Chapter 12

Topological Excitations

Topological excitations in a superfluid are singular defects of a superfluid order parameter. The unique feature of topological excitations lies in the fact that they can move freely in space without changing their characteristics and that they are robust against external perturbations. The properties of topological excitations are independent of material-specific parameters and charecterized by a set of integers called topological charges. Topological excitations are best classified in terms of homotopy theory. This theory gives what type of topological excitations can exist in which type of order parameter manifolds, and describes what happens if two topological defects coexist and how they coalesce or disintegrate. In this chapter, we present a brief introduction of homotopy theory and touch upon a rich variety of topological excitations in superfluid systems.

12.1 Homotopy Theory

12.1.1 *Homotopic relation*

To describe topological defects, we develop a mathematical tool with which to identify objects that can transform into each other in a continuous manner as belonging to the same equivalent class. Such a classification can be carried out using homotopy theory that classifies continuous maps. Let $f : X \to Y$ and $g : X \to Y$ be two continuous maps between topological spaces X and Y. These two maps are called homotopic and denoted as $f \sim g$ if there exists a continuous map

$$F : X \times [0,1] \to Y \tag{12.1}$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for $^\forall x \in X$. Viewed as a function of t, $F(x,t)$ gives a continuous family of maps from X to Y; F is called the homotopy for f and g.

The homotopic relation satisfies the following three conditions that are necessary and sufficient for this relation to be an equivalence one. Let $f, g, h : X \to Y$ be continuous maps. Then

$$\begin{array}{lll} \text{(i)} & f \sim f & \text{(symmetric);} \\ \text{(ii)} & \text{If } f \sim g, \text{ then } g \sim f & \text{(reflexive);} \\ \text{(iii)} & \text{If } f \sim g \text{ and } g \sim h, \text{ then } f \sim h & \text{(transitive).} \end{array} \tag{12.2}$$

Condition (i) can be shown by taking $F(x,t) = f(x)$. To show (ii), let $F(x,t)$ be the homotopy for $f \sim g$. Then, $F(x, 1-t)$ gives the homotopy for $g \sim f$. To show condition (iii), let $F : X \times I \to Y$ and $G : X \times I \to Y$ be the homotopies for $f \sim g$ and $g \sim h$, respectively, where $I \equiv [0,1]$. Then,

$$H(x,t) \equiv \begin{cases} F(x, 2t) & 0 \le t \le \frac{1}{2} \\ G(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \tag{12.3}$$

gives the homotopy for $f \sim h$.

Let f and g be maps for $X \to Y$ and $Y \to Z$, respectively. Then, the composite map $g \cdot f : X \to Z$ is defined as $g(f(x))$. Let f, f' and g, g' be mappings for $X \to Y$ and $Y \to Z$, respectively. If $f \sim f'$ and $g \sim g'$, then $g \cdot f \sim g' \cdot f'$. To show this, let $F : X \times I \to Y$ and $G : Y \times I \to Z$ be homotopies for f, f' and g, g', respectively. Then, $H(x,t) \equiv G(F(x,t),t)$ defines a continuous map $H : X \times I \to Z$ such that

$$\begin{aligned} H(x,0) &= G(f(x),0) = g \cdot f(x), \\ H(x,1) &= G(f'(x),1) = g' \cdot f'(x). \end{aligned} \tag{12.4}$$

Thus, $g \cdot f$ and $g' \cdot f'$ are homotopic under H.

A constant map $c : X \to Y$ is defined as a map whose image is an arbitrary fixed element of Y. The identity map $i_d : X \to X$ is defined as a map such that $i_d(x) = x$ for $^\forall x$. A topological space X is said to be contractible if the identity map i_d is homotopic to a constant map c:

$$X \text{ is contractible} \iff i_d \sim c. \tag{12.5}$$

If a topological space X is contractible, any continuous map $f : X \to Y$ is homotopic to a constant map. In fact, if X is contractible, there exists some homotopy $F : X \times I \to X$ such that $F(x,0) = x$ and $F(x,1) = \text{constant} \equiv x_0 \in X$ for $^\forall x \in X$. Then, a homotopy for $f \sim c$ is given by $G(x,t) \equiv f(F(x,t))$, for $G(x,0) = f(F(x,0)) = f(x)$ and $G(x,1) = f(F(x,1)) = f(x_0) = \text{constant} \in Y$.

12.1.2 *Fundamental group*

Let a continuous map $\varphi : I \to X$ be called a path whose initial point and terminal point are given by $\varphi(0)$ and $\varphi(1)$, respectively. The inverse φ^{-1} of φ is defined as

$$\varphi^{-1}(s) \equiv \varphi(1-s). \tag{12.6}$$

If the terminal point of a path φ_1 coincides with the initial point of another path φ_2, *i.e.*, $\varphi_1(1) = \varphi_2(0)$, we can define the product of the two paths $\varphi_1 \cdot \varphi_2$ as

$$\varphi_1 \cdot \varphi_2(t) \equiv \begin{cases} \varphi_1(2s) & 0 \leq s \leq \frac{1}{2} \\ \varphi_2(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases} \tag{12.7}$$

A loop ℓ is a special path such that the initial and terminal points coincide:

$$\ell(0) = \ell(1) \equiv x_0, \tag{12.8}$$

where x_0 is called the base point. Let $\ell_1(s)$ and $\ell_2(s)$ be two loops sharing the same base point. If there exists a homotopy $F : I \times I \to X$ such that

$$\begin{aligned} F(s,0) &= \ell_1(s), \\ F(s,1) &= \ell_2(s), \\ F(0,t) &= F(1,t) = x_0, \end{aligned} \tag{12.9}$$

then ℓ_1 and ℓ_2 are said to be homotopic. The constant loop ℓ_c is the one in which $\ell_c(s) = x_0$ for $\forall s \in I$. We can classify all the loops that share the same base point into a set of equivalent classes called homotopy classes:

$$[\ell_1], [\ell_2], \cdots\cdots, \tag{12.10}$$

where $[\ell_i]$ comprises all loops that are homotopic to ℓ_i. For each homotopy class $[\ell_i]$, we may choose an arbitrary loop in $[\ell_i]$ to represent the homotopy class because all elements belonging to the same class are homotopic to each other.

The product of two homotopy classes $[\ell_1]$ and $[\ell_2]$ is defined as

$$[\ell_1] \cdot [\ell_2] = [\ell_1 \cdot \ell_2], \tag{12.11}$$

where $\ell_1 \cdot \ell_2$ denotes the product of two loops in which ℓ_1 is first traversed and then ℓ_2 is traversed as shown in Fig. 12.1. Members of $[\ell_1 \cdot \ell_2]$, which are homotopic to $\ell_1 \cdot \ell_2$, need not return to the base point x_0 en route to the terminal point [see the dashed curve in Fig. 12.1]. With definition (12.11),

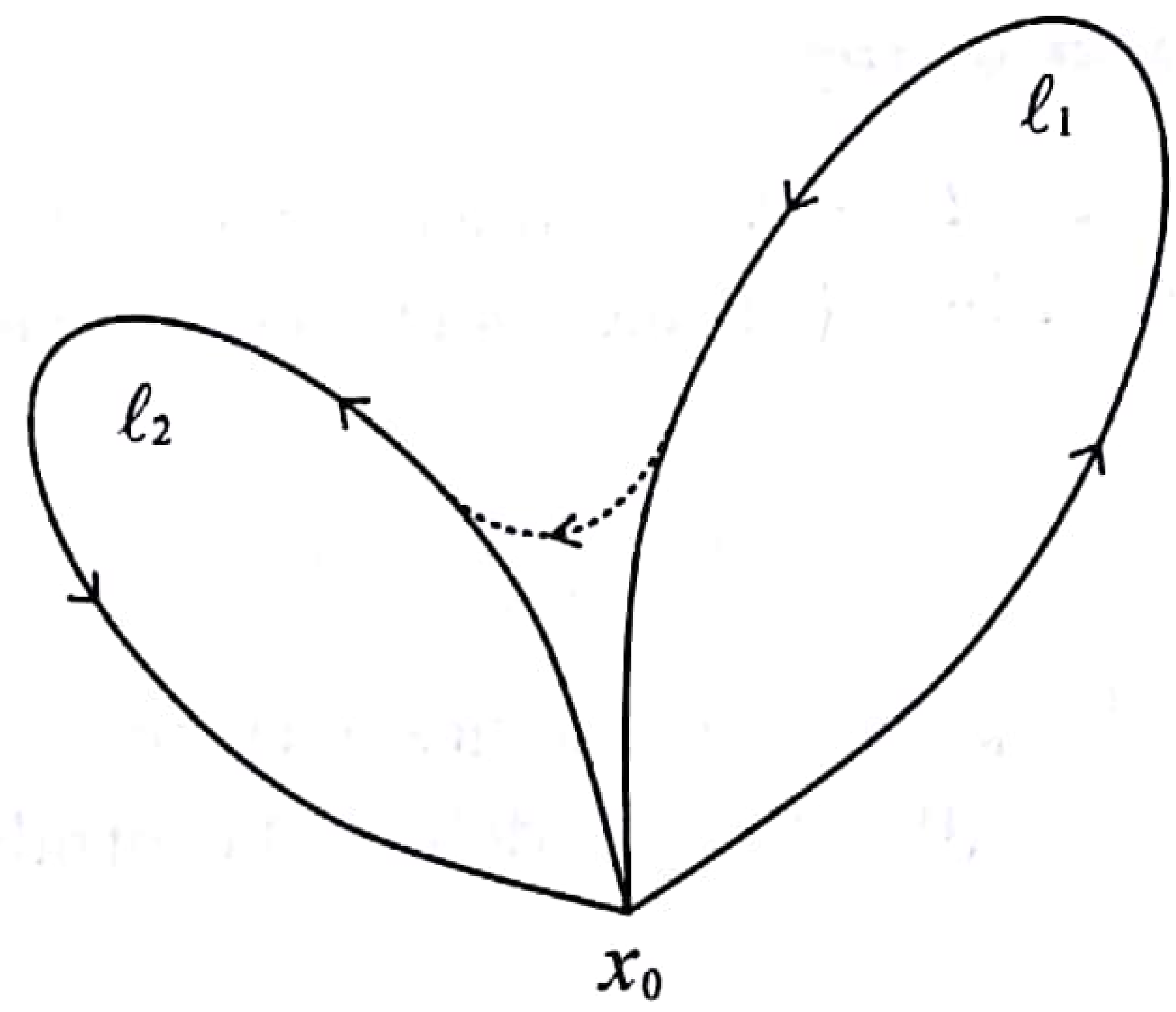

Fig. 12.1 A product of two loops $\ell_1 \cdot \ell_2$ in which ℓ_1 is first traversed and then ℓ_2 is traversed. Loops homotopic to $\ell_1 \cdot \ell_2$ need not return to the base point x_0 en route to the terminal point, as indicated by the dashed curve.

a set of homotopy classes form a group. In fact, they satisfy the associative law

$$([\ell_1] \cdot [\ell_2]) \cdot [\ell_3] = [\ell_1] \cdot ([\ell_2] \cdot [\ell_3]); \tag{12.12}$$

those loops that are homotopic to the constant loop c form the identity element $[c]$:

$$[c] \cdot [\ell] = [\ell] \cdot [c] = [\ell] \text{ for any } [\ell]; \tag{12.13}$$

the inverse $[\ell^{-1}]$ of $[\ell]$ is the homotopy class that consists of inverse loops [see Eq. (12.6)] of $[\ell]$:

$$[\ell^{-1}] \cdot [\ell] = [\ell] \cdot [\ell^{-1}] = [c]. \tag{12.14}$$

The group defined above is called the fundamental group or the first homotopy group and denoted as $\pi_1(X, x_0)$.

