2021年5月6日

$G = \nu \frac{e^2}{h}$, $\nu \in \mathbb{Z}$.

与 Monopole 的关系

$\int \vec{B} \cdot d\vec{s} = \mathbb{Z}$

同伦群.

几何相. , Berry Phase.

Zak Phase.

$\boxed{\gamma_n = \frac{2\pi}{a} X_{nn}}$ → polarization.

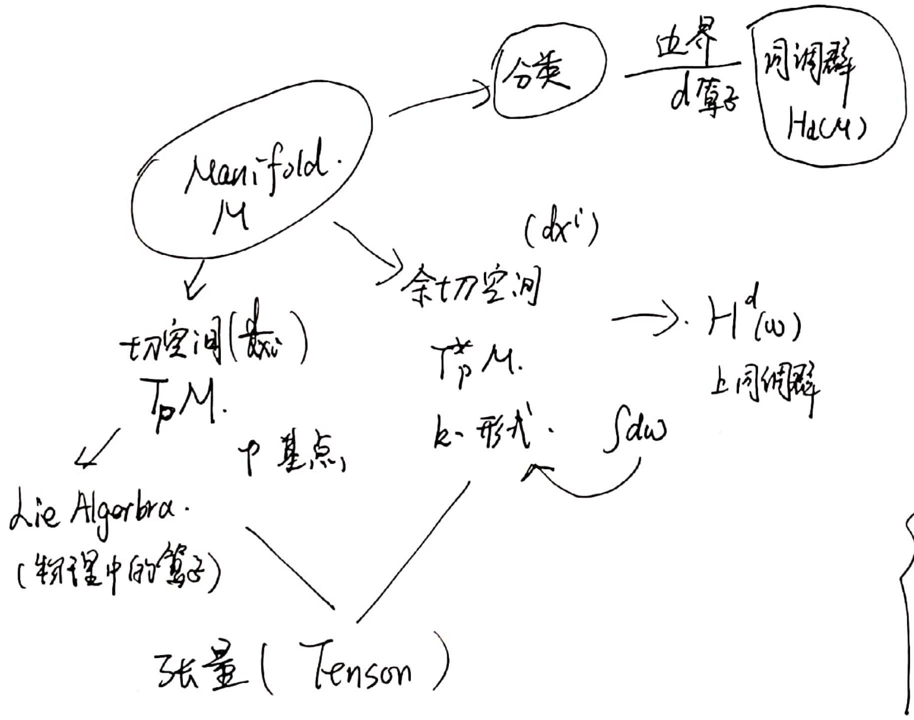
都是做一类积分 $\int d\omega$, 无法直接使用 Stokes Theorem.

Wu. C.N Yang. , Monopole.

$$\begin{cases} \vec{B} = \nabla \times \vec{A} \\ \vec{A} \sim \vec{A} + \nabla \phi \end{cases}$$

本课目的, 介绍对这些物理量的数学背景。

$\int_{\Sigma} d\omega = \int_{\partial \Sigma} \omega$



$\Rightarrow$ 分类.

为什么?

对 $\langle v, d\omega \rangle$

$= \langle \partial v, \omega \rangle$

本质, 这两者都是等价的。

$$\begin{cases} A \to A + \nabla \phi \\ \omega \to \omega + d\eta \\ d\omega \to d\omega \\ v \to v + \partial v \end{cases}$$

A. Zee.

① "A tensor is something transform like a tensor."

② Multi linear

群
- 群表示论
- 群同态与群同构 (Nother) → 同伦群 (群元 = mapping) → Lie群 / 对称性 ↑ 用于以上模型.
  - Topo 分类.
  - ↓ 与和对称性有关。

同伦等价

$\pi_d(G)$

$: S^d \xrightarrow{f} G$

目的: 理解 1) Topological Defect (70年代)

2) Topological Band. (近15年).

3) BEC.

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1) manifold.

局部同胚于 $\mathbb{R}^n$ 的空间

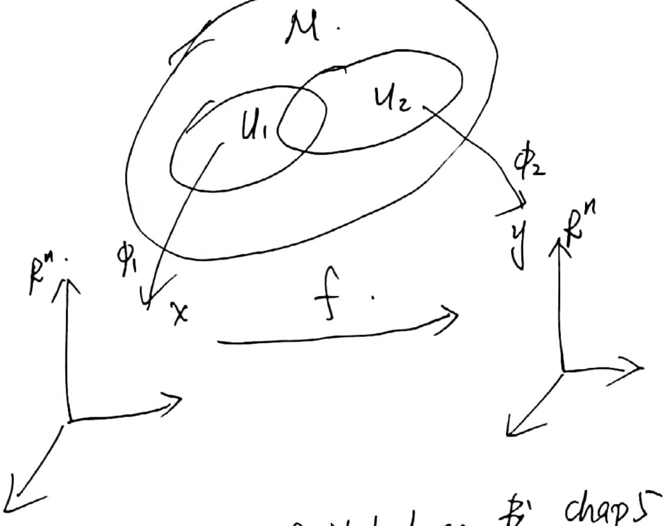
(~~Isomorphism~~)

微分流形 Homeomorphism

定义: M is a manifold

~~坐标~~: 尽量避免使用坐标

物理规律与坐标无关



存在 $p \in U_1 \cap U_2$.

