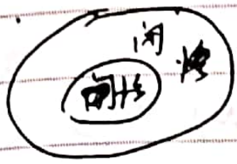


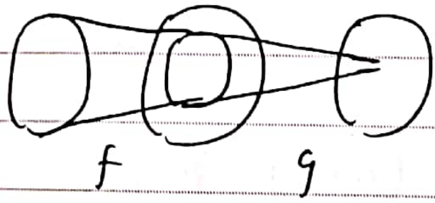
$\alpha = P\,dx + Q\,dy$

$d\alpha = 0$ 等 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

1)正合序列

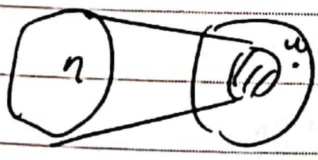
2)为什么做这个分类？

$A \xrightarrow{f} B \xrightarrow{g} C$



$\operatorname{im} f = \ker g$

$$\int_V d\omega = \langle V, d\omega\rangle = \langle \partial V, \underset{\omega + d\eta}{\underset{\downarrow}{\omega}}\rangle = \langle \partial V, \omega\rangle + \underset{0}{\underset{\parallel}{\langle \partial V, d\eta\rangle}}$$



群与同伦群

manifold

$\vec{x}^i \quad f(\vec{x}^i)$

$f: \vec{x}^i \longrightarrow \mathbb{R}$

两个空间 $\begin{cases} \text{切空间 } T_p(M) \Rightarrow \frac{\partial}{\partial x^i} \longrightarrow \text{Lie代数} \\ \text{余切.. } T_p^*(M) \Rightarrow dx^i \longrightarrow \text{Caculus} \end{cases}$

余切 ⇒ $\begin{cases} \text{1元微分} & \text{线} & \oint \vec{F}\cdot d\vec{l} & & \text{1 form} \in \Omega^1 \\ \text{2元 - :} & \text{面} & \oint \vec{B}\cdot d\vec{S} & \oint \vec{E}\cdot d\vec{S} & \text{2 form} \in \Omega^2 \\ \text{3元 —} & \text{体} & \oint f\,dx\,dy\,dz & \oint f\,dV & \text{3 form} \in \Omega^3 \end{cases}$

K正合序列

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

d: 本质上是 linear operator

Stokes定理

$$\oint_V d\omega = \oint_{\partial V} \omega = \oint d(\omega + d\eta) \in \mathbb{Z}$$

$$\oint \vec{E}\cdot d\vec{s} = Q_+ - Q_- \in \mathbb{Z}$$

推广

eg. $\oint \frac{1}{2\pi i} \mathrm{Tr}(U^{-1} dU) \xleftarrow{\text{推广}} \frac{1}{2\pi i}\oint \frac{dz}{z}$

$\oint \mathrm{Tr}(U^{-1}dU \wedge U^{-1}dU) \leftarrow$

$\downarrow$

$\oint \mathrm{Tr}[(U^{-1}dU)^n]$

理论: $f: X \to Y$

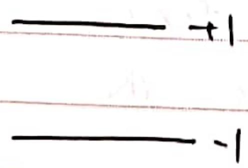
$\pi(X, Y) \simeq \mathbb{Z}_2$ $\quad$ π: 对f分类

$\pi(X, Y) \simeq \mathbb{Z}$

Homotopy group

$\left\{\begin{array}{l}\text{Topo Insulator} \\ \text{Topo SC}\end{array}\right. \longrightarrow$ Lie 代数

Fermi surface



+1

-1

$H = U^{+} \Lambda U$

$\downarrow$

$\tilde{H} = U^{+}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} U$

$\tilde{H}^{+}\tilde{H} = 1 \longrightarrow$ Lie gp

群论 $\left\{\begin{array}{l}\text{表示论} \quad g: \mapsto T(g) \\ \text{同态、同构基本定理}\end{array}\right.$

