群论

1) $(G, *)$

2) 抽象

3) 用在 $g_i \in G \to T(g_i)$

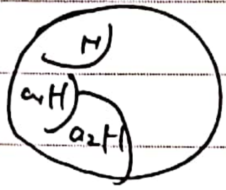
$g_i g_j = g_k \Rightarrow T(g_i)T(g_j) = T(g_k)$ 群表示

4) 不可约表示与简并度

5) 子群 正规子群 ⇒ 分类

$H \subset G$ $H \triangleleft G \to$ 定义商群 $G/H = Q$ 等价类

coset $= \{[e], [a_1], [a_2], \cdots\}$

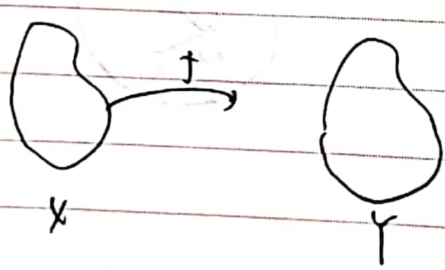


今天核心：G/H 以及其应用

例子 1) Topo defect：70年代，BKT相变（1971），液晶，BEC

2) Topo Band insulator：动量空间 ⇒ Bloch eq.

同伦



$\psi: \vec{x} \to \mathbb{C}^N$ $\psi(\vec{x})$

$H: \vec{k} \to H(\vec{k})$

同伦：有多少种不等价的映射

同伦群：1) 群 基本群

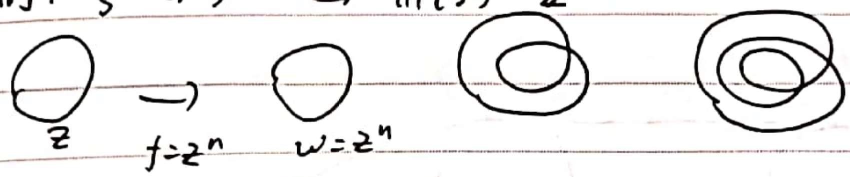
2) 群元 $= f$

3) 恒等元 $c(x) = y_0 \in Y$

$\pi_1(S^2) = 0$.

$\pi_n(X): S^n \to X$

特例: $S^1 \to S^1 \Rightarrow \pi_1(S^1)=\mathbb{Z}$



$\frac{1}{2\pi i}\oint \frac{dz}{z}=1$ $\quad \oint \frac{1}{2\pi i}\frac{dw}{n}=n$

数学公式，名词 $\pi_n(X)$

1) $\pi_n(X\times Y)=\pi_n(X)\times\pi_n(Y)$

2) $\pi_0(X) = \begin{cases} 0 & X \text{ is connected} \\ \mathbb{Z} & X\text{分成 } n \text{ 块} \end{cases}$

点 $\to X$

3) $\pi_n(G/H)=?$

计算 $\pi_n(G/H)$ long exact sequence (长正合序列)

基础：自然同态 $H \triangleleft G$ $\quad H \hookrightarrow G \xrightarrow{\pi} G/H$

$i(h)=h$

$$\begin{array}{ccccccc} & \multicolumn{3}{c}{\overbrace{\hspace{6cm}}^{\text{已知}}} & & \text{待求} & \\ \to & \pi_2(H) & \to & \pi_2(G) & \to & \pi_2(G/H) & \to \\ \to & \pi_1(H) & \to & \pi_1(G) & \to & \pi_1(G/H) & \to \\ \to & \pi_0(H) & \to & \pi_0(G) & \to & \pi_0(G/H) & \end{array}$$

例: $0 \to \pi_1(G) \to \pi_1(G/H) \to 0$ $\quad \pi_1(H)=0 \quad \pi_0(H)=0$

短正合序列

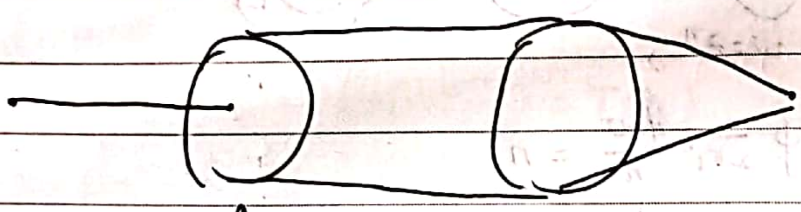
$\pi_1(G) \cong \pi_1(G/H)$

两个定理/结论 (Short exact sequences)