有 $\begin{cases} x = \phi_1(p) \\ y = \phi_2(p) \end{cases} \Rightarrow p = \phi_1^{-1}(x)$

$\Downarrow$

$y = \phi_2(\phi_1^{-1}(x))$

or $y = \phi_2 \circ \phi_1^{-1}(x)$

or $f = \phi_2 \circ \phi_1^{-1}$

~~若且~~ 且 映射是连续函数.

且存在连续的导数.

Ref. ① Nakahara 书 chap5.

② Gauge, field, knots. Geometry

切空间/切向量：

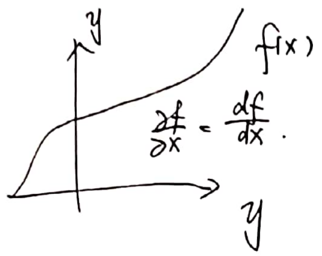
$T(\vec{x})$. 温度场

$E(\vec{x})$. 电场

$f: M \to R$ 的映射

光滑函数

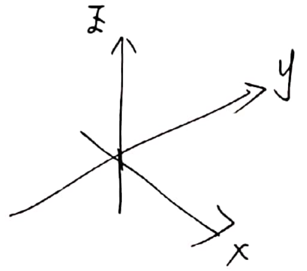
① 1) R^2 平面：



$\frac{\partial f}{\partial x} = \frac{df}{dx}$.

2) ② 三维空间：

$z = h(x,y)$



则其导数为 $\left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right)$.

1) 定义 $\hat{v} = \frac{\partial}{\partial x}$，其效果为 $\hat{v}f = \frac{\partial f}{\partial x}$.

or $\hat{v}: C^{\infty}(M) \to R$.

eg. ① $L_x = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$

② $e^{a\frac{\partial}{\partial x}} f(x) = f(x+a)$. 平移算子.

2). 定义 $\hat{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$，则

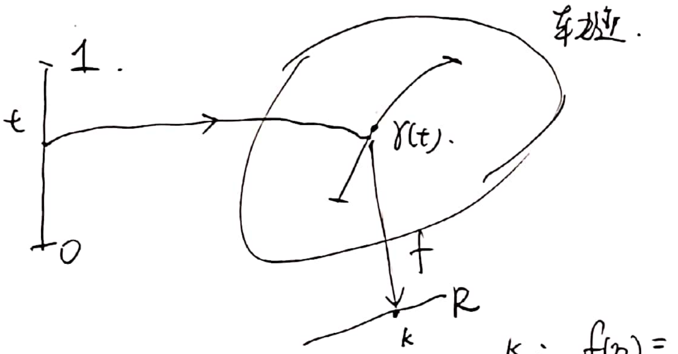
$\hat{v}f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

若假设

$p = (x_1, \dots, x_n)$

$q(t+dt)$, $p(t)$.

考虑 manifold M，其有 $f: M \to R$.



$\frac{df}{dt} = \frac{f_q - f_p}{dt}$ 是坐标无关.

$= f' \frac{d\gamma}{dt}$ ← 速度.

$= \frac{\partial f}{\partial x_i} \cdot \frac{dx^i}{dt}$

$k: f(p) = f(\gamma(t)) = f \circ \gamma(t) = \left(v^i \frac{\partial}{\partial x_i} f\right)$

$= \hat{v}(f)$

$\hat{V}_p = \sum_i V^i \left(\frac{\partial}{\partial x^i}\right)_p$ or $\sum_i \frac{dx^i}{dt} \cdot \frac{\partial}{\partial x^i}$ 切向量 T_pM 切空间

性质：

① $\hat{V}$ 和坐标无关，$\hat{V} = \sum_i V^i \frac{\partial}{\partial x^i} = \sum_i V^i(y) \cdot \frac{\partial}{\partial y^i}$

$$y = y(x)$$

$$\frac{\partial x^i}{\partial t} = \frac{\partial y^j}{\partial t} \cdot \frac{\partial x^i}{\partial y^j}$$