群定义：$(G, *)$

$G=\{g_i\}$ g_i 群元

1) 单元/恒等元 e $g_i * e = g_i$

2) 逆元

$$\begin{cases} g_i * g_j = e \\ g_j = g_i^{-1} \end{cases}$$

3) 封闭性 $g_i \in G$ $g_j \in G \Rightarrow g_i * g_j \in G$

4) 结合律 $(g_i * g_j) * g_k = g_i * (g_j * g_k)$

例 $(\mathbb{R}, +)$

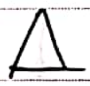
$(\mathbb{Z}, +)$

U矩阵 $(n\times n)$ = $U(n)$群 $(U(n), \cdot)$

$\mathbb{Q}(\sqrt{2}) = \{a+\sqrt{2}b \mid a, b \in \mathbb{Q}\}$

$(a+\sqrt{2}b)(a'+\sqrt{2}b') = (aa'+2bb') + \sqrt{2}(ab'+ba')$

对称操作 C_{4v}



对称操作 C_{3v}

群 乘法表

1) $\mathbb{Z}_2 = \{1, -1\}$ $\mathbb{Z}_4 = \{1, -1, i, -i\}$

$\mathbb{Z}_2 = \{1, \sigma_x\}$

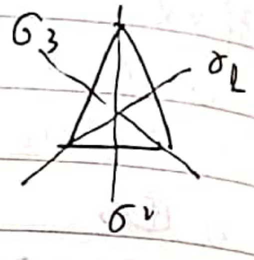
	1	-1
1	1	-1
-1	-1	1

	1	σ_x
1	1	σ_x
σ_x	σ_x	1

每一行、每一列 无相同元素 $\Rightarrow$ 重排定理

C_{3v}

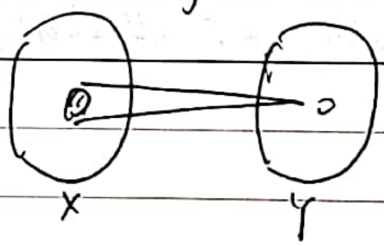
	1	C_3	C_3^2	σ_1	σ_2	σ_3
1	1	C_3	C_3^2	σ_1	σ_2	σ_3
C_3	C_3	C_3^2	1	σ_2	σ_3	σ_1
C_3^2	C_3^2	1	C_3	σ_3	σ_1	σ_2
σ_1	σ_1	σ_2	σ_3	1	C_3^2	C_3
σ_2	σ_2	σ_3	σ_1	C_3	1	C_3^2
σ_3	σ_3	σ_1	σ_2	C_3^2	C_3	1

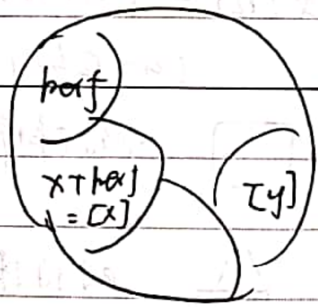


子群 $H \subseteq G$

子群 → 陪集

Linear algebra ⇒ 小空间 + 等价类





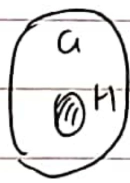
$[x] = \{x + x' \mid x' \in \ker f\}$

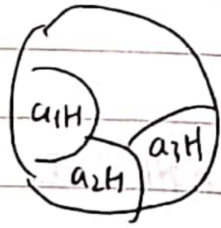
$f(x+x') = f(x)$

因为 $f(x') = 0$

若 $[x] \neq [y]$，则 $f(x) \neq f(y)$

陪集 → 等价类 → 群分类





$H \to \ker(f)$

G 按 H 分类

. Lagrange 定理.

~~阶~~ $|H|/|G|$

左陪集 Left coset aH

右陪集 Right coset Ha

$G = \bigcup (a_i H) = \bigcup (H a_i)$

特殊：左=右 正规子群 $H \triangleleft G$, $G = A \times B$ $\Longrightarrow$ 商群 $G/H = \{[e],[a_1],\dots\}$

$\Pi(X,Y) = G/\sim$ = 非等价的 $X \to Y$ mapping

群 → 子群 → 陪集 → 商群 → 分类

↳ 表示

$g_i H = H g_i$

$g_j g_i H = g_j H g_i = H g_j g_i$

若H的能谱为

— — — —

— — —

— $g_i \psi_4 = e^{i\theta_i}\psi_4$

— — —

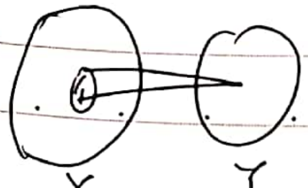
— — ← 2d 表示

— ← 1d 表示

$g_i \psi_{nq} = \sum_k C^{iq}_{nk} \psi_{nk}$ 导致了空间中态的混合

群表示的物理意义：简并度 ⟷ 矩阵的维数

群表示：同态/同构



X Y

$X/\ker(f) \simeq \mathrm{Im}(f) \simeq Y$

↳ 等价类

$\dim \ker f + \dim Y = \dim X$

$\ker(f) \xhookrightarrow{\text{嵌入}} X \xrightarrow{\text{投影/同态}} Y \simeq \mathrm{im}(f)$