Two points x_0 and x_1 in X are said to be arcwise connected if there exists a continuous map $f : I \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. A topological space X is said to be arcwise connected if any two pairs in X are arcwise connected. Then, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic to each other:

$$\pi_1(X, x_0) \cong \pi_1(X, x_1). \tag{12.15}$$

In fact, by assumption, there exists a path α that connects x_1 to x_0 [see Fig. 12.2]. Then, for every loop ℓ that has the base point at x_0, loop $\alpha \cdot \ell \cdot \alpha^{-1}$ has the base point at x_1. This correspondence can be utilized to introduce a map from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ via

$$[\ell] \to [\alpha \cdot \ell \cdot \alpha^{-1}]. \tag{12.16}$$

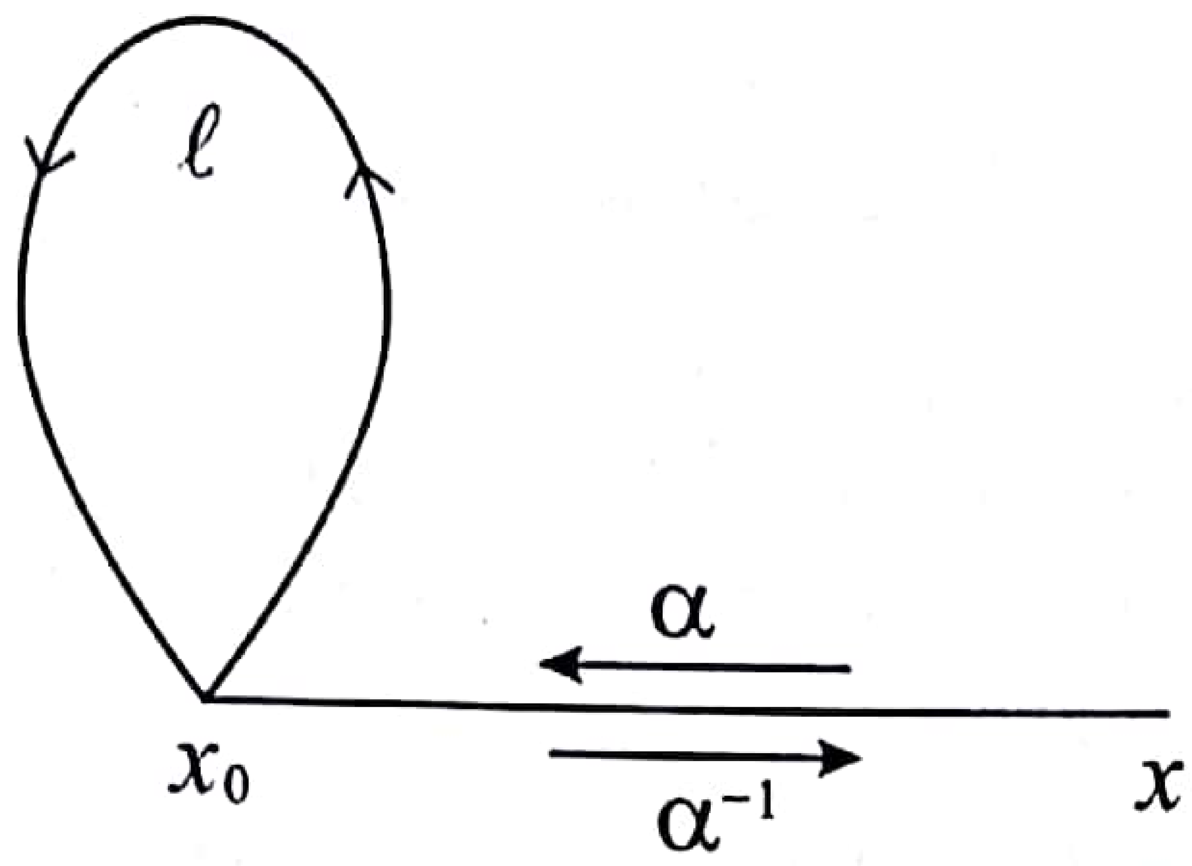

Fig. 12.2 For every loop ℓ that has the base point at x_0, the loop $\alpha \cdot \ell \cdot \alpha^{-1}$ has the base point at x_1.

This map is a homomorphism, because the product $[\ell_1] \cdot [\ell_2]$ is mapped into

$$\begin{aligned}[\ell_1] \cdot [\ell_2] = [\ell_1 \cdot \ell_2] &\to [\alpha \cdot \ell_1 \cdot \ell_2 \cdot \alpha^{-1}] \\ &= [(\alpha \cdot \ell_1 \cdot \alpha^{-1}) \cdot (\alpha \cdot \ell_2 \cdot \alpha^{-1})] \\ &= [\alpha \cdot \ell_1 \cdot \alpha^{-1}] \cdot [\alpha \cdot \ell_2 \cdot \alpha^{-1}]. \end{aligned} \tag{12.17}$$

To prove the isomorphism (12.15), we must show that the mapping in (12.16) is a bijection. Since the map has the inverse $[\ell'] \to [\alpha^{-1} \cdot \ell' \cdot \alpha] \in \pi_1(X, x_0)$ for any $[\ell'] \in \pi_1(X, x_1)$, the map (12.16) is surjective. If $[\ell'_1]$ and $[\ell'_2]$ are two different homotopy classes of $\pi_1(X, x_1)$, their inverses $[\alpha^{-1} \cdot \ell'_1 \cdot \alpha]$ and $[\alpha^{-1} \cdot \ell'_2 \cdot \alpha]$ are different classes of $\pi_1(X, x_0)$, because if they belong to the same class, there exists some homotopy $F : I \times I \to X$ such that $F(s, 0) = \alpha^{-1} \cdot \ell'_1 \cdot \alpha$ and $F(s, 1) = \alpha^{-1} \cdot \ell'_2 \cdot \alpha$. Then, $\alpha \cdot F(s, t) \cdot \alpha^{-1}$ is a homotopy for ℓ'_1 and ℓ'_2; this contradicts our initial assumption. By *reductio ad absurdum*, the mapping (12.16) is injective, and hence, it is bijective. Unless otherwise stated, we assume that X is arcwise connected and denote $\pi_1(X, x_0)$ simply as $\pi_1(X)$ because all fundamental groups that have different base points will be isomorphic to each other. If $\pi_1(X)$ consists only of the identity element, the topological space X is called simply connected.

Let the n-sphere S^n be defined as

$$S^n = \left\{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\right\}, \tag{12.18}$$

where S^1 is a unit circle and S^2, a unit sphere. Because $\pi_1(S^1)$ concerns a mapping from a loop to the unit circle, it is isomorphic to the additive group of integers:

$$\pi_1(S^1) \cong \mathbb{Z}, \tag{12.19}$$

where each element of integers corresponds to the number of times the unit circle is wound by the map. Because any loop on S^2 can be contracted to a point, we have

$$\pi_1(S^2) \cong \{[c]\}. \tag{12.20}$$

This is also denoted simply as $\pi_1(S^2) = 0$.

The fundamental group of the direct product of two arcwise connected topological spaces X and Y is isomorphic to the direct product of their individual fundamental groups:

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y). \tag{12.21}$$

In fact, each element of $\pi_1(X \times Y)$ is expressed as $[(\ell_X, \ell_Y)]$, where ℓ_X and ℓ_Y are loops belonging to X and Y, respectively, and each pair of loops defines a map $(\ell_X, \ell_Y)\colon I \to X \times Y$. If $\ell_X \sim \ell'_X$ and $\ell_Y \sim \ell'_Y$, then $(\ell_X, \ell_Y) \sim (\ell'_X, \ell'_Y)$, and therefore, there exists a one-to-one correspondence between $[(\ell_X, \ell_Y)]$ and $([\ell_X], [\ell_Y])$, which proves (12.21).

12.1.3 *Higher homotopy groups*

We consider a map $\varphi_n : I^n \to X$, where I^n is the unit n-cube:

$$I^n = \{(s_1, s_2, \cdots, s_n) |\, 0 \le s_i \le 1 \text{ for } i = 1, 2, \cdots, n\}. \tag{12.22}$$

Let ∂I^n be the boundary of I^n defined by

$$\partial I^n = \left\{(s_1, s_2, \cdots, s_n) |\; s_i = 0 \text{ or } 1 \text{ for } {}^{\exists}i\right\}. \tag{12.23}$$

Similar to the case of π_1, we require that all points on the boundary be mapped to a single base point x_0; then, φ_n is called an n-loop at x_0.

Two maps α and β are said to be homotopic to each other and denoted as $\alpha \sim \beta$ if there exists a continuous map $H : I^n \times I \to X$ such that

$$\begin{aligned} H(s_1, s_2, \cdots, s_n, 0) &= \alpha(s_1, s_2, \cdots, s_n), \\ H(s_1, s_2, \cdots, s_n, 1) &= \beta(s_1, s_2, \cdots, s_n), \\ H(s_1, s_2, \cdots, s_n, t) &= x_0 \text{ for } {}^{\forall}t \in I \text{ if } (s_1, s_2, \cdots, s_n) \in \partial I^n. \end{aligned} \tag{12.24}$$

The homotopic relation allows us to classify all n-loops at x_0 into homotopy classes $[\alpha]$.

The inverse α^{-1} of α is defined by

$$\alpha^{-1}(s_1, s_2, \cdots, s_n) = \alpha(1 - s_1, s_2, \cdots, s_n). \tag{12.25}$$

The product of two n-loops α and β is defined by

$$\alpha \cdot \beta = \begin{cases} \alpha(2s_1, s_2, \cdots, s_n) & 0 \le s_1 \le \frac{1}{2} \\ \beta(2s_1 - 1, s_2, \cdots, s_n) & \frac{1}{2} \le s_1 \le 1. \end{cases} \tag{12.26}$$

It can be seen that the homotopy classes $[\alpha], [\beta], \cdots$ form a group called the n-th homotopy group $\pi_n(X, x_0)$, where the multiplication law is defined by

$$[\alpha] \cdot [\beta] \equiv [\alpha \cdot \beta]. \tag{12.27}$$

Similarly to Eq. (12.15), if X is arcwise connected, two homotopy groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic to each other, and therefore, we may omit the base point and write $\pi_n(X, x_0)$ simply as $\pi_n(X)$. A relationship similar to (12.21) holds for the n-th homotophy group:

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y). \tag{12.28}$$

12.2 Order Parameter Manifold

12.2.1 *Isotropy group*

Let ψ be an order parameter of the system and let G be a Lie group whose arbitrary element g transforms ψ in such a manner to keep the free-energy functional invariant. We also introduce the isotropy group H whose arbitrary element h makes ψ invariant, that is, $h\psi = \psi$. The order parameter manifold M is defined as the coset of H in G:

$$M = G/H. \tag{12.29}$$

For example, in s-wave superconductors, superfluid helium-4, or spin-polarized gaseous BECs, a global phase change $\psi \to e^{i\phi}\psi$ does not change the free energy of the system. Thus, $G = U(1)$. In this case, only the identity element makes ψ invariant. Thus, $H = \{1\}$ and $M = U(1)$.

