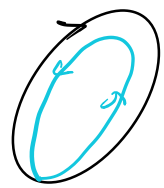
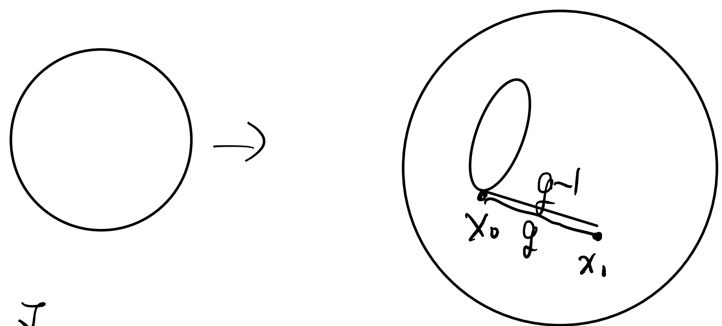
$((l_1 \cdot l_2) \cdot l_3 = l_1 \cdot (l_2 \cdot l_3)$ 结合律.

逆元 $[l \cdot l^{-1}] = [c]$



x_0, x_1 是基点,

$S^1 \to Y$ $\quad \pi_1(Y, x_0)$ $\quad \pi_1(Y, x_1)$ $\quad g \cdot l \cdot g^{-1}$



1> $\pi_1(S^1) \cong \mathbb{Z}$

2> $\pi_1(S^2) = 0$

3> $\pi_n(X, X \otimes Y) = \pi_n(X) \otimes \pi_n(Y)$

下一节课 Topo defect 应用. $\quad \pi_n(G/H)$

Review

群论

1>. $(G, *)$

2>. 抽象

表示 $\Downarrow$

3>. 同态 $\quad g_i \in G \to T(g_i)$ $\quad$, $g_i g_j = g_k \Rightarrow T(g_i) T(g_j) = T(g_k)$

4). 不可约表示和简并度.

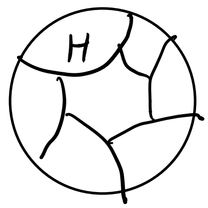
5). 子群与正规子群、分类

$H \subset G$ $\quad\quad$ $H \triangleleft G$ $\Rightarrow$ 商群 $G/H = Q$

正规子群

$\downarrow$

coset



$= \{[e], [a_1], \cdots, [a_{n-1}]\}$

今天(5月27)核心, G/H 及其应用.

例子: 1) Topological defect 70年代. | BKT 相变. (1971年)

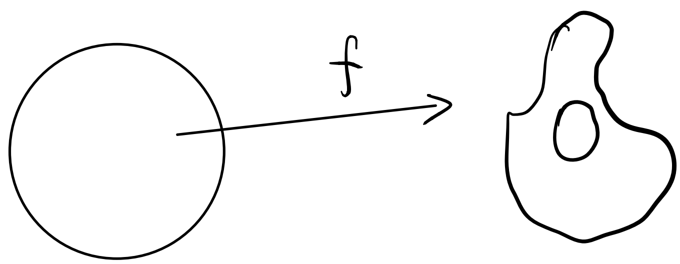
液晶 (Liquid crystal)

BEC

2). Topological Band Insulator

动量空间 $\Rightarrow$ Bloch Equation

目标:



有多少种不同的映射 f

$\psi: \vec{x} \to C^N$ $\quad \psi(\vec{x})$

提供分类表格.

$\Downarrow$ 正合序列

$H: \vec{k} \to H(\vec{k})$

同伦群:

$\pi_n(X): S^n \to X$

1) 群 $\quad$ 2). 群元是映射 f 的同伦等价类 $[f]$

3) 恒等元: $c(x) = y_0 \in Y$

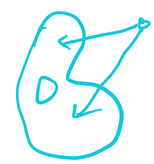
例1: $S^1 \to S^1$ $\quad \pi_1(S^1) = \mathbb{Z}$ $\quad$ winding Number

$S^1 \to S^2$ $\quad \pi_1(S^2) = 0$ $\qquad \frac{1}{2\pi i}\oint \frac{dz}{z^n} = n$

数学名词/公式

1) $\pi_n(X \otimes Y) = \pi_n(X) \otimes \pi_n(Y)$

2) $\pi_0(X) = \begin{cases} 0, & X \text{ is connected} \\ \mathbb{Z}, & X \text{ 分成N块} \end{cases}$



3). $\pi_n(G/H) = ?$

计算 $\pi_n(G/H)$: 长正合序列 (Long exact sequence)