群表示 $g_i \to T(g_i)$ 同态：保运算，不保结构

$T(g_i)T(g_j) = T(g_i g_j)$

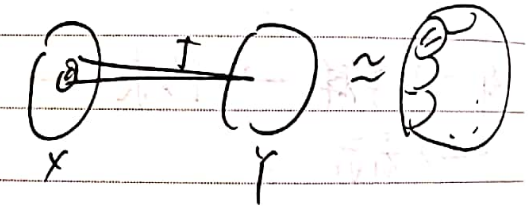
e.g. $T(g_i) = I$

$H \hookrightarrow G \to G/H$

总结:

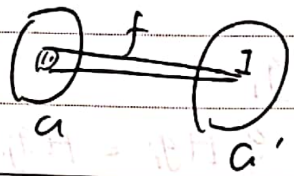
Linear algebra $f: X \to Y$

$X/\ker f \simeq \mathrm{im} f \simeq Y$.



群 $f: G \to G'$ 为群同态

$G/\ker f \simeq \mathrm{im} f \simeq G'$



同伦群以及在物理中的应用

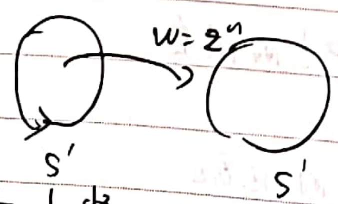
Ref. 1)(重点) Ueda 书 chap. 12

2) Nakahara 书 chap. 4

3) 补充 Soft matter physics Chap. 12 软物质 (Liquid crystal)

目的: 求/区分 $X \to Y$ 不等价的 mapping 的个数

e.g. $S^1 \to S^1$



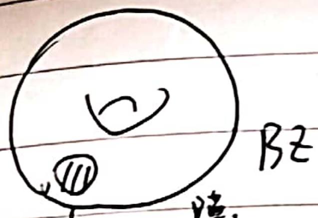
$\frac{1}{2\pi i}\oint \frac{dz}{z} = 1$ $\qquad \frac{1}{2\pi i}\oint \frac{dw}{w} = n\frac{1}{2\pi i}\oint\frac{dz}{z} = n$

$S^1 \to S^1$ 个数 $n \in \mathbb{Z}$

古典: $G/H = \pi(X,Y)$

mapping $\to G$

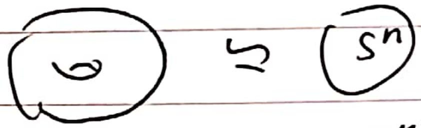
不能直接计算(微扰合)



↳ 能隙关闭/打开的区间

TB $-2t\cos k \simeq -2t\left[1-\frac{k^2}{2}\right] = tk^2-2t$

one-point compactification



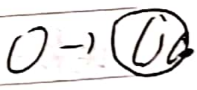
因此 $\pi(X,Y) \longrightarrow \underline{S^n \to Y}$

记作 $\pi_n(Y)$

同伦群, 群元 = mapping

群的四元素

1) e : constant mapping $c(x) = y_0 \in Y$



2) 逆 : $\varphi^{-1}(s) = \varphi(1-s)$

3) 封闭 : $\varphi_1 * \varphi_2(s) = \begin{cases} \varphi_1(2s) & 0 \le s \le 1/2 \\ \varphi_2(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$

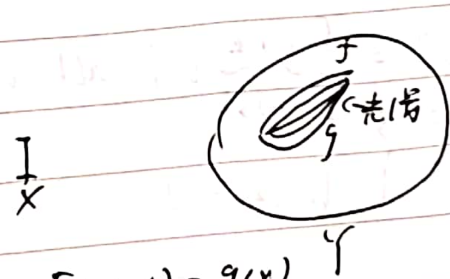
4) 结合律

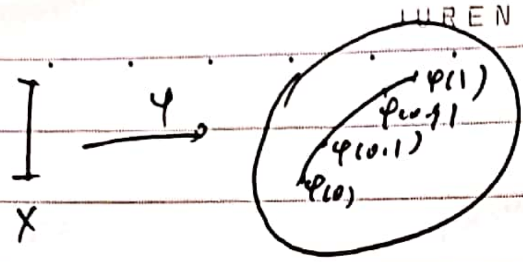
什么叫同伦?