For the case of spinor BECs in the absence of an external magnetic field, the free energy is invariant under the global gauge transformation and an orbitrary rotation in spin space. Thus,

$$G = U(1)_\phi \times SO(3)_{\mathbf{S}}, \tag{12.30}$$

where the subscripts ϕ and $\mathbf{S}$ indicate that the group operation acts on the gauge (*i.e.*, the global phase) and spin, respectively. In the presence of a magnetic field, the isotropy of space is broken; however, the rotation about the direction, say the z-axis, of the magnetic field still keeps the free energy invariant. Thus,

$$G = U(1)_\phi \times U(1)_{S_z}, \tag{12.31}$$

where $U(1)_{S_z}$ describes the group of rotations of spin about the z-axis.

The isotropy group of a spinor BEC is a subgroup of G in Eq. (12.30); therefore, we can list possible candidates without investigating individual phases. The rotation group $SO(3)$ is the group of rotations in three dimensions, where each rotation is specified by the rotation axis (determined by the polar and azimuthal angles) and the angle of rotation about the axis. The group is therefore characterized by three continuous parameters. The $SO(3)$ group has a continuous subgroup that is the $U(1)$ group of rotations about a symmetry axis, and the following five discrete subgroups:

- $\mathbb{C}_n$: the cyclic group of rotations through angle $2\pi k/n$, where $k = 0, 1, 2, \cdots, n-1$.
- $\mathbb{D}_n$: the dihedral group consisting of $\mathbb{C}_n$ and n reflections about the axes symmetrically placed on a plane perpendicular to the symmetry axis of the $\mathbb{C}_n$.
- $\mathbb{T}$: the tetrahedral group with 12 elements consisting of the identity; three π rotations about the axes connecting midpoints of opposite sides; and four $\pi/3$ and $2\pi/3$ rotations about the axes connecting each vertex and the center of the opposite face.
- $\mathbb{O}$: the octahedral group with 24 elements consisting of the identity; three $\pi/2$, π, and $3\pi/2$ rotations about the axes connecting centers of opposite surfaces; four $2\pi/3$ and $4\pi/3$ rotations about the axes connecting opposite vertices; and six π rotations about the axes connecting midpoints of opposite sides.
- $\mathbb{I}$: the icosahedral group with 60 elements, consisting of the identity; six $2\pi/5$, $4\pi/5$, $6\pi/5$, and $8\pi/5$ rotations about the axes connecting opposite vertices; ten $2\pi/3$ and $4\pi/3$ rotations about the centers of opposite faces; and fifteen π rotations about the axes connecting the midpoints of opposite sides.

Many of these subgroups indeed have physical counterparts: $U(1)$, $\mathbb{C}_2$, $\mathbb{D}_2$, and $\mathbb{T}$ correspond to the ferromagnetic, antiferromagnetic, polar, and cyclic phases, respectively. We also expect that $\mathbb{O}$ finds a physical counterpart for a spin-3 BEC.

12.2.2 *Spin-1 BEC*

An element g of G in Eq. (12.30) corresponds to a transformation of the order parameter ψ as

$$g\psi = e^{i\theta} U(\alpha, \beta, \gamma)\psi, \tag{12.32}$$

where $e^{i\theta} \in U(1)_\phi$ and $U(\alpha, \beta, \gamma) = e^{if_z\alpha} e^{if_y\beta} e^{if_z\gamma} \in SO(3)_\mathrm{S}$. For the case of a spin-1 ferromagnetic phase with $\psi = (1, 0, 0)^T$, the general order parameter is given by Eq. (6.98). Here, we note that the gauge angle θ and the rotation angle γ appear as a linear superposition, so that elements of the isotropy group can be expressed as

$$h\psi = e^{i\gamma}U(0,0,\gamma)\psi = \psi. \tag{12.33}$$

Therefore, the isotropy group is given by

$$H^{\mathrm{F}} = U(1)_{\phi+S_\gamma}, \tag{12.34}$$

where the subscript $\phi + S_\gamma$ indicates that elements of the isotropy group describe the combined operations of the gauge transformation and the spin rotation about the direction of the spin. The order parameter manifold of the ferromagnetic phase is given by

$$M^{\mathrm{F}} = \frac{U(1)_\phi \times SO(3)_{\mathrm{S}}}{U(1)_{\phi+S_\gamma}} = SO(3)_{\phi,\mathrm{S}}. \tag{12.35}$$

For the case of the spin-1 polar phase, the general order parameter is given by Eq. (6.107), which does not depend on γ. Moreover, it is invariant under the combined transformation of gauge $\phi = \theta \to \theta + \pi$ and spin $\beta \to \beta + \pi$ that constitutes the nontrivial element of the two-element group $\mathbb{Z}_2$, the other element being the identity. Thus, the isotropy group is given by

$$H^{\mathrm{P}} = U(1)_{S_\gamma} \times (\mathbb{Z}_2)_{\phi,S_\beta}, \tag{12.36}$$

and therefore, the order parameter manifold is

$$M^{\mathrm{P}} = \frac{U(1)_\phi \times SO(3)_{\mathrm{S}}}{U(1)_{S_\gamma} \times (\mathbb{Z}_2)_{\phi,S_\beta}} = \frac{U(1)_\phi \times S^2_{\mathrm{S}}}{(\mathbb{Z}_2)_{\phi,S_\beta}}, \tag{12.37}$$

where we use the rotation $SO(3)/U(1) = S^2$.

12.2.3 *Spin-2 BEC*

Next, we consider the case of a spin-2 BEC. For the ferromagnetic phase, we again obtain Eq. (12.35). For the uniaxial nematic phase, the same argument as the case of the spin-1 polar phase applies and we obtain Eq. (12.37). For the biaxial nematic (or antiferromagnetic) phase, the continuous $U(1)$ symmetry in Eq. (12.36) is broken; however, the system possesses an additional inversion symmetry about the axis perpendicular to the quantization axis of the spin, and consequently, the isotropy group becomes the quaternion group $\mathbb{Q}$. Thus,

$$M^{\mathrm{biaxial}} = \frac{U(1)_\phi \times SO(3)_{\mathrm{S}}}{\mathbb{Q}}. \tag{12.38}$$

Because the quaternion group is non-Abelian, the biaxial nematic phase of a spin-2 BEC exhibits nontrivial topological excitations.

The isotropy group of the cyclic phase of a spin-2 BEC is the tetrahedral group:

$$H^{\text{cyclic}} = (\mathbb{T})_{\phi,\mathbf{S}}. \tag{12.39}$$

This can be understood as follows. The fundamental building block of the cyclic phase is a spin-singlet trio that geometrically forms a regular triangle [Ueda and Koashi (2002)]. The possible polyhedral subgroups are the tetrahedron and icosahedron. Now, the order parameter of the spin-2 BEC is described by five complex numbers. Because of the normalization condition and the global gauge invariance, only four complex numbers are free parameters that specify four vertices of the tetrahedron [Barnett, *et al.* (2006)]. The order parameter manifold is therefore given by

$$M^{\text{cyclic}} = \frac{U(1)_\phi \times SO(3)_{\mathbf{S}}}{(\mathbb{T})_{\phi,\mathbf{S}}}. \tag{12.40}$$

The tetrahedral group has 12 elements that are divided into three conjugacy classes. It is non-Abelian and gives rise to non-Abelian vortices [Kobayashi, *et al.* (2009)].

Table 12.1 lists order parameter manifolds M and their homotopy groups $\pi_n(M)$ for some typical systems.

12.3 Classification of Defects

12.3.1 *Domains*

By convention, $\pi_0(M)$ gives the number of "domain walls" that separate the order parameter manifold M. If $\pi_0(M) = 0$, M is said to be connected. If $\pi_0(M) = 1$, M is divided into two disconnected regions, and so on.

12.3.2 *Line defects*

12.3.2.1 *U*(1) *vortex*

Line defects such as quantized vortices in superfluids or disclinations in liquid crystals are classified by the first homotopy group $\pi_1(M)$ that describes a mapping from a loop in real space onto the order parameter manifold M. Let us consider, for example, a scalar order parameter $\psi(\mathbf{r}) = |\psi(\mathbf{r})|e^{i\phi(\mathbf{r})}$ of a superfluid. If the amplitude is constant, the order parameter is specified by the phase $\phi(\mathbf{r})$ that ranges from 0 and 2π. The order parameter

Table 12.1 List of order parameter manifolds M and their homotopy groups π_n, where $\mathbb{Q}$ and $\mathbb{T}^*$ are the quaternion group and the binary tetrahedral group, respectively; $\mathbb{R}P^2$ and $\mathbb{R}P^3$ are the two- and three-dimensional real projective spaces, respectively.

	M	π_1	π_2	π_3	π_4
planar spin	$U(1)$	$\mathbb{Z}$	0	0	0
Heisenberg spin	S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$
nematics	$\mathbb{R}P^2 \cong \frac{S^2}{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$
biaxial nematics	$\frac{SU(2)}{\mathbb{Q}}$	$\mathbb{Q}$	0	0	$\mathbb{Z}_2$
ferromagnetic BEC	$SO(3)_{\phi,S_\gamma} \cong \mathbb{R}P^3_{\phi,S_\gamma}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
spin-1 polar BEC	$\frac{S^2_{\mathbf{S}} \times U(1)_\phi}{(\mathbb{Z}_2)_{\phi,\mathbf{S}}}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$
cyclic BEC	$\frac{SO(3)_{\mathbf{S}} \times U(1)_\phi}{\mathbb{T}_{\phi,\mathbf{S}}}$	$\mathbb{T}^*$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
^{3}He-A (dipole-free)	$\frac{S^2 \times SO(3)}{\mathbb{Z}_2}$	$\mathbb{Z}_4$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
^{3}He-A (dipole-locked)	$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
^{3}He-B (dipole-free)	$S^1 \times SO(3)$	$\mathbb{Z} \times \mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$
^{3}He-B (dipole-locked)	$S^1 \times S^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$

manifold is therefore described by a unit circle called 1-sphere S^1. In this case, $\pi_1(S^1)$ describes a mapping from a loop in real space to S^1 according to the correspondence [see Fig. 12.3]