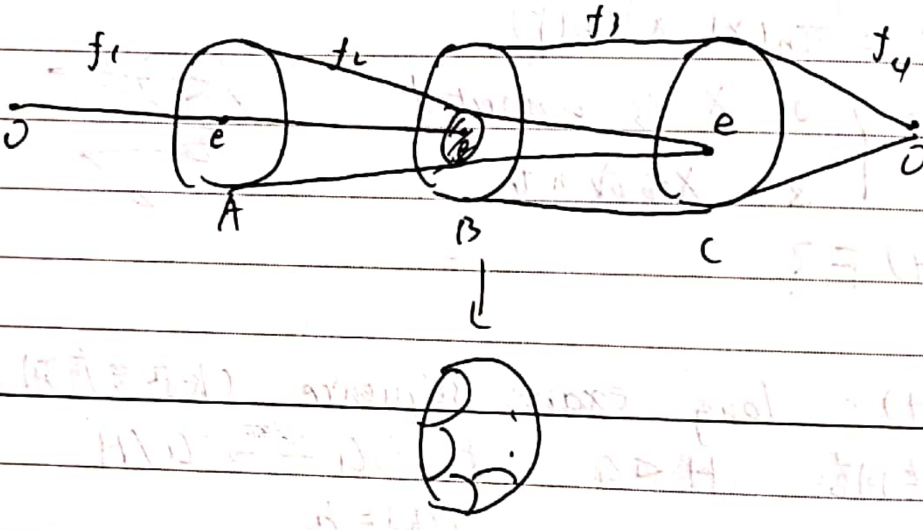
1) $0\to A\to B\to 0$ 正合 $\Rightarrow A\cong B$

2) $0 \to A \to B \to C \to 0$ 正合 $\Rightarrow$ $B = A \otimes C$ or $A \cong B/C$

证: 1)



2) $0 \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} C \xrightarrow{f_4} 0$



$B/\ker(f_3) \cong C$

$\ker(f_3) = \mathrm{Im}(f_2)$

$\cong A$ (单射)

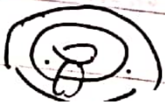
$\Rightarrow B/A \cong C$

1) 计算 f $\Rightarrow$ 形式，同调群，余切空间
2) 计算 同伦群

应用:

$S^n \to S^m$ 映射 $\Rightarrow \pi_n(S^m)$

$\pi_1(S^1 \times S^1) = \pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$



	π_1	π_2	π_3	π_4	π_5	π_6
S^1	$\mathbb{Z}$	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$
S^5	0	0	0	0	$\mathbb{Z}$	

$\pi_n(S^n)=\mathbb{Z}$

$S^3\to S^2$ Hopf map

$S^n\to S^m$

① $n<m$ 0

② $n>m$ 不平庸

如何计算 $\pi_n(S^n)=\mathbb{Z}$ $\Longrightarrow$ 可证明 $c_n\int\varepsilon_{i_1\cdots i_n}x^{i_1}dx^{i_2}dx^{i_3}\cdots dx^{i_n}$

基本公式 Stokes 定理

1) $\pi_1(S^1)$ $\Longrightarrow$ $\frac{1}{2\pi i}\oint\frac{dz}{z}$

令 $z=x+iy$ $|z|=1$

$$\underbrace{\frac{1}{2\pi i}\oint\frac{(xdy+ydx)}{x^2+y^2}}_{=0}+\frac{1}{2\pi}\oint(xdy-ydx)=\frac{1}{2\pi}\int\varepsilon_{ij}x^idx^j$$

绕度数

$$\int_{D^2}dx^1dx^2=\pi=\int_{D^2}d\omega=\int_{\partial D^2}\omega=\frac{1}{2}\int_{S^1}(x^1dx^2-x^2dx^1)$$

$$d\omega=dx^1\wedge dx^2$$

$$\omega=\frac{1}{2}(x^1dx^2-x^2dx^1)$$

$$1=\frac{1}{2\pi}\int_{S^1}(x^1dx^2-x^2dx^1)=\frac{1}{2\pi}\int\varepsilon_{ij}x^idx^j$$