$f: X \to Y$

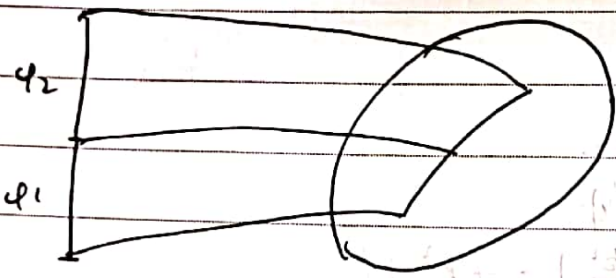
$g: X \to Y$

$\exists\ F(x,t)$ $\quad F(x,0) = f(x),\ F(x,1) = g(x)$

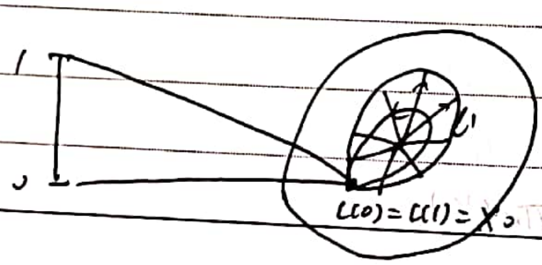




$\varphi^{-1}(0.1)=\varphi(1-0.1)=\varphi(0.9)$



path



x_0: base point 基点

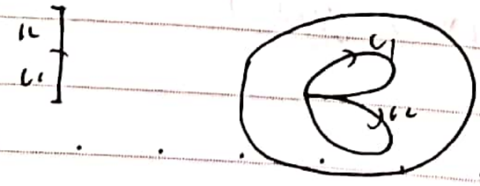
$\exists F(x,t)$ 光滑 $F(x,0)=c(x)$ $F(x,1)=c'(x)$

注意: Topo insulator: $X\to S^n$
$Y\to$ Lie algebra 动量空间

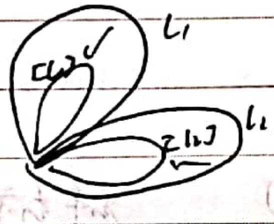
Topo defect: $X\to\mathbb{R}^n$
$Y\to S^m$ 实空间

定义 $[c]=\{c'\simeq c \mid \text{all } c'\}$

$$c_1\cdot c_2=\begin{cases}c_1(2s)\\ c_2(2s-1)\end{cases}$$



$[l_1 \cdot l_2] = [l_1] \cdot [l_2]$



$(l_1 \cdot l_2) \cdot l_3 = l_1 \cdot (l_2 \cdot l_3)$

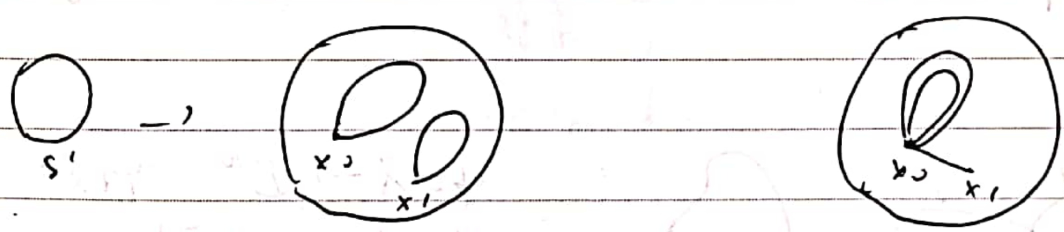
$[(l_1 \cdot l_2) \cdot l_3] = ([l_1] \cdot [l_2]) \cdot [l_3]$

逆元 $[l \cdot l^{-1}] = [c]$ $\quad l \cdot l^{-1} \neq c$

$S^1 \to Y$

$\pi_1(Y)$

$\pi_1(Y, x_0) \quad \pi_1(Y, x_1)$



$\pi_1(Y, x_0) \simeq \pi_1(Y, x_1)$

$\downarrow$

$q \cdot l \cdot q^{-1}$

1) $\pi_1(S^1) \simeq \mathbb{Z}$

2) $\pi_1(S^2) = 0$

3) $\pi_n(X \times Y) = \pi_n(X) \times \pi_n(Y)$

下节课：Topological defect $\pi_n(G/H)$