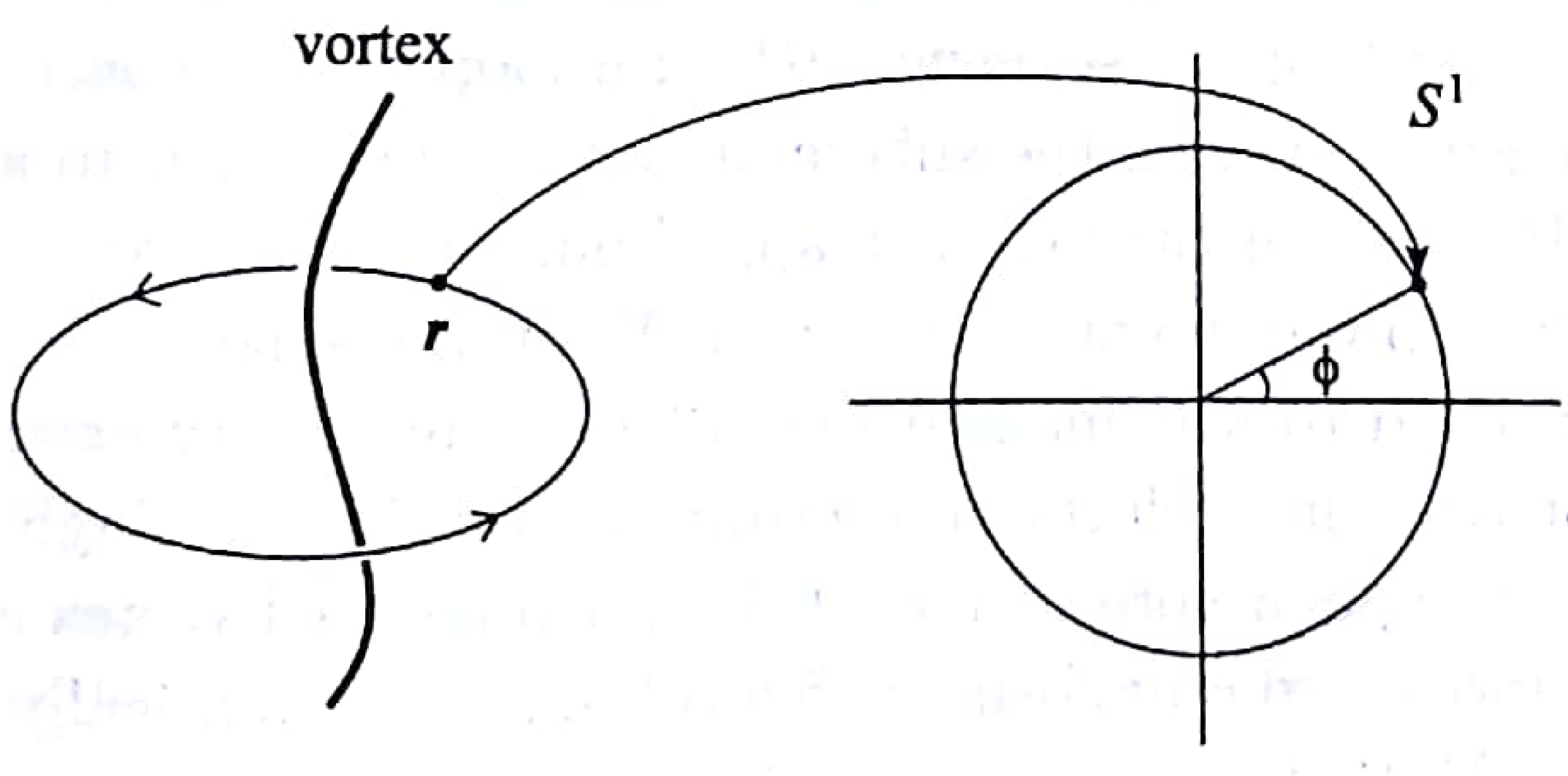

Fig. 12.3 Mapping from a loop in real space to the order parameter manifold S^1 according to the correspondence $\psi:\mathbf{r} \to \arg\psi(\mathbf{r}) = \phi(\mathbf{r})$. $\pi_1(S^1)$ classifies how many times this mapping covers S^1; therefore, $\pi_1(S^1) \cong \mathbb{Z}$

$$\psi : \mathbf{r} \to \arg\psi(\mathbf{r}) = \phi(\mathbf{r}). \tag{12.41}$$

If $\pi_1(S^1)$ gives n, the map covers S^1 n times. Physically, if $n = 0$, the loop contains no vortex. If $n = 1$, the loop contains a singly quantized vortex, and so on. Here, n is referred to as the winding number or topological charge of a vortex.

Thus, $\pi_1(S^1)$ classifies how many times the mapping (12.41) covers S^1; therefore, $\pi_1(S^1) \cong \mathbb{Z}$. The fact that $\pi_1(S^1)$ is isomorphic to the additive group of integers implies that two vortices with winding numbers m and n can coalesce into a single vortex with winding number $m + n$. Conversely, a vortex with winding number n can disintegrate into two vortices with winding numbers k and $n - k$. Whether these processes actually occur depends on the dynamics and energetics of an individual system.

Because any loop on a 2-sphere can be contracted to a point, we obtain $\pi_1(S^2) = 0$. More generally, $\pi_1(S^n) = 0$ for $n \geq 2$. Because the order parameter manifold of a three-dimensional ferromagnet is S^2, $\pi_1(S^2) = 0$, implying that this system cannot support a topologically stable line defect.

12.3.2.2 *SO*(3) *vortex*

The order parameter manifold of a ferromagnetic BEC is $SO(3)$, as shown in Eq. (12.35), and the fundamental group is

$$\pi_1(SO(3)) \cong \mathbb{Z}_2. \tag{12.42}$$

This can be understood if we represent an element of $SO(3)$ by a point within a sphere of radius π. A given point P in the sphere designates both the rotation axis $\overrightarrow{OP}$ and the rotation angle $\phi = \overline{OP}$, where the overbar denotes the length of the segment OP. If a loop in real space is mapped onto a loop, say C, within the sphere, it can be contracted to a point [see Fig. 12.4]. However, if the loop is mapped onto a curve that connects two diametrically opposite points, say A and A', on the sphere, the curve cannot be contracted to a point [see Fig. 12.4]. Thus, there exists only one type of nontrivial line defects in a ferromagnetic BEC, a singly quantized vortex called the polar-core vortex. A doubly quantized vortex can be continuously deformed to a uniform configuration, as schematically illustrated in Fig. 12.5. Mathematically, this is because $\pi_1(SO(3))$ is isomorphic to $\mathbb{Z}_2 = \{0, 1\}$—the additive group of integers modulo 2. Spin textures belonging to class 0 can continuously transform to a uniform spin configuration, whereas those belonging to class 1 involve singly quantized vortices that are topologically stable. Because $1 + 1 = 0$ (mod 2), the coalescence of two singly quantized vortices is homotopic to a uniform spin configuration. The order parameter of a polar-core vortex can be obtained by setting $\theta = \gamma$ and $\alpha = \phi$ in Eq. (6.98):

$$\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \sqrt{n} \begin{pmatrix} e^{-i\phi} \cos^2 \frac{\beta}{2} \\ \frac{1}{\sqrt{2}} \sin\beta \\ e^{i\phi} \sin^2 \frac{\beta}{2} \end{pmatrix}. \tag{12.43}$$

The core of this vortex is filled by the "polar" (*i.e.*, $m = 0$) component; hence, it is called the polar-core vortex. It is a nonsingular vortex.

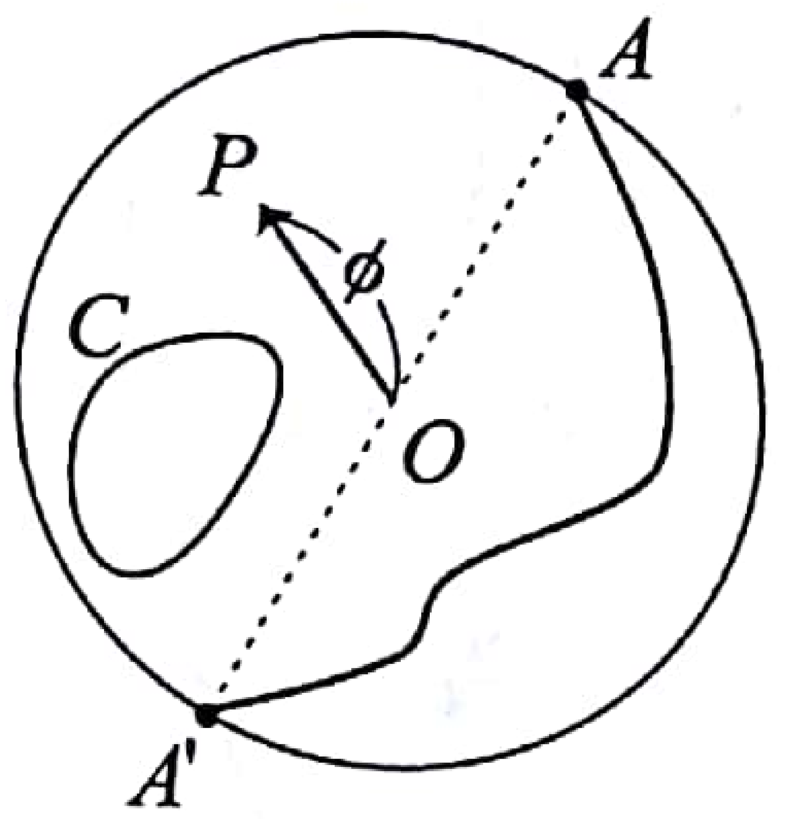

Fig. 12.4 Representation of an element of $SO(3)$ by a point P that lies within a sphere of radius π. The direction $\overrightarrow{OP}$ designates the rotation axis and the distance $\overline{OP}$ indicates the rotation angle ϕ. Note that two diametrically opposite points A and A' represent the same rotation because a π rotation about $\overrightarrow{OA}$ is equivalent to that about $\overrightarrow{OA'}$. A curve connecting A and A' represents a nonsingular polar-core vortex [see the text].

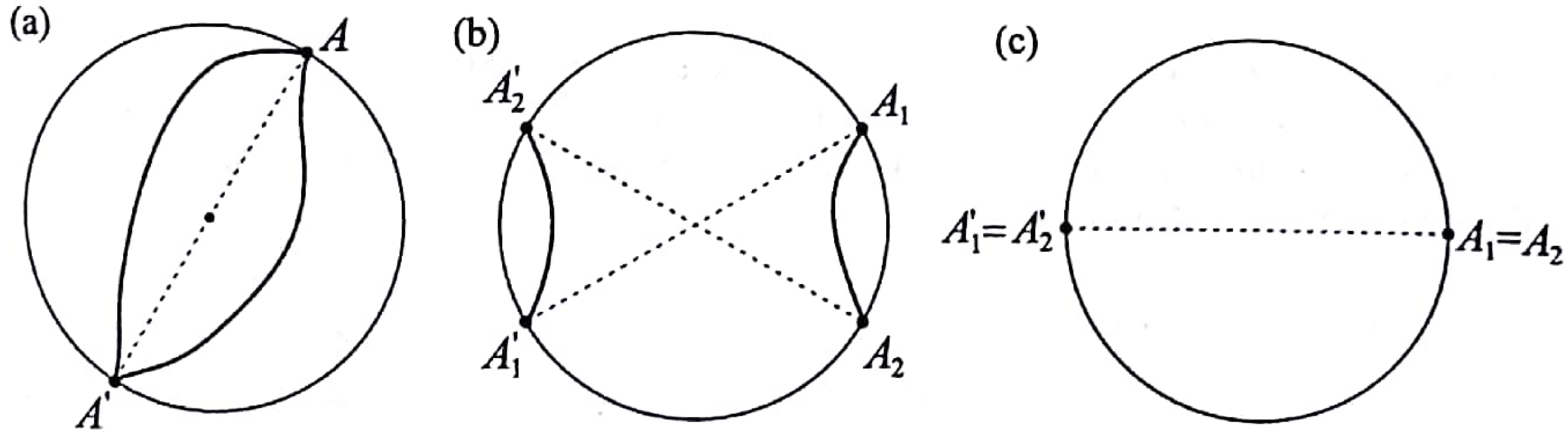

Fig. 12.5 A doubly quantized vortex (a) can be decomposed into a pair of singly quantized vortices that are diametrically connected (b); they can be continuously contracted to two equivalent single points (c).