线性空间

$a = \sum_i a_i e_i$

e_i 线性无关

② $V^i(p)$ $V^i: M \to \mathbb{R}$

③ $\hat{V}$ 是线性空间中的元素，V^i 是系数，$\frac{\partial}{\partial x^i}$ 是基矢

④ $\frac{\partial}{\partial x^i}$ 是线性独立的 线性：$\frac{\partial}{\partial x^i}$

$$x^i = A^{ij} y_j$$

$$\frac{\partial}{\partial x^i} = (A^{-1})^{i\alpha} \frac{\partial}{\partial y_\alpha}$$

⑤ 具有可加性

⑥ 若 $\hat{V}, \hat{W}$ 都属于 T_pM，则 $\hat{V} + \hat{W}$ 都属于 T_pM

$\hat{V}\hat{W}$ 不一定属于 T_pM

但 $[\hat{V}, \hat{W}]$ 属于 T_pM

余切空间：

$df = f_Q - f_P$ 与坐标无关

$= \sum_i \left(\frac{\partial f}{\partial x^i} \cdot dx^i\right)$ $\Leftrightarrow$ 也可以写成线性空间的形式

$= \sum_i \frac{\partial f}{\partial y^i} \cdot \frac{\partial y^i}{\partial x_\alpha} \cdot \frac{\partial x^\alpha}{\partial y^j} dy^j$

$= \sum_i \frac{\partial f}{\partial y^i} \cdot dy^i$

Basis：dx^i

系数：$\left(\frac{\partial f}{\partial x^i}\right)$

dx^i 是线性空间中的独立基矢

余切空间与切空间中的对偶关系.

$$\begin{cases} \hat{v} = \sum_i v^i \frac{\partial}{\partial x_i} = (v^1, v^2, \cdots, v^n) \in T_pM. \\ w = \sum_i \frac{\partial f}{\partial x_i} dx^i = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right) \in T_p^*M. \end{cases}$$

内积 (inner product) $\langle a, b \rangle = \sum_i a_i^* b_i$

$\langle w, \hat{v} \rangle = v^i \frac{\partial f}{\partial x_i} = V(p)$

另一种表示:

$w = df = \frac{\partial f}{\partial x_i} dx^i$

$\hat{v} = v^i \frac{\partial}{\partial x_i}$

取 $e^i = dx^i$

$e_j = \frac{\partial}{\partial x_j}$

$\langle w, v \rangle = v^i \frac{\partial}{\partial x_j} \langle dx^i, \frac{\partial}{\partial x_j} \rangle$

$= v^i \frac{\partial}{\partial x^i}$

材料: Schutz《数学物理中的几何方法》chap 5.

讨论了很多微分形式的应用. eg ① 热力学. $du = Tds - pdv$

$d(du) = d^2u = 0$

$0 = -dp \wedge dv + dT \wedge ds$.

$\begin{cases} P(V, S) \\ T(V, S). \end{cases}$

Maxwell Equation.

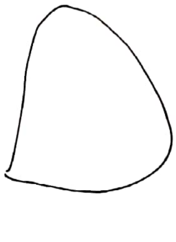
$-\frac{\partial P}{\partial S} = \frac{\partial T}{\partial S}$.

② 数学. $df = Pdx + Qdy$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

③: $\begin{cases} w = ~~d~~ \sum_i dp_i \wedge dq_i \\ dw = 0 \end{cases}$

$\int_a^b f(x)dx$ 闭曲线上的积分.



可以用 T_p^*M 做什么？

1> 积分.

2>. Line Intergral $d=1$

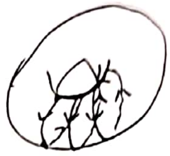
Surface Intergral $d=2$.

Volume. $d=3$.

> Tensor

$d=1$ eg. $\int f(x)dx$

or $\int f_1dx + f_2dy + f_3dz$



All dxdy is a tensor

$dx \wedge dy$

保证积分结果与坐标无关.

$d=2$ eg.

$\int f(x,y)dxdy$

$\int$ ~~$f_1(x,y)d$~~ $f_1dxdy + f_2dydz + f_3dzdx$.

引入张量（具体）.

C_n^0 0-form : $C_p^\infty(M)$ 函数. $\Omega^0(M)$

C_n^1 1-form : $\sum_i f_i dx_i$ $\Omega^1(M)$

C_n^2 2-form : $\sum_{ij} f^{ij} dx^i \wedge dx^j$ $\Omega^2(M)$

C_n^3 3-form : $\sum_{ijk} f^{ijk} dx^i \wedge dx^j \wedge dx^k$ $\Omega^3(M)$.

$\equiv \Omega = \bigoplus_{k=0}^{n} \Omega^n(M)$.