基础: 自然同态 $H \triangleleft G$

$$H \xhookrightarrow{i} G \xrightarrow{\pi} G/H \qquad i(h) = h$$

$$\begin{array}{ccccccc}
\multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{\text{已知}}} & & \text{待求} & & \\
\vdots & & \vdots & & \vdots & & \\
\to \pi_2(H) & \to & \pi_2(G) & \to & \pi_2(G/H) & \to & \\
\to \pi_1(H) & \to & \pi_1(G) & \to & \pi_1(G/H) & \to & \\
\to \pi_0(H) & \to & \pi_0(G) & \to & \pi_0(G/H) & &
\end{array}$$

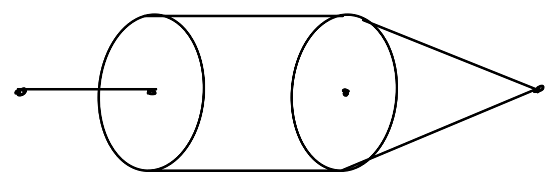
$$\boxed{\begin{array}{c} 0 \to \pi_1(G) \to \pi_1(G/H) \to 0 \\ \Leftrightarrow \pi_1(G) \cong \pi_1(G/H) \end{array}}$$

两个定理/结论

1) $0 \to A \to B \to 0$ 正合 $\Rightarrow A \simeq B$

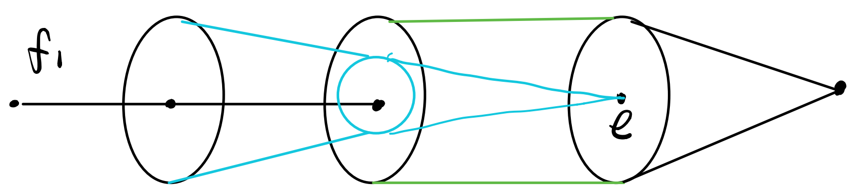
2) $0 \to A \to B \to C \to 0$ 正合 $\Rightarrow B = A \oplus C$, or $B/A \simeq C$

1)



$A \simeq B$

2). $0 \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} C \xrightarrow{f_4} 0$



$B/\ker(f_3) \simeq C \qquad C = \ker(f_4)$

$\ker(f_3) = \mathrm{Im}(f_2)$

1) 计算 $f \Rightarrow$
- 形式
- 同调群
- 余切空间

2) 同伦群
- $\pi_n(G/H)$
- $\pi_n(X \otimes Y)$

应用: $S^n \to S^m$ 映射 $\Rightarrow \pi_n(S^m)$ $\qquad \pi_n(S^n) = \mathbb{Z}$

↑ Hopf map: $S^3 \to S^2$

	π_1	π_2	π_3	π_4	π_5	π_6
S^1	$\mathbb{Z}$	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$.	$\mathbb{Z}\otimes\mathbb{Z}_2$.
S^5	0	0	0	0	$\mathbb{Z}$	

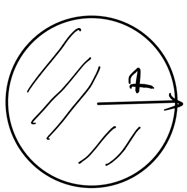
如何计算 $\pi_n(S^n)=\mathbb{Z}$ $\Longrightarrow$ 可证明 $C_n\int\varepsilon_{i_1,\dots,i_n}x^{i_1}dx^{i_2}\cdots dx^{i_n}\neq 0$.

基本公式：stokes定理.

$$\pi_1(S^1)=\frac{1}{2\pi i}\oint\frac{dz}{z}=\frac{1}{2\pi i}\oint\frac{xdx+dy}{x^2+y^2}+\frac{1}{2\pi}\oint(xdy-ydx)$$

令 $z=x+iy$，取 $|z|=1$

$$=\frac{1}{2\pi}\int\varepsilon^{ij}x^idx^j$$



$$S=\int_{D^2}dx^1dx^2=\pi.=\int_{D^2}d\omega=\int_{\partial D^2}\omega$$

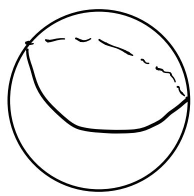
$$d\omega=dx^1\wedge dx^2$$

$$\omega=\frac{1}{2}(x^1dx^2-x^2dx^1)$$

$$1=\frac{1}{2\pi}\int_{S^1}(x^1dx^2-x^2dx^1)$$

$$=\frac{1}{2\pi}\int\varepsilon_{ij}x^idx^j$$

$$=\frac{1}{2\pi}\oint d\varphi=\frac{\int\text{角积分}}{\text{总角}}.$$



$$= \int_{D^3} dx^1 dx^2 dx^3 = \frac{4}{3}\pi$$

$$= \int_{D^3} d\omega, \quad \omega = \frac{1}{3}(x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2)$$