12.3.2.3 *Half-quantum (Alice) vortex*

Let us next consider the polar phase of a spin-1 BEC, where the order parameter manifold is given by Eq. (12.37). Since S^2 makes a trivial contribution to π_1, we obtain from Eq. (12.21)

$$\pi_1(M^{\mathrm{P}}) \cong \mathbb{Z}. \tag{12.44}$$

Thus, the polar phase can have multiply quantized vortices. However, due to the combined $\mathbb{Z}_2$ symmetry of the spin and gauge shown in Eq. (12.37), the circulation is quantized in units of $h/2M$ rather than h/M. To show this, we consider the order parameter of the polar phase in Eq. (6.105):

$$\psi_{\mathrm{P}} = \sqrt{n}e^{i\theta}\begin{pmatrix} -\frac{e^{-i\alpha}}{\sqrt{2}}\sin\beta \\ \cos\beta \\ \frac{e^{i\alpha}}{\sqrt{2}}\sin\beta \end{pmatrix}. \tag{12.45}$$

As we circumnavigate the loop shown in Fig. 12.6, we let θ, α, and β vary

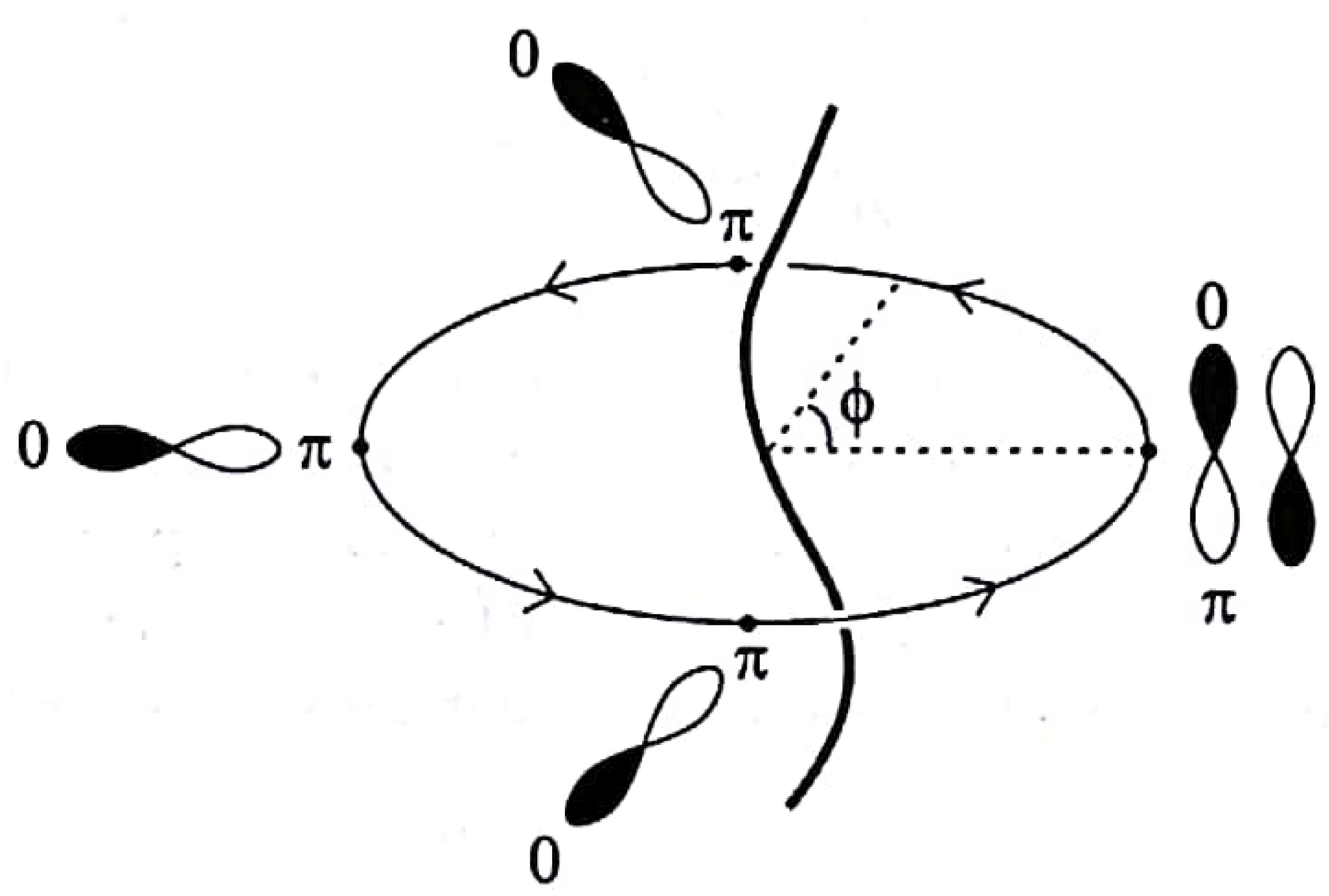

Fig. 12.6 Half-quantum (Alice) vortex of a spin-1 polar BEC. The order parameter rotates through π about a horizontal axis as one circumnavigates a loop. The single-valuedness of the order parameter is satisfied if the π rotation ($\beta \to \beta+\pi$) is accompanied by a gauge transformation by π ($\theta \to \theta + \pi$).

as

$$\theta = \frac{\phi}{2}, \quad \alpha = \text{constant}, \quad \beta = \frac{\phi}{2}. \tag{12.46}$$

Then, the superfluid velocity is calculated from Eq. (6.100) to give

$$\mathbf{v}_{\mathrm{s}}^{\mathrm{P}} = \frac{\hbar}{2M}\nabla\phi, \tag{12.47}$$

and thus, the circulation is quantized in units of $h/2M$ which is one half of the usual h/M. This vortex is therefore referred to as a half-quantum vortex or Alice vortex.

12.3.3 *Point defects*

12.3.3.1 *'t Hooft–Polyakov monopole*

Point defects such as monopoles and hedgehogs are characterized by $\pi_2(M)$ that classifies mappings from a sphere in real space onto an order parameter manifold M. Suppose that we enclose a point-like object O with a two-dimensional sphere Σ and consider a mapping from a point $\mathbf{r}$ on the sphere onto the order parameter manifold M. The second homotopy group $\pi_2(M)$ classifies such a map. For example, if $M = S^2$, $\pi_2(S^2)$ classifies the mapping according to the number of times the mapping covers S^2 [see Fig. 12.7]. Thus,

$$\pi_2(S^2) = \mathbb{Z}. \tag{12.48}$$

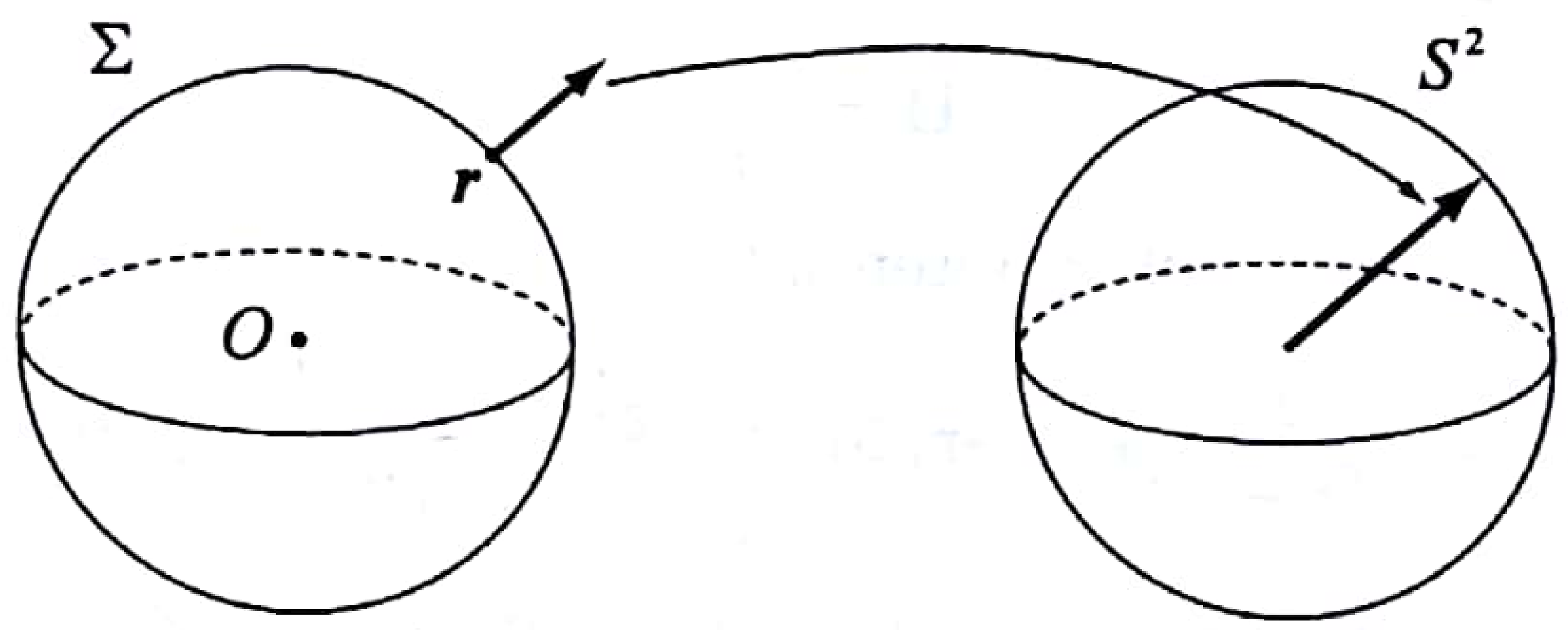

Fig. 12.7 Mapping from a two-dimensional sphere Σ in real space onto a 2-sphere S^2 in the order-parameter manifold. This mapping defines the second homotopy group $\pi_2(S^2)$.

The order parameter manifold M of the polar phase of a spin-1 BEC is given by Eq. (12.37) which involves the factor S^2 and gives

$$\pi_2(M^{\mathrm{P}}) = \mathbb{Z}.$$

Therefore, the system can possess a point defect. To understand the nature of this defect, we rewrite the order parameter (12.45) as

$$\psi_{\mathrm{P}} = \sqrt{\frac{n}{2}} e^{i\theta} \begin{pmatrix} -d_x + id_y \\ \sqrt{2} d_z \\ d_x + id_y \end{pmatrix}, \tag{12.49}$$

where

$$d_x = \sin\beta\cos\alpha, \quad d_y = \sin\beta\sin\alpha, \quad d_z = \cos\beta. \tag{12.50}$$

If we set $\theta = 0$ and

$$\mathbf{d}(\mathbf{r}) = \frac{\mathbf{r}}{r}, \tag{12.51}$$

the spin texture becomes spherical. The point defect of the type (12.50) is called a hedgehog or 't Hooft–Polyakov monopole with topological charge 1. This point defect, however, is unstable against deformation into the Alice ring [see Sec. 12.3.5.2].