$=\frac{1}{2\pi}\oint d\phi$ 取 $x^1=\cos\phi$ $x^2=\sin\phi$ $=\frac{\int\text{角积分}}{\text{总角}}$

$$\int_{D^3}dx^1dx^2dx^3=\frac{4}{3}\pi$$

$=\int_{D^3}d\omega$, $\omega=\frac{1}{3}(x^1dx^2\wedge dx^3+x^2dx^3\wedge dx^1+x^3dx^1\wedge dx^2)$

$$1 = \frac{1}{4\pi}\int_{S^2} x^1 dx^2\wedge dx^3 + x^2 dx^3\wedge dx^1 + x^3 dx^1\wedge dx^2$$

$\Rightarrow$ 定义 $\vec{x} = (x^1, x^2, x^3)$

$$\begin{cases} x^1 = \sin\theta\cos\phi \\ x^2 = \sin\theta\sin\phi \\ x^3 = \cos\theta \end{cases} \qquad d\Omega = \sin\theta\, d\theta\, d\phi$$

$$1 = \frac{1}{4\pi}\oint \sin\theta\, d\theta\, d\phi = \frac{1}{4\pi}\oint d\Omega$$

$\varepsilon_{ijk} x^i dx^j\wedge dx^k \xrightarrow{\text{推广}} \varepsilon_{i_1\cdots i_n} dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_n}$

$x^i = x^i(\theta,\phi)$

$$x^i\left(\frac{\partial x^j}{\partial\theta}\frac{\partial x^k}{\partial\phi} - \frac{\partial x^j}{\partial\phi}\frac{\partial x^k}{\partial\theta}\right) d\theta\, d\phi$$

$\downarrow$ 利用 $\vec{a}\cdot(\vec{b}\times\vec{c}) = \varepsilon_{ijk}a^i b^j c^k$

$\vec{x}\cdot\left(\frac{d\vec{x}}{d\theta}\times\frac{d\vec{x}}{d\phi}\right) d\theta\, d\phi$ 明确意义：立体角

体积

$\frac{1}{3}\vec{x}\cdot(d_\theta\vec{x}\times d_\phi\vec{x}) = \frac{1}{3}|\vec{x}|^3 d\Omega$

Ueda 书 P318 eq 12.64

$$N_2 = \frac{1}{4\pi}\oint \vec{m}\cdot(\vec{m}_\theta\times\vec{m}_\phi)\, d\theta\, d\phi$$

$\mathbb{Z}^2 \to \mathbb{Z}^2$ mapping

可以证明：

$$\begin{cases} m_x = \sin n\theta\, \cancel{\sin}\overset{\cos}{} m\phi \\ m_y = \sin n\theta\, \cancel{\cos}\overset{\sin}{} m\phi \\ m_z = \cos n\theta \end{cases}$$

$$N_2 = n\cdot m$$

$S^3 \to S^3$ $\quad \pi_3(S^3) = \frac{1}{12\pi^2}\int \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma\delta}\, n_\alpha\, \partial_i n_\beta\, \partial_j n_\gamma\, \partial_k n_\delta$

1) ε_{ijk} 与 $\varepsilon_{\alpha\beta\gamma\delta}$ 的起源

2) $12\pi^2$ 为什么 π^2.

超球 $x_1^2+x_2^2+x_3^2+x_4^2 \le 1$

$$V=\int dx_1dx_2dx_3dx_4=\frac{\pi^2}{2}\cdot N, \quad \text{统3 N-六}$$

$$=\int_{S^3}\frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}x^\alpha dx^\beta\wedge dx^\gamma\wedge dx^\delta$$

$$=\int\frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{ijk}\partial_i x^\beta\partial_j x^\gamma\partial_k x^\delta\, d\vec{\theta}$$

$$\Longleftrightarrow \quad N=\frac{1}{12\pi^2}\int\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{ijk}\partial_i x^\beta\partial_j x^\gamma\partial_k x^\delta\, d\vec{\theta}$$