$$1 = \frac{1}{4\pi} \int_{S^2} x^1 dx^2 \wedge dx^3 + x^3 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$$

$$\begin{cases} x_1 = \sin\theta\cos\phi \\ x_2 = \sin\theta\sin\phi \\ x_3 = \cos\theta \end{cases}$$

$$= \frac{1}{4\pi} \oint \sin\theta \, d\theta \, d\phi = \frac{1}{4\pi} \oint d\Omega$$

$$\varepsilon^{ijk} x^i dx^j \wedge dx^k \rightarrow \text{推广 } \varepsilon^{i_1 i_2 \cdots i_n} x^{i_1} dx^{i_2} \cdots dx^{i_n}.$$

$$x^i = x^i(\theta, \varphi)$$

$$x^i \left(\frac{\partial x^j}{\partial \theta} \cdot \frac{\partial x^k}{\partial \phi} - \frac{\partial x^j}{\partial \phi} \cdot \frac{\partial x^k}{\partial \theta} \right) d\theta \, d\varphi$$

利用 $\vec{a} \cdot (\vec{b} \times \vec{c}) - \varepsilon_{ijk} a^i b^j c^k$

$$\vec{X} \cdot \left(\frac{d\vec{x}}{d\theta} \times \frac{d\vec{x}}{d\varphi} \right) d\theta \, d\varphi$$ ，立体角.

Ueda 书 P_{381}, eq 12.64

$$N_2 = \frac{1}{4\pi} \oint \vec{m} \cdot (\vec{m}_\theta \times \vec{m}_\varphi) \, d\theta \, d\varphi = n \cdot l$$

$$\begin{cases} m_x = \sin(n\theta)\cos(l\phi) \\ m_y = \sin(n\theta)\sin(l\phi) \\ m_z = \cos(n\theta) \end{cases}$$

$S^3 \rightarrow S^3 : \pi_3(S^3)$

eq 12.127

$\Rightarrow$ Page 320

$$= \frac{1}{12\pi^2} \int \varepsilon_{ijk} \varepsilon_{\alpha\beta\gamma\delta} n_\alpha (\partial_i n_\beta)(\partial_j n_\gamma)(\partial_k n_\delta)$$

1) ε_{ijk} 与 $\varepsilon_{\alpha\beta\gamma\delta}$ 起源

2). $12\pi^2$ 为什么 π^2

$$V = \int dx_1\, dx_2\, dx_3\, dx_4 = \frac{\pi^2}{2}$$

$$= \int_{S^3} \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta}\, x^\alpha dx^\beta \wedge dx^\gamma \wedge dx^\delta$$

$$= \int \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{ijk}\, \partial_i x^\beta \partial_j x^\gamma \partial_k x^\delta \cdot d\Theta$$

$$\Leftrightarrow \quad 1 = \frac{1}{12\pi^2} \int \varepsilon_{\alpha\beta\gamma\delta} \varepsilon_{ijk} (\qquad)$$

Lie 群的分类

1) 所有 $N\times N$ 可逆矩阵构成群. 要求 $\det(A_i) \neq 0$

$A_i^{-1} A_i = 1$

2) 所有 $N\times N$ Unitary martix 构成 $U(N)$ 群.

3) $U(N) + \det(U) = 1 \Rightarrow SU(N)$ 群.

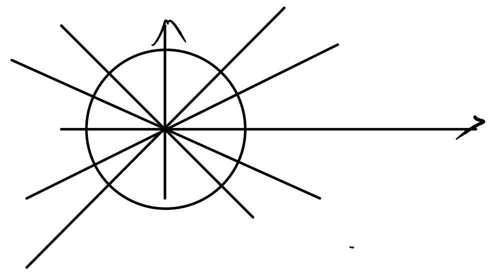
4) 正交矩阵 $O(N)$ 群.

5) $O(N) + \det(O) = 1 \Rightarrow SO(N)$ 群.

6) $Sp(2N)$ 群 辛群.