12.3.3.2 *Dirac monopole*

Another type of point defect, the Dirac monopole, was originally envisaged as a magnetic analogue of the quantized electric charge. In analogy with the Poisson equation for an electric field, we assume that the magnetic field $\mathbf{B}$ obeys

$$\nabla \cdot \mathbf{B} = 4\pi g \delta(\mathbf{r}), \tag{12.52}$$

where g denotes the strength of the magnetic monopole. The solution of this equation is given by

$$\mathbf{B} = g\frac{\mathbf{r}}{r^3} \tag{12.53}$$

and the corresponding vector potential is

$$\mathbf{A} = \frac{g}{r(r-z)}(y, -x, 0) = -\frac{g(1+\cos\theta)}{r\sin\theta}\mathbf{e}_\varphi(\mathbf{r}), \tag{12.54}$$

where (r, θ, φ) are the polar coordinates and $\mathbf{e}_\varphi(\mathbf{r})$ is the unit vector along the φ direction at position $\mathbf{r}$. The vector potential in Eq. (12.54) reproduces the magnetic field in Eq. (12.53) except on the positive z-axis on which the magnetic field calculated from Eq. (12.54) exhibits a singularity that is called the Dirac string:

$$\text{rot}\mathbf{A} = g\frac{\mathbf{r}}{r^3} - 4\pi g \delta(x)\delta(y)\theta(z)\mathbf{e}_z, \tag{12.55}$$

where $\theta(z)$ is the unit-step function and $\mathbf{e}_z$ is the unit vector along the z-direction.

The Dirac monopole can be created in the ferromagnetic phase of a spin-1 BEC [Ruostekoski and Anglin (2003)]. Substituting $\theta - \gamma \to -\varphi, \alpha \to \varphi$, and $\beta \to \theta$ in Eq. (6.98), we obtain

$$\xi_{\mathrm{F}} \equiv \frac{1}{\sqrt{n}}\psi_{\mathrm{F}} = \begin{pmatrix} e^{-2i\varphi}\cos^2\frac{\theta}{2} \\ \frac{e^{-i\varphi}}{\sqrt{2}}\sin\theta \\ \sin^2\frac{\theta}{2} \end{pmatrix}. \tag{12.56}$$

The superfluid velocity

$$\mathbf{v}_s = -\frac{i\hbar}{M}\xi_{\mathrm{F}}^\dagger\nabla\xi_{\mathrm{F}} = -\frac{\hbar}{M}(1+\cos\theta)\nabla\varphi = -\frac{\hbar(1+\cos\theta)}{Mr\sin\theta}\mathbf{e}_\varphi \tag{12.57}$$

takes the same form as the vector potential (12.54) of the Dirac monopole with the identification $g = \hbar/M$. The magnetization is calculated to be

$$\langle \mathbf{F} \rangle = \sum_{\alpha,\beta} \xi_\alpha^* \mathbf{F}_{\alpha\beta} \xi_\beta = (\mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi) \sin\theta + \mathbf{e}_z \cos\theta, \quad (12.58)$$

which looks like a hedgehog. However, when $\theta = 0$, φ cannot be determined from $\langle \mathbf{F} \rangle$. This is consistent with the fact that the $m = 1$ component of ξ_F is singular except for $\theta = \pi$. The Dirac monopole is attached to a line singularity along the positive z-axis. Equation (12.56) shows that this singularity is a doubly quantized vortex. In accordance with the general argument of the $SO(3)$ vortex shown in Fig. 12.5, the Dirac monopole can be continuously deformed into a nonsingular spin texture. In fact, if we replace θ by $\theta(1-t) + \pi t$ in Eq. (12.56), the order parameter deforms continuously from the Dirac monopole at $t = 0$ to a nonsingular texture $\xi_\mathrm{F} = (0, 0, 1)^\mathrm{T}$ at $t = 1$.

12.3.4 *Skyrmions*

The third homotoopy group $\pi_3(M)$ classifies topological objects that extend over the entire three-dimensional space. These objects are called Skyrmions or particle-like solitons. Examples include the Shankar Skyrmion[1] and various types of knots.

12.3.4.1 *Shankar Skyrmion*

The order parameter manifold of a ferromagnetic BEC is $SO(3)$ which supports a topological object called the Shankar Skyrmion because $\pi_3(SO(3)) = \mathbb{Z}$. [Shankar (1977)] The order parameter is characterized by the direction of the spin and the rotation angle about that direction. Because of the spin-gauge symmetry, the rotation angle is related to the overall phase of the order parameter. We can specify both of them with a single vector $\boldsymbol{\Omega}$ whose direction and magnitude give the direction of the spin and the rotation angle, respectively. Each element of $\pi_3(SO(3)) = \mathbb{Z}$ may be realized by rotating the order parameter $(1, 0, 0)^T$ at each position $\mathbf{r}$ through angle $f(r)n$ about the direction

$$\boldsymbol{\Omega} = \frac{\mathbf{r}}{r} f(r) n, \quad n \in \mathbb{Z}, \quad (12.59)$$

[1] Despite the widespread nomenclature of the Shankar monopole, it is, in fact, a skyrmion.

where $f(0) = 2\pi$ and $f(\infty) = 0$. The condition $f(\infty) = 0$ ensures that the order parameter is uniform at spatial infinity. By identifying all points at spatial infinity, the three-dimensional space is compactified to the 3-sphere S^3. The order parameter of the Shankar Skyrmion for the case of $n = 1$ is shown in Fig. 12.8. The unitary operation for the rotation specified in

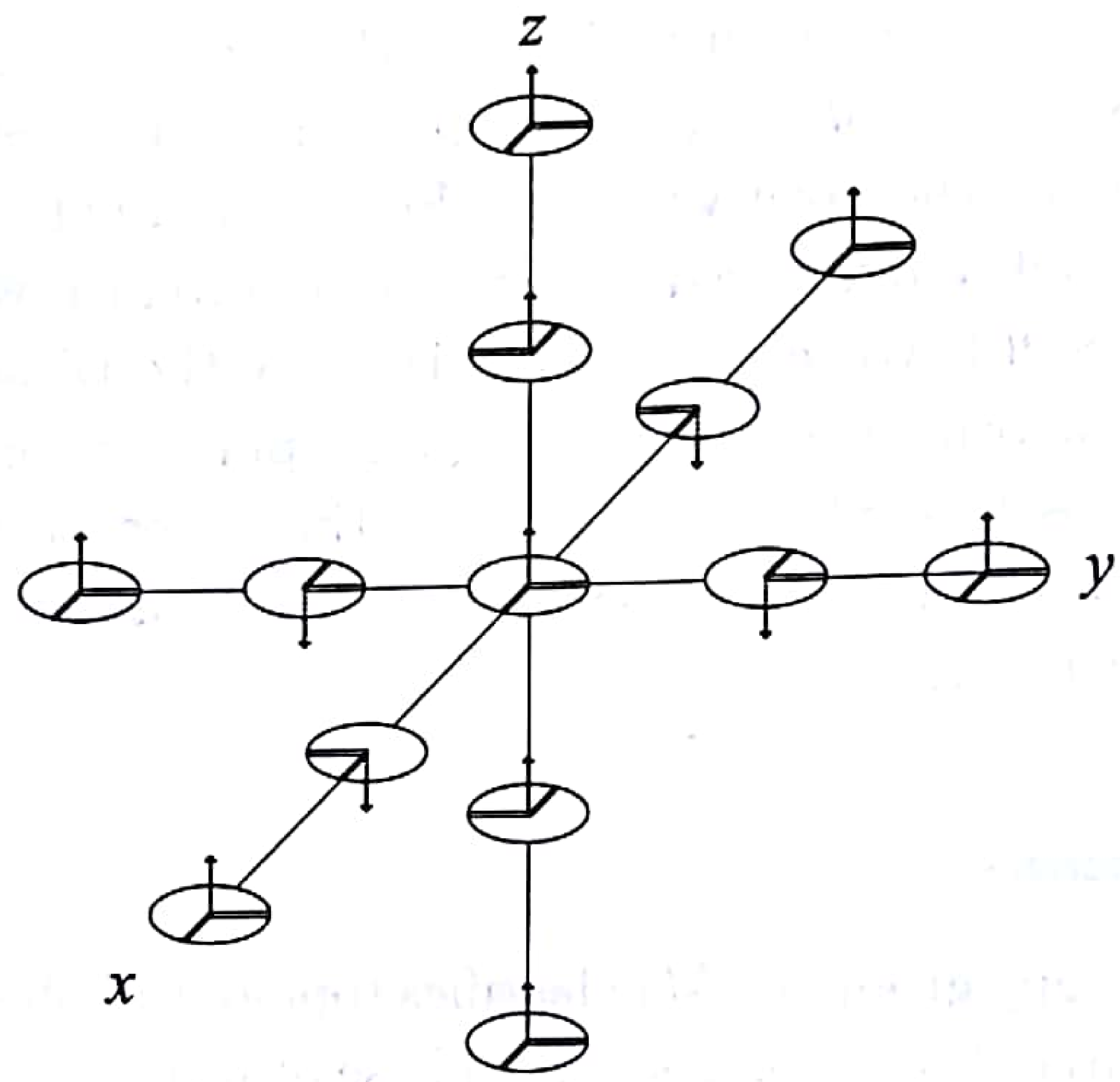

Fig. 12.8 Shankar Skyrmion with $n = 1$ in Eq. (12.59). In each circle, the arrow indicates the direction of the spin and the other two bars indicate how the overall phase of the order parameter changes over space. At spatial infinity, the direction of the spin and the phase of the order parameter are uniform; thus, the three-dimensional space is compactified to the 3-sphere S^3.