⋮

(n-1)-form : $\sum_i f^{(1,2,\cdots,n-1)} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1} + f^{(1,2,\cdots,n-2,n)} dx^1 \wedge dx^2 \wedge \cdots dx^{n-2} \wedge dx^n + \cdots$

C_n^n n-form : . . . .

(0, k) 形式 对 $\Omega^k(M)$

$\begin{cases} A \otimes B \\ u \in A,\ v \in B. \\ uv \in A \otimes B \end{cases}$

满足多重线性关系

e.g. (1,2) 张形式.

$T_pM \otimes (T_p^*M \otimes T_p^*M.)$

$(v^i \frac{\partial}{\partial x^i}) \otimes \frac{\partial f}{\partial x_i} \cdot dx_i \otimes \frac{\partial f}{\partial x_\alpha} \cdot dx_\alpha$.

$-(v^i, \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_\alpha}) \frac{\partial}{\partial x^i} \otimes dx_i \otimes dx_\alpha$.

(0, k) 形式.

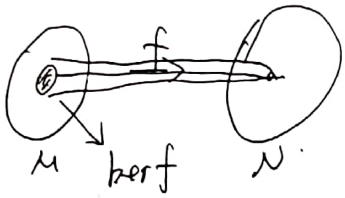
长正合序列

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

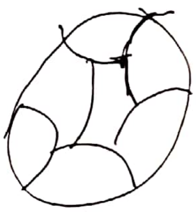
$f\in C_p^\infty(M)$ $\quad df = (\frac{\partial f}{\partial x_i})dx^i$ $\quad d\omega$

$$\oint \langle \psi | \nabla_R | \psi\rangle dR = \oint \langle\psi|d|\psi\rangle$$

线性代数



f =



构造.

$$R^n \xrightarrow{f^{n,n-1}} R^{n-1} \xrightarrow{f^{n-1,n-2}} R^{n-2} \xrightarrow{f^{n-2,n-3}} \cdots \xrightarrow{f^{2,1}} R^1 \xrightarrow{f^{1,0}} 0.$$

~~存在 kerf~~

对 $x\in \ker f$ 有 $f(x)=0$. 陪集分解.

$$f([x]) = f(x) = y.$$

~~dimension~~

$$\dim(M) = \dim(\mathrm{im} f) + \dim(\ker f)$$

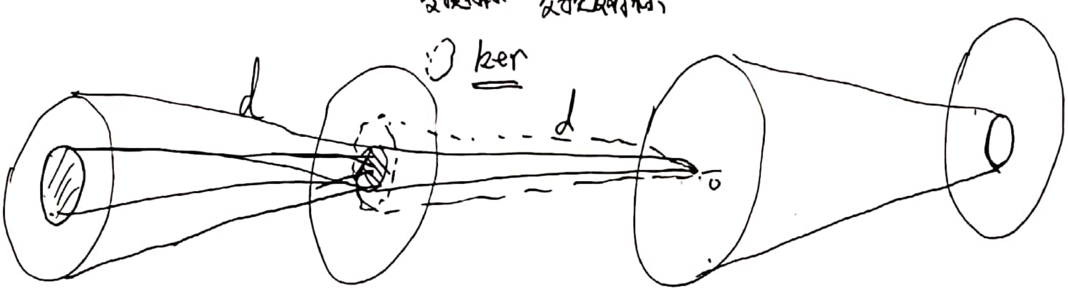
$$\Omega^0 \xrightarrow{d} \Omega^1 \to \Omega^2 \to \Omega^3$$

f $\quad df = \frac{\partial f}{\partial x_i}dx^i$ $\quad d(df)=0$

$$d(\frac{\partial f}{\partial x_i}dx_i) = \frac{\partial^2 f}{\partial x_i\partial x_j} dx_i\wedge dx_j = 0, \quad d^2=0$$

交换对称　交换反对称

∴ ker



exact form : 恰当形式 $\alpha = d\eta \to d^2\eta = 0$

closed form : 闭形式 $d\alpha = 0$.

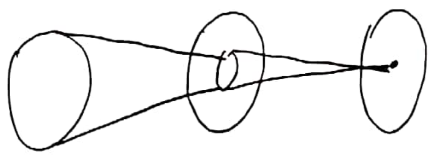
$$\begin{cases} \alpha = Pdx + Qdy. \\ d\alpha = 0 \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \end{cases}$$

1) 正合序列

2) why

$A \xrightarrow{f} B \xrightarrow{g} C$

If. $\operatorname{Im} f = \ker g$



$$\int_V d\omega = \langle V, d\omega \rangle = \langle \partial V, \omega \rangle$$

or $\omega + d\eta = \langle \partial V, \omega \rangle + \langle \partial V, d\eta \rangle$

$= \langle V, d\omega \rangle + 0$