7) Projective Space $\begin{cases} \text{real projective space} & RP^n \\ \text{complex - - - - -} & CP^n \end{cases}$

$2l\ u\ c\ 4$



$RP^1 \simeq S^1/\mathbb{Z}_2 \simeq S^1$

$RP^2 \simeq S^2/\mathbb{Z}_2$

符号：$U \equiv \lim_{N\to\infty} U(N)$

$$Sp \equiv \lim_{N\to\infty} Sp(N)$$

$$SO = \lim_{N\to\infty} SO(N)$$

$$O = \lim_{N\to\infty} O(N)$$

自由度 　　　　H：厄密

$U(N)$：　$U = e^{iH}$，　$N + \frac{N(N-1)}{2} \times 2 = N^2$ 自由度

$SU(N)$：　$\det(U) = 1$ or $\det e^{iH}) = 1$ or $\mathrm{tr}(H) = 0$

$$\Rightarrow N^2 - 1$$

$O(N)$：　$O = e^{H}$，H 反对称的，$H^T = -H$，　$\frac{N(N-1)}{2}$

$SO(N)$：　　$\frac{N(N-1)}{2}$

$Sp(N)$：

$U(N)$

	π_1	π_2	π_3	π_4	
$U(1)$	$\mathbb{Z}$	0	0	0	
$U(2)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	
[illegible]	[illegible]	[illegible]	[illegible]		

$U(3)$	0	0	$\mathbb{Z}$	0
$U(4)$	0	0	$\mathbb{Z}$	0
$U(5)$	0	0	$\mathbb{Z}$	0

$SO(N)$

	π_1	π_2	π_3	π_4
$SO(2)$	$\mathbb{Z}$	0	0	0
$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
$SO(4)$	$\mathbb{Z}_2$	0	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}_2\times\mathbb{Z}_2$
$SO(5)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Bott periodicity / Bott周期性

$$\pi_k[O(n)] \cong \pi_k(SO(n)) = \begin{cases} 0, & k = 2,4,5,6 \\ \mathbb{Z}_2, & k = 0,1 \\ \mathbb{Z} & k = 3,7 \end{cases} \mod (8)$$

$n \geq k+2$ or $n \to \infty$

RP

	π_1	π_2	π_3
RP^1	$\mathbb{Z}$	0	0
RP^2	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
RP^3	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Bott periodicity

$$\pi_k(Sp(n)) = \begin{cases} 0 \\ \mathbb{Z}_2 \\ \mathbb{Z}. \end{cases}$$

1) 液晶. $\uparrow \leftrightarrow \downarrow$ or $x \sim -x$

2). Fermion / Boson / Anyon.

$H \lhd G$

1) 正规子群.

2) 李代数 , $H = \ker(f)$: 同态核.

3) 本质上是等价关系 $G/N \to H$

4) Fibre bundle $H \xhookrightarrow{i} G \xrightarrow{\pi} G/H$ } 自然同态.

$\pi \cdot i = 0$

$G/H = M$,

1) order parameter group

2). Goldstone

3). stablizer group

4) little group. (Wigner)

5). orbital group

6). Quatiant group

$G/H = M \Rightarrow \pi_n(M)$

☆ $G \Rightarrow$ 自由能F保持不变/自由能不变.

$H \Rightarrow H = \{ h \in G \mid h\psi = \psi \}$

Eg.

7月中旬

BEC spin-1.

$S = 0, +1, -1$

$\psi_0, \psi_1, \psi_{-1}$

$$G = SU(3) \times U(1)$$

$$g\psi = e^{i\phi}\, U(\alpha, \beta, \gamma)\psi$$

Ground state

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$h|\psi\rangle = |\psi\rangle$ 不变.

$$h = e^{i\phi} U(0, 0, \gamma) = e^{i\phi} e^{iS_z\gamma}, \quad S_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$H = U(1)$$

$$M = U(1) \times SU(3)/U(1) = SO(3)$$

	π_1	π_2	π_3	π_4
SO(3)	Z_2	0	Z	Z_2

求 $M = SU(3) \times U(1)/Z_2 =$

$$\underset{0}{\pi_3(Z_2)} \to \underset{Z}{\pi_3(G)} \to \underset{Z}{\pi_3(G/Z_2)} \to$$

$$
\begin{array}{ccccccc}
 & \pi_2(\mathbb{Z}_2) & \longrightarrow & \pi_2(G) & \to & \pi_2(G/\mathbb{Z}_2) & \to \\
 & 0 & & 0 & & 0 & \\
 & \pi_1(\mathbb{Z}_2) & \longrightarrow & \pi_1(G) & \to & \pi_1(G/\mathbb{Z}_2) & \to \\
 & 0 & & \mathbb{Z}_2 \oplus \mathbb{Z} & & & \\
\to & \pi_0(\mathbb{Z}_2) & \longrightarrow & \pi_0(G) & \to & \pi_0(G/\mathbb{Z}) & \\
 & \mathbb{Z}_2 & & 0 & & &
\end{array}
$$

$\pi^* = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\pi_1(SO(3) \times U(2)) = \mathbb{Z}_2 \times \mathbb{Z}$

$\pi_2(SO(3)$