Eq. (12.59) is given by

$$\hat{U}(\mathbf{\Omega}) = \exp(-i\hat{\mathbf{f}} \cdot \mathbf{\Omega}), \tag{12.60}$$

where $\hat{\mathbf{f}} = (\hat{f}_x, \hat{f}_y, \hat{f}_z)$. For the spin-1 case, we obtain

$$\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} = \hat{U}(\mathbf{\Omega}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{bmatrix} \left(\cos \frac{f(r)n}{2} - i\cos\theta \sin \frac{f(r)n}{2}\right)^2 \\ -\sqrt{2}i \left(\cos \frac{f(r)n}{2} - i\cos\theta \sin \frac{f(r)n}{2}\right) \sin \frac{f(r)n}{2} \sin\theta e^{i\phi} \\ -\sin^2 \frac{f(r)n}{2} \sin^2\theta e^{2i\phi} \end{bmatrix}, \tag{12.61}$$

where (r, θ, ϕ) are the polar coordinates of $\mathbf{r}$. The spin vector (f_x, f_y, f_z) at $\mathbf{r}$ is therefore calculated to give

$$\begin{aligned}
f_x &= \sqrt{2}\mathrm{Re}\{(\psi_1 + \psi_{-1})\psi_0^*\} \\
&= 2\sin\frac{f(r)n}{2}\sin\theta\left(\sin\frac{f(r)n}{2}\cos\theta\cos\phi + \cos\frac{f(r)n}{2}\sin\phi\right), \\
f_y &= -\sqrt{2}\mathrm{Im}\{(\psi_1 - \psi_{-1})\psi_0^*\} \\
&= 2\sin\frac{f(r)n}{2}\sin\theta\left(\sin\frac{f(r)n}{2}\cos\theta\sin\phi - \cos\frac{f(r)n}{2}\cos\phi\right), \\
f_z &= |\psi_1|^2 - |\psi_{-1}|^2 = 1 - 2\sin^2\frac{f(r)n}{2}\sin^2\theta. \qquad (12.62)
\end{aligned}$$

12.3.4.2 *Knots*

Another example of topological excitations characterized by the third homotopy group are knots. Knots are distinguished from other topological excitations in that they are characterized by the linking number rather than the winding number. Knots are hosted by the polar phase of a spin-1 BEC, whose order parameter manifold is given by Eq. (12.37), of which the S^2 part makes a nontrivial contribution to π_3:

$$\pi_3(S^2) = \mathbb{Z}. \qquad (12.63)$$

The topological charge given by Eq. (12.63) is called the Hopf charge whose physical meaning can be understood as follows. Suppose that we trace a circle in a BEC along which the spinor order parameter points in one common direction $\mathbf{f}_1$ [see circle C_1 in Fig. 12.9]. We take another circle C_2 along which the order parameter points in another common direction $\mathbf{f}_2$ [see circle C_2]. If C_1 and C_2 link once, the linking number is 1. The linking number may be positive or negative depending on the relative orientation of the two loops.

A knot is a topological object in which a loop is embedded into itself in a nontrivial manner. The simplest example is the trefoil knot shown in Fig. 12.10. Technically speaking, the preimage of a single point on S^2 is called a simple unknotted loop, and therefore, the configuration in Fig. 12.9 is also called an unknot of a pair of closed loops with linking number 1. A topological object of the type shown in Fig. 12.9 is shown to be created in the polar phase of a spin-1 BEC [Kawaguchi, *et al.* (2008)].

The homotopy classification of topological excitations is summarized in Table 12.2.

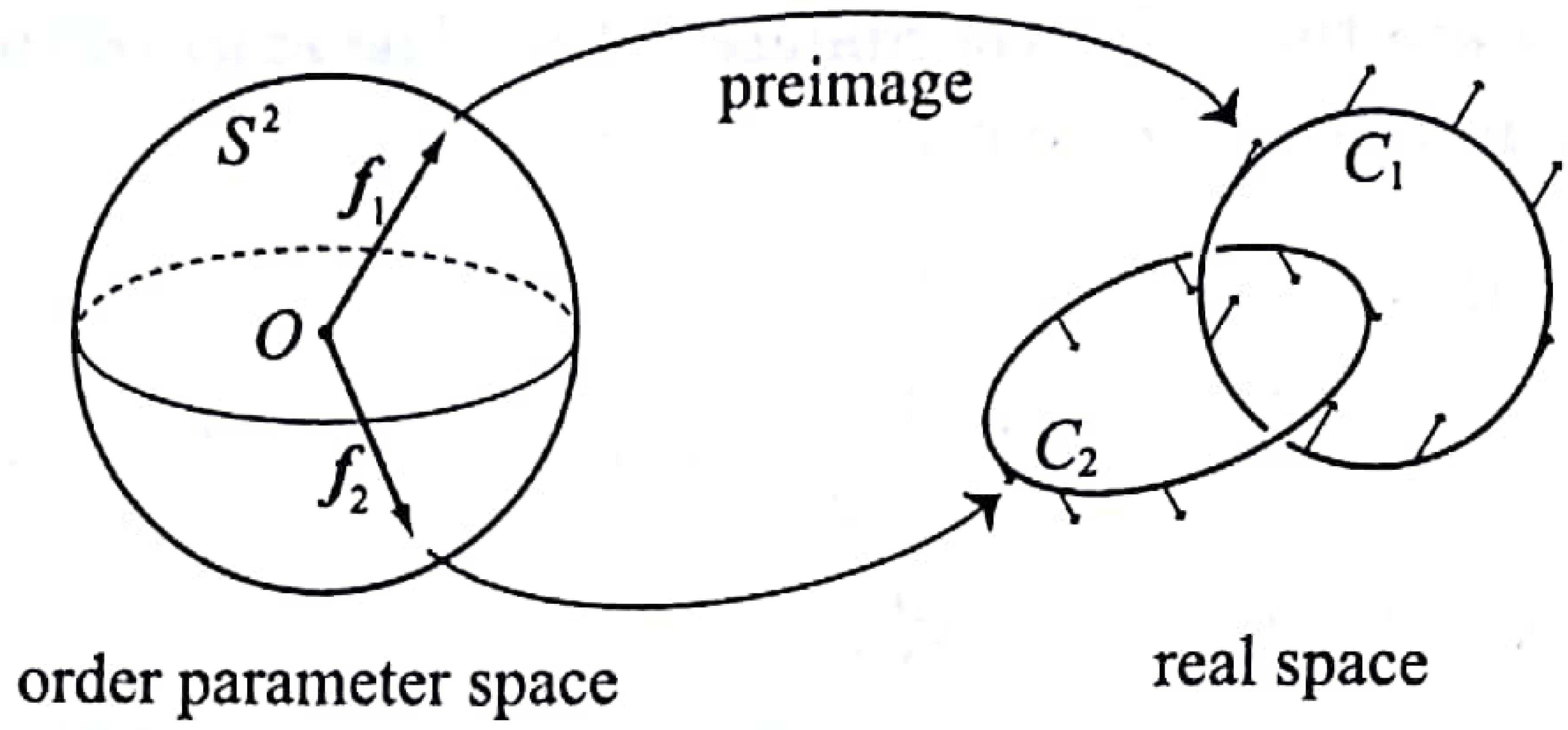

Fig. 12.9 Hopf charge (linking number). The preimage of each point in the order parameter space is a loop in real space. If C_1 and C_2 link once, the linking number is 1.

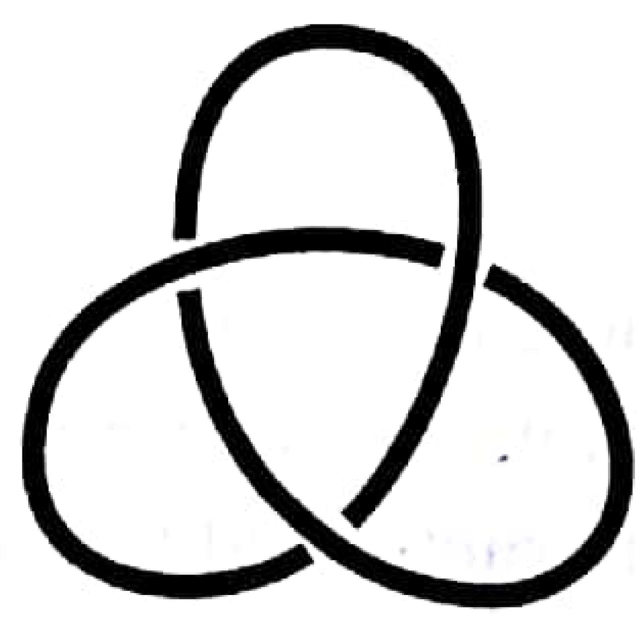

Fig. 12.10 Trefoil knot.

Table 12.2 Homotopy classification of defects and solitons.

π_n	defects	solitons
π_0	domain walls	
π_1	vortices	nonsingular domain walls
π_2	monopoles	2D Skyrmions
π_3	Skyrmions	Skyrmions, knots
π_4	instantons	$SO(3)$ instantons

12.3.5 *Influence of different types of defects*

12.3.5.1 *Influence of π_1 on π_2*

When two different types of topological objects coexist, they may influence each other in a nontrivial manner. As an example, consider a spin-1 polar phase in which two monopoles with charge one merge in the presence of a half-quantum vortex, as shown in Fig. 12.11 (a). The half-quantum vortex introduces a branch cut for field lines of monopoles, so that field lines on one side of the branch cut cannot merge with those on the other side by crossing it. Thus, if the two monopoles approach each other along path 1, the order-parameter field must deform, as shown in Fig. 12.11 (b). Because the number of field lines emanating from the center is twice as many as that of a monopole with charge one, the combined object has a topological charge of two. On the other hand, when the two monopoles merge along path 2, the order-parameter manifold can deform into the one shown in Fig. 12.11 (c). Because all the field lines close upon themselves and can shrink to a point, the topological charge is zero, indicating that the two monopoles annihilate in pairs. The latter example indicates that the charge of the monopole changes its sign as it moves around the half-quantum vortex along path 2. This example illustrates how the coalescence of topological objects is influenced by the presence of a different type of topological excitation.

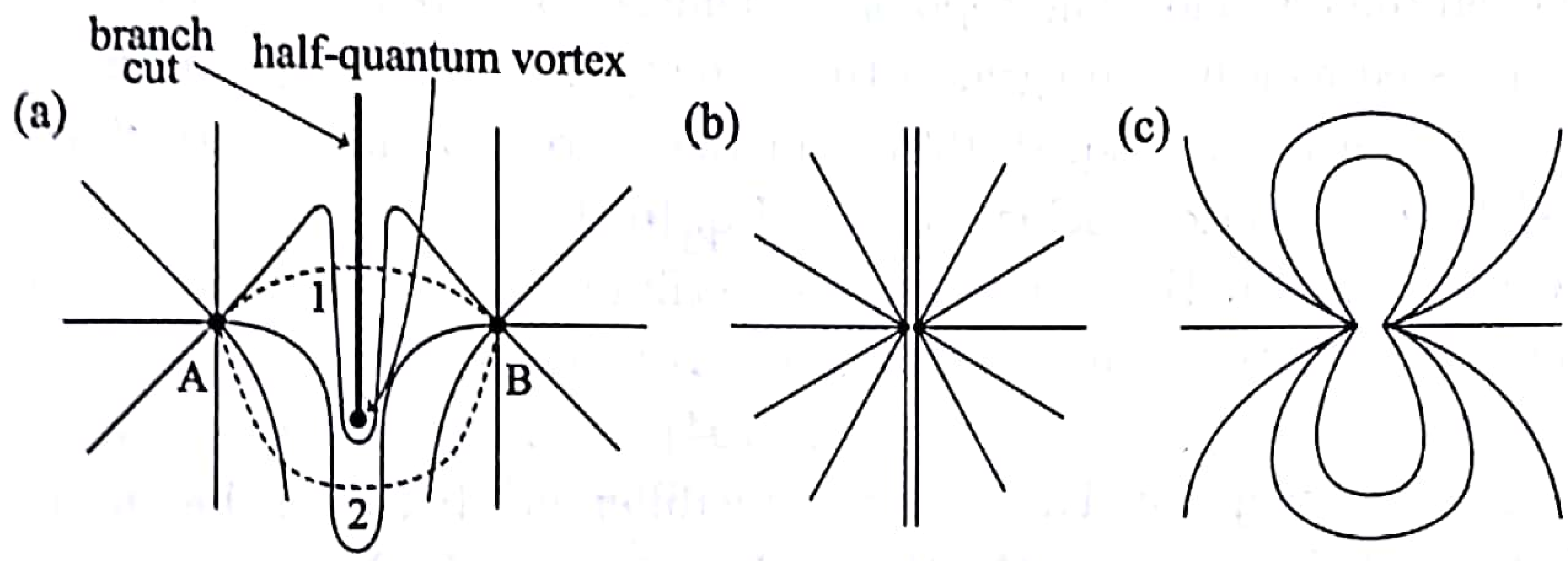

Fig. 12.11 (a) Coalescence of two monopoles with charge one in the presence of a half-quantum vortex. (b) If they merge along path 1, the topological charge of the combined object is two. (c) If they merge along path 2, the topological charge of the combined object is zero because the field lines can continuously shrink to a uniform configuration.