Lie 群分类 $\Rightarrow$ 矩阵

1) 所有 $n\times n$ 可逆矩阵构成一个群

$A_i^{-1}A_i=I \quad \Rightarrow \quad \det(A_i)\neq 0$

$(A_iA_j)A_k=A_i(A_jA_k)$

2) 所有 $N\times N$ Unitary matrix 构成 $U(N)$ 群

$\det(U_i)=e^{i\theta}$

3) $U(N)$ + $\det T=1 \Rightarrow SU(N)$ 群

4) 正交矩阵 $O(N)$ 群 $A^TA=I \quad \det A=\pm 1$

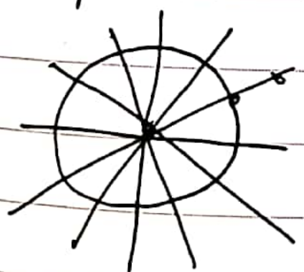
5) $Sp(2n)$ 群：辛群

Projective space ($x\sim \lambda x$ 等价关系)

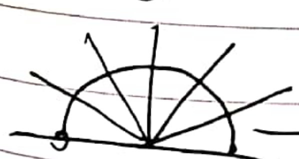
real projective space RP^n

complex — — — CP^n $\quad \psi\sim c\psi \quad c\sim\mathbb{C}$

$\mathbb{C}^n\sim c\cdot\mathbb{C}^n$



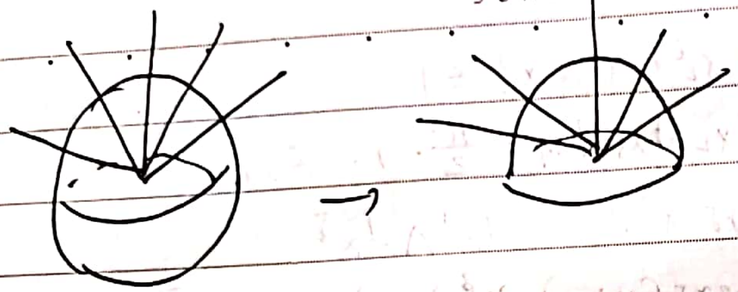
$\downarrow$



$\longrightarrow$ 任何等价的点与之总有一个交点

$$RP^1\simeq S^1/\mathbb{Z}_2\simeq S^1$$

$[0,\pi)$. $[0,2\pi)$



$RP^2 \simeq S^2/\mathbb{Z}_2$

符号 $U=\lim_{N\to\infty} U(N)$

$Sp=\lim_{N\to\infty} Sp(N)$

$O=\lim_{N\to\infty} O(N)$

$SO=\lim_{N\to\infty} SO(N)$

自由度

$U(N)$: $U=e^{iH}$ $\quad U^\dagger U=1 \Rightarrow e^{iH}e^{-iH^\dagger}=1$

$\boxed{S^1: x_1^2+x_2^2=1}$ $\quad N+[1+2+\cdots+(N-1)]\times 2 = N^2$ 个

$SU(N)$: $\det(e^{iH})=1 \Rightarrow Tr(H)=0 \Rightarrow N^2-1$ 个

$O(N)$: $O=e^{H}$ $\quad O^TO=1 \Leftrightarrow e^{H^T}e^{H}=1 \Leftrightarrow \left| \begin{array}{l} H^T=-H \\ H_{ij}\in\mathbb{R} \end{array}\right.$

$1+2+\cdots+(N-1)=\frac{N}{2}(N-1)$

$SO(N)$: $\frac{N}{2}(N-1)$

$Sp(N)$

	π_1	π_2	π_3	π_4
$U(1)$	$\mathbb{Z}$	0	0	0
$U(2)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$U(3)$	0	0	$\mathbb{Z}$	0
$U(4)$	0	0	$\mathbb{Z}$	0
$U(5)$	0	0	$\mathbb{Z}$	0

$U(N)=SU(N)\times U(1)$

$\det U(N)=e^{i\theta}$

$$\pi_3(U(3)) = \mathbb{Z} = \pi_3(U(1)\times SU(3)) = \underbrace{\pi_3(U(1))}_{=0} \times \pi_3(SU(3))$$

$$\Longrightarrow \pi_3(SU(3)) = \mathbb{Z}$$

正交群

~~SO(2)~~	π_1	π_2	π_3	π_4
$SO(2)$	$\mathbb{Z}$	0	0	0
$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$SO(4)$	$\mathbb{Z}_2$	0	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$SO(5)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Bott periodicity / Bott 周期

$$\pi_k(O(n)) = \pi_k(SO(n)) = \begin{cases} 0 & k=2,4,5,6 \\ \mathbb{Z}_2 & k=0,1 \qquad \bmod(8) \\ \mathbb{Z} & k=3,7 \end{cases}$$

