y^\alpha} \left(\frac{\partial y^\alpha}{\partial y^\beta}\right) dy^\beta$

$= \sum \frac{\partial f}{\partial y^\alpha} dy^\alpha$

dx^i 线性无关

对偶关系

$\hat{V} = \sum_i V^i \frac{\partial}{\partial x^i} = (V^1, \cdots, V^n) \quad \in T_p(M)$

$\omega = \sum_i \frac{\partial f}{\partial x^i} dx^i = \left(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}\right) \quad \in T_p^*(M)$

$\omega(\hat{V})$ 内积 inner product

$\langle a, b\rangle = a \cdot b = a_i e^i \cdot b^\alpha e_\alpha \quad e^i \cdot e_\alpha = \delta^i_\alpha$

$\check{V} + \check{V}$

另一种表示 $\quad \omega = df = \frac{\partial f}{\partial x^i} dx^i \quad V = V^i \frac{\partial}{\partial x^i}$

取 $e^i = dx^i \quad e_i = \frac{\partial}{\partial x^i}$

$\langle \omega, V\rangle = V^i \frac{\partial f}{\partial x^j} \langle dx^i, \frac{\partial}{\partial x^j}\rangle = V^i \frac{\partial f}{\partial x^i} \langle e^i, e_j\rangle = V^i \frac{\partial f}{\partial x^i}$

$\langle \omega, \hat{V}\rangle = V^i \frac{\partial f}{\partial x^i} = V(f) = \langle df, V\rangle$

应用: Schutz 数学物理中的几何方法 (Chap 5)

微分形式的应用 / 物理

热力学: $dU = -pdV + TdS$

$d(dU) = -dp\wedge dV + dT\wedge dS$

$p = p(V,S) \quad T = T(V,S)$

$-\frac{\partial p}{\partial S} = \frac{\partial T}{\partial V}$

数学: $df = P\,dx + Q\,dy$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

经典力学: $\omega = \sum dp^i \wedge dq^i$

$d\omega = 0$

可以用 T_pM 做什么? $\Rightarrow$ tensor $\Rightarrow$ 得到结合和坐标无关

1) 结合

2) line integral, Surface integral, Volume integral

3)

引入张量(具体): 形式

0-form: $C^\infty(M)$ 函数 $\quad \Omega^0(M) \quad C_n^0$

1-form: $\sum_i f_i dx^i \quad \Omega^1(M) \quad C_n^1 \quad T_p^*(M)$

2-form $\sum_{ij} f_{ij}\, dx^i\wedge dx^j \quad \Omega^2(M) \quad C_n^2 \quad T_p^*(M)\otimes T_p^*(M)$

3-form $\sum_{ijk} f_{ijk}\, dx^i\wedge dx^j\wedge dx^k \quad C_n^3$

⋮

(n-1)-form

n-form $f_{1\dots n}\, dx^1\wedge\dots\wedge dx^n \quad C_n^0$

$\Omega = \bigoplus_{k=0}^{n} \Omega^k(M) \quad (0,k)$ 形式

以后 $A\otimes B \quad u\in A \quad v\in B \quad uv\in A\otimes B$

多重线性

$w^i v^j \to w^{i\alpha}\tilde{u}_\alpha Q^{j\beta}\tilde{v}_\beta = (w^i \otimes Q^j)\,\tilde{u}^\alpha \tilde{v}^\beta$

$dx^i \wedge dx^j \to w^{i\alpha} w^{j\beta} dy^\alpha \wedge dy^\beta$ 乘法: $\wedge$

举例 (1,2) 维 tensor $T_pM \otimes T_p^*M \otimes T_p^*M$

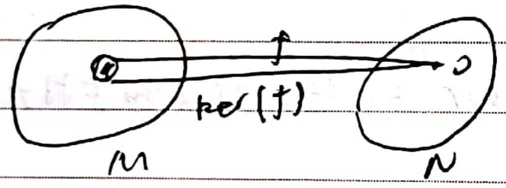
$(V^\sigma \frac{\partial}{\partial x^\sigma}) \otimes (\frac{\partial f}{\partial x^i} dx^i) \otimes (\frac{\partial f}{\partial x^\alpha} dx^\alpha) = V^\sigma \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^\alpha} (\frac{\partial}{\partial x^\sigma} \otimes dx^i \otimes dx^\alpha)$

(0,k) 形式

$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots \to \Omega^n(M) \xrightarrow{d} 0$

$G = \frac{ne^2}{h} \quad \oint \langle\psi|\nabla_R|\psi\rangle dR = \oint \langle\psi|d|\psi\rangle$

线性代数



$f([x]) = f(x) = y \qquad [x] = \{x + x' \mid x' \in \ker f\}$

$\dim M = \dim(\mathrm{im} f) + \dim(\ker f)$

$f \in C^\infty(M) \xrightarrow{d} df = (\frac{\partial f}{\partial x^i}) dx^i \xrightarrow{d} \cdots$ 大正合序列

$\omega = g_i dx^i$

$\Omega^0 \to \Omega^1 \to \Omega^2$

$f \qquad df = \frac{\partial f}{\partial x^i} dx^i \qquad d(df) = 0$

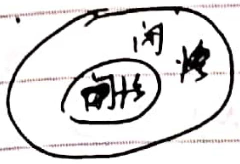
$d(\frac{\partial f}{\partial x^i} dx^i) = \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i + \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = 0$

$\omega \neq \frac{\partial f}{\partial x^i} dx^i$ $\qquad \Omega^n$

$d\omega = 0$

~~exat~~ exact form: 恰当形式 $\alpha = d\eta \Rightarrow d^2\eta = 0$

closed form: 闭形式 $d\alpha = 0$



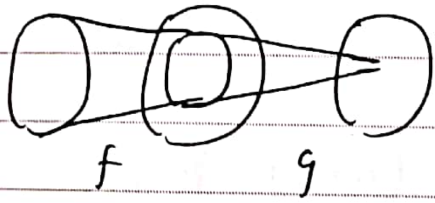
$\alpha = P\,dx + Q\,dy$

$d\alpha = 0$ 得 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

1) 正合序列

2) 为什么做这个分类?

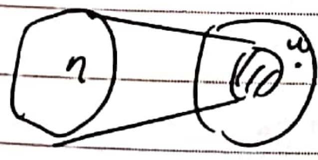
$A \xrightarrow{f} B \xrightarrow{g} C$



$\mathrm{im} f = \ker g$

$\int_V d\omega = \langle V, d\omega\rangle = \langle \partial V, \omega\rangle = \langle \partial V, \omega\rangle + \langle \partial V, d\eta\rangle$

$\omega \to \omega + d\eta$, $\langle \partial V, d\eta\rangle = 0$



群与同伦群

manifold

$\vec{x}'$ $f(\vec{x})$

$f: \vec{x}' \longrightarrow \mathbb{R}$

两个空间
- 切空间 $T_p(M) \Longrightarrow \frac{\partial}{\partial x_i} \longrightarrow$ Lie 代数
- 余切.. $T_p^*(M) \Longrightarrow dx^i \longrightarrow$ Caculus

余切 $\Longrightarrow$
- 1元微分 线 $\oint \vec{F}\cdot d\vec{l}$, 1 form $\in \Omega^1$
- 2元 面 $\oint \vec{B}\cdot d\vec{S}$, $\oint \vec{E}\cdot d\vec{S}$, 2 form $\in \Omega^2$
- 3元 体 $\oint f\,dx\,dy\,dz$ $\oint f\,dV$ 3 form $\in \Omega^3$