12.3.5.2 *Alice ring*

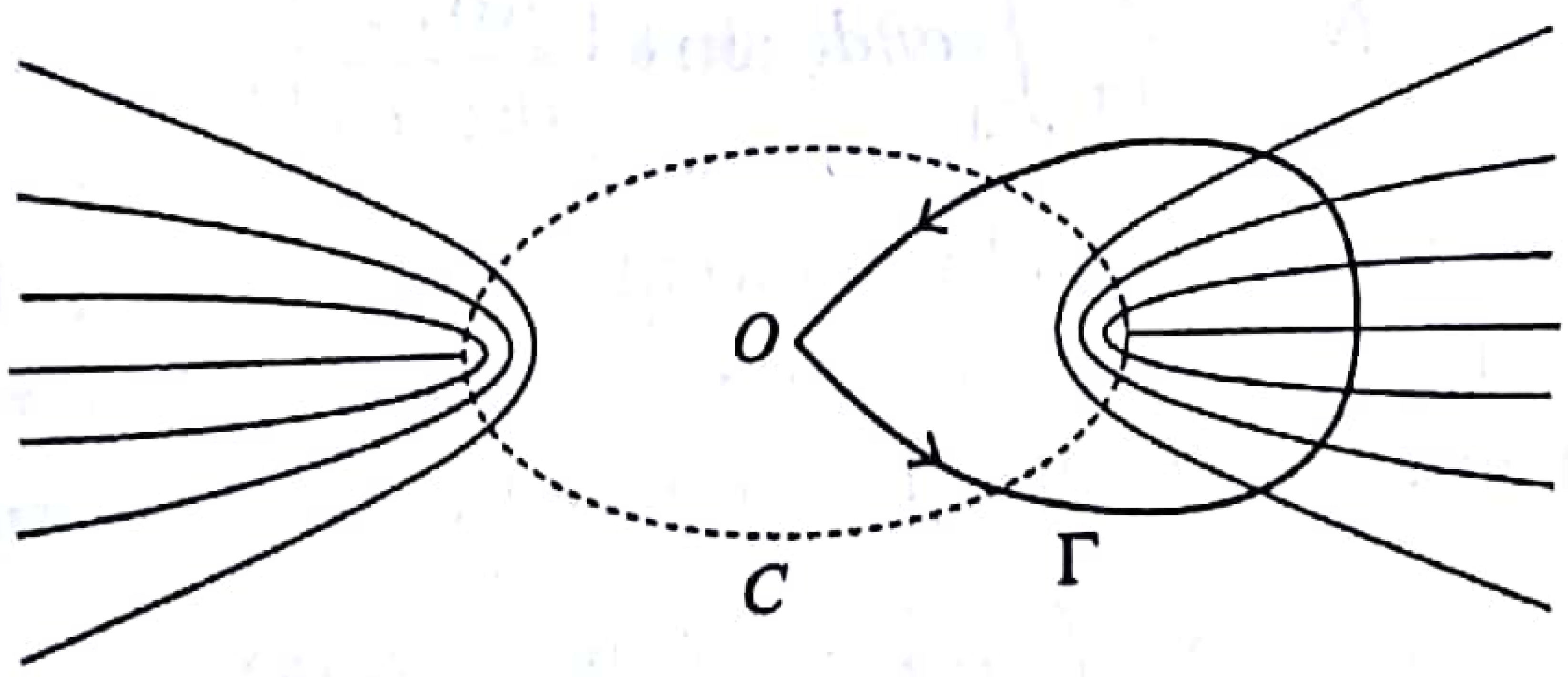

Fig. 12.12 Alice ring comprised of continuously distributed half-quantum vortices along a contour C. Far from the origin, it looks like a monopole.

The Alice ring is a combined object of two defects characterized by π_1 and π_2 [see Fig. 12.12]. Far from the origin it appears to be a point defect ($\pi_2 = 1$); however, along the ring C on which the order parameter is singular, it appears to be a continuous distribution of line defects with $\pi_1 = 1$. This composite topological object is called an Alice ring. The Alice ring was predicted to be realized in an optically trapped spin-1 ^{23}Na BEC [Ruostekoski and Anglin (2003)].

12.3.6 *Topological charges*

12.3.6.1 *Line defects: π_1*

The topological charge can be expressed algebraically as a function of the order-parameter field. The topological charge corresponding to $\pi_1(S^1)$ can be expressed as a line integral of the velocity field. The superfluid current density $\mathbf{j}$ is given by Eq. (6.100), and the corresponding expression for a spin-1 ferromagnetic BEC is given by Eq. (6.101).

Unlike a scalar BEC, where the superfluid velocity is irrotational, the superfluid velocity for the ferromagnetic BEC has nonvanishing $\nabla \times \mathbf{v}_s^F$ due to the Berry phase, as shown in Eq. (6.104). As a consequence, the circulation alone is not quantized; however, the difference between the circulation and the contribution from the Berry phase is quantized.

12.3.6.2 *Point defects: π_2*

The topological charge N_2 of a point defect can be calculated as follows. By definition, N_2 gives the number of times the unit vector $\mathbf{m}(\mathbf{r})$ representing the order parameter wraps S^2 [see Fig. 12.7]. Expressing the components of this vector as $m_x = \sin\alpha\cos\beta$, $m_y = \sin\alpha\sin\beta$, and $m_z = \cos\alpha$, N_2 is given by

$$N_2 = \frac{1}{4\pi}\int_\Sigma d\theta d\phi \sin\theta \left| \frac{\partial(\alpha,\beta)}{\partial(\theta,\phi)} \right|, \tag{12.64}$$

where θ and ϕ are the polar and azimuthal angles of $\mathbf{r}$ and the last term on the right-hand side is the Jacobian of the transformation of the coordinates. The right-hand side can be directly expressed in terms of $\mathbf{m}$ as

$$N_2 = \frac{1}{4\pi}\int_\Sigma d\theta d\phi\, \mathbf{m}\cdot\left(\frac{\partial \mathbf{m}}{\partial\theta} \times \frac{\partial \mathbf{m}}{\partial\phi}\right). \tag{12.65}$$

It follows from this that the topological charge of the hedgehog is $N_2 = 1$.

The topological charge N_2 corresponding to $\pi_2(S^2)$ can be expressed as a surface integral of a vector quantity $\mathbf{j}$:

$$N_2 = \frac{1}{4\pi} \int \mathbf{j} \cdot d\mathbf{S}, \tag{12.66}$$

where

$$\mathbf{j} = \frac{1}{2}\epsilon_{ijk} m_i (\nabla m_j \times \nabla m_k). \tag{12.67}$$

We can use Gauss' law to rewrite the right-hand side of Eq. (12.66) as a volume integral:

$$N_2 = \int \rho_{\mathrm{m}} d\mathbf{r}, \tag{12.68}$$

where

$$\rho_{\mathrm{m}}(\mathbf{r}) = \frac{1}{4\pi} \nabla \cdot \mathbf{j}(\mathbf{r}) \tag{12.69}$$

gives the density distribution of point singularities.

12.3.6.3 *Skyrmions:* π_3

The topological charge of the third homotopy group can be introduced in a manner similar to that of the second homotopy group. We consider a continuous map $U(\mathbf{r}) : \mathbf{R}^3 \to SU(2) \cong S^3$ that satisfies the boundary condition $U(\mathbf{r}) \to \hat{1}$ (identity map) as $|\mathbf{r}| \to \infty$. Then, $\mathbf{R}^3$ is compactified to S^3 and $U(\mathbf{r})$ describes mapping $S^3 \to S^3$. The topological charge of the third homotopy group can be calculated by using the Hopf map in which the $SU(2)$ matrix U is expressed in terms of two complex numbers z_1 and z_2 as

$$U(\mathbf{r}) = \begin{pmatrix} z_2 & -z_1^* \\ z_1 & z_2 \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1, \tag{12.70}$$

where z_1 and z_2 are related to $\mathbf{r}$ as follows. We first use $\mathbf{r}$ and $f(r)$ in Eq. (12.59) to define a four-component unit vector as

$$\mathbf{n} = \frac{\mathbf{r}}{r} \sin\frac{f(r)n}{2}, \quad n_4 = \cos\frac{f(r)n}{2}, \tag{12.71}$$

where $\mathbf{n} = (n_1, n_2, n_3)$. Then, $z_1 \equiv n_1 + in_2$ and $z_2 \equiv n_3 + in_4$ satisfy the normalization condition $|z_1|^2 + |z_2|^2 = 1$. Introducing $Z = (z_2, z_1)^T$, the spin vector is given by

$$\mathbf{f} = Z^\dagger \sigma Z, \tag{12.72}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. It can be shown that Eq. (12.72) reproduces Eq. (12.62). The fact that a local $U(1)$ transformation $Z \to e^{i\gamma}Z$ does not change $\mathbf{f}$ implies that S^3 is a $U(1)$ fibre bundle over S^2; physically, this means that the preimage of a point in the order parameter manifold is a closed loop [see Fig. 12.9]. We can use Z to introduce an $SU(2)$-valued gauge potential

$$A_j = \frac{i}{2}(Z^\dagger \partial_j Z - (\partial_j Z^\dagger)Z) \tag{12.73}$$

and the associated field tensor

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{12.74}$$

Then, the Hopf charge is given by the integral of the Chern–Simons term over $\mathbf{R}^3$:

$$N_3 = \frac{1}{4\pi^2}\int d\mathbf{r}\epsilon_{ijk}F_{ij}A_k. \tag{12.75}$$

Expressing the right-hand side in terms of $(\mathbf{n}, n_4)$, we can rewrite the Hopf charge as the Jacobian of the transformations:

$$N_3 = \frac{1}{12\pi^2}\int d\mathbf{r}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma\delta}n_\alpha\partial_i n_\beta\partial_j n_\gamma\partial_k n_\delta, \tag{12.76}$$

where ϵ_{ijk} and $\epsilon_{\alpha\beta\gamma\delta}$ are the completely antisymmetric tensors of the third and fourth orders, respectively, and the Roman letters run over x, y, z and the Greek letters run over $1, 2, 3, 4$. Substituting Eq. (12.71) in Eq. (12.76) and performing integration, we obtain $N_3 = n$. Thus, we find that the number of times the spin rotates as it goes from the origin to spatial infinity [see Eq. (12.59)] is equal to the Hopf charge.