$n \geq k+2$

物理中，$n\to\infty$，总是能满足

	π_1	π_2	π_3
RP^1	$\mathbb{Z}$	0	0
RP^2	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
RP^3	$\mathbb{Z}_2$	0	$\mathbb{Z}$

1) 液晶 $\uparrow\ \downarrow$ 斜向 $\uparrow\ \downarrow$

2) Fermion / Boson / Anyon

$H \lhd G$

1) 正规子群
2) Linear algebra $H \simeq \ker f$ 同态核
3) 对应上是等价关系 $G/\sim H$
4) Fibre bundle $H \hookrightarrow G \xrightarrow{\pi} G/H$ } 自然同态
   $\pi \circ i = 0$

$G/H = M$

1) order parameter space
2) Goldstone 定理
3) Stablizer group
4) Little group
5) Orbital group
6) Quotient space

$G/H = M \Rightarrow \pi_n(M)$

哈密顿量 $\mathcal{H}$：找 G 和 H？

$G: \Rightarrow F$ 不变 F：自由能

$H \Rightarrow H = \{h \in G \mid h\psi = \psi\}$

故：$H \lhd G$ $H = \ker f$

$\pi_k(U(N))$ → 不稳定
$\pi_k(U(N))$ → Topo bands

BEC spin-1

$S_z = 0, +1, -1$

$\psi_0, \psi_1, \psi_{-1}$ $SO(3)\times U(1)$

$G = SO(3) \times U(1)$

$g\psi = e^{i\phi}\, \underbrace{U(\alpha,\beta,\gamma)}_{e^{iF_x\alpha} e^{iF_y\beta} e^{iF_z\gamma} \in SO(3)} \psi$ Euler angle

Ground state $|\psi\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$h|\psi\rangle = |\psi\rangle$ 不变

$e^{i\phi} U(0,0,\gamma) = e^{i\phi} e^{i f_z \gamma}$ $\quad f_z = \begin{pmatrix}1 & & \\ & 0 & \\ & & -1\end{pmatrix}$

$$= \begin{pmatrix} e^{i(\phi+\gamma)} & & \\ & e^{i\phi} & \\ & & e^{i(\phi-\gamma)} \end{pmatrix}$$

$\phi = -\gamma$ $\quad \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$H = U(1)$

$M = U(1) \times SO(3)/U(1)$

	π_1	π_2	π_3	π_4
$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

$\mathbb{R}^1 \sim S^1$

$\mathbb{R}^2 \sim S^2$ 单位紧致化

$\mathbb{R}^3 \sim S^3$

求 $M = SO(3) \times U(1)/\mathbb{Z}_2 = G/\mathbb{Z}_2$

$$\begin{array}{ccccccc}
 & \pi_3(\mathbb{Z}_2) & \to & \pi_3(G) & \to & \pi_3(G/\mathbb{Z}_2) & \\
 & 0 & & \mathbb{Z} & & \mathbb{Z} & \\
 & \pi_2(\mathbb{Z}_2) & \to & \pi_2(G) & \to & \pi_2(G/\mathbb{Z}_2) & \to \\
 & 0 & & 0 & & 0 & \\
\to & \pi_1(\mathbb{Z}_2) & \to & \pi_1(G) & \to & \pi_1(G/\mathbb{Z}_2) & \to \\
 & 0 & & \mathbb{Z}_2 \times \mathbb{Z} & & & \\
\to & \pi_0(\mathbb{Z}_2) & \to & \pi_0(G) & \to & \pi_0(G/\mathbb{Z}_2) & \to \\
 & \mathbb{Z}_2 & & 0 & & &
\end{array}$$

$T' = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$0 \to A \to B \to C \to 0$

$\pi_1(SO(3) \times U(1)) = \mathbb{Z}_2 \times \mathbb{Z}$

$\pi_2(SO(3) \times U(1)) = 0$

$\pi_3(SO(3) \times U(1)) = \mathbb{Z}$

$B/A \simeq C$