4

HOMOTOPY GROUPS

The idea of homology groups in the previous chapter was to assign a group structure to cycles that are not boundaries. In homotopy groups, however, we are interested in continuous deformation of maps one to another. Let X and Y be topological spaces and let $\mathcal{F}$ be the set of continuous maps, from X to Y. We introduce an equivalence relation, called 'homotopic to', in $\mathcal{F}$ by which two maps $f, g \in \mathcal{F}$ are identified if the image $f(X)$ is continuously deformed to $g(X)$ in Y. We choose X to be some *standard* topological spaces whose structures are well known. For example, we may take the n-sphere S^n as the standard space and study all the maps from S^n to Y to see how these maps are classified according to homotopic equivalence. This is the basic idea of homotopy groups.

We will restrict ourselves to an elementary study of homotopy groups, which is sufficient for the later discussion. Nash and Sen (1983) and Croom (1978) complement this chapter.

4.1 Fundamental groups

4.1.1 Basic ideas

Let us look at figure 4.1. One disc has a hole in it, the other does not. What characterizes the difference between these two discs? We note that any loop in figure 4.1(b) can be continuously shrunk to a point. In contrast, the loop α in figure 4.1(a) cannot be shrunk to a point due to the existence of a hole in it. Some loops in figure 4.1(a) may be shrunk to a point while others cannot. We say a loop α is homotopic to β if α can be obtained from β by a *continuous* deformation. For example, any loop in Y is homotopic to a point. It turns out that 'homotopic to' is an equivalence relation, the equivalence class of which is called the homotopy class. In figure 4.1, there is only one homotopy class associated with Y. In X, each homotopy class is characterized by $n \in \mathbb{Z}$, n being the number of times the loop encircles the hole; $n < 0$ if it winds clockwise, $n > 0$ if counterclockwise, $n = 0$ if the loop does not wind round the hole. Moreover, $\mathbb{Z}$ is an additive group and the group operation (addition) has a geometrical meaning; $n + m$ corresponds to going round the hole first n times and then m times. The set of homotopy classes is endowed with a group structure called the fundamental group.

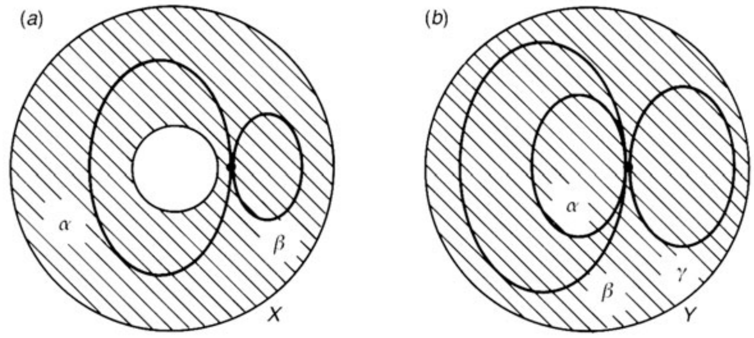


Figure 4.1. A disc with a hole (a) and without a hole (b). The hole in (a) prevents the loop α from shrinking to a point.

4.1.2 Paths and loops

Definition 4.1. Let X be a topological space and let $I = [0, 1]$. A continuous map $\alpha : I \to X$ is called a **path** with an initial point x_0 and an end point x_1 if $\alpha(0) = x_0$ and $\alpha(1) = x_1$. If $\alpha(0) = \alpha(1) = x_0$, the path is called a **loop** with **base point** x_0 (or a loop at x_0).

For $x \in X$, a **constant path** $c_x : I \to X$ is defined by $c_x(s) = x$, $s \in I$. A constant path is also a constant loop since $c_x(0) = c_x(1) = x$. The set of paths or loops in a topological space X may be endowed with an algebraic structure as follows.

Definition 4.2. Let $\alpha, \beta : I \to X$ be paths such that $\alpha(1) = \beta(0)$. The product of α and β, denoted by $\alpha * \beta$, is a path in X defined by

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \le s \le 1 \end{cases} \tag{4.1}$$

see figure 4.2. Since $\alpha(1) = \beta(0)$, $\alpha * \beta$ is a continuous map from I to X. [Geometrically, $\alpha * \beta$ corresponds to traversing the image $\alpha(I)$, in the first half, then followed by $\beta(I)$ in the remaining half. Note that the velocity is doubled.]

Definition 4.3. Let $\alpha : I \to X$ be a path from x_0 to x_1. The inverse path α^{-1} of α is defined by

$$\alpha^{-1}(s) \equiv \alpha(1-s) \qquad s \in I. \tag{4.2}$$

[The inverse path α^{-1} corresponds to traversing the image of α in the opposite direction from x_1 to x_0.]

Since a loop is a special path for which the initial point and end point agree, the product of loops and the inverse of a loop are defined in exactly the same way.

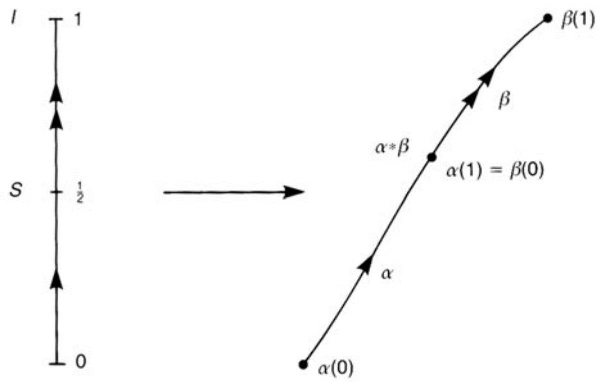


Figure 4.2. The product $\alpha * \beta$ of paths α and β with a common end point.

It seems that a constant map c_x is the unit element. However, it is not: $\alpha * \alpha^{-1}$ is not equal to c_x! We need a concept of homotopy to define a group operation in the space of loops.

4.1.3 Homotopy

The algebraic structure of loops introduced earlier is not so useful as it is. For example, the constant path is not exactly the unit element. We want to classify the paths and loops according to a neat equivalence relation so that the equivalence classes admit a group structure. It turns out that if we identify paths or loops that can be deformed continuously one into another, the equivalence classes form a group. Since we are primarily interested in loops, most definitions and theorems are given for loops. However, it should be kept in mind that many statements are also applied to paths with proper modifications.

Definition 4.4. Let $\alpha, \beta : I \to X$ be loops at x_0. They are said to be **homotopic**, written as $\alpha \sim \beta$, if there exists a continuous map $F : I \times I \to X$ such that

$$\begin{gathered} F(s, 0) = \alpha(s), \qquad F(s, 1) = \beta(s) \qquad \forall s \in I \\ F(0, t) = F(1, t) = x_0 \qquad \forall t \in I. \end{gathered} \tag{4.3}$$

The connecting map F is called a **homotopy** between α and β.

It is helpful to represent a homotopy as figure 4.3(a). The vertical edges of the square $I \times I$ are mapped to x_0. The lower edge is $\alpha(s)$ while the upper edge is $\beta(s)$. In the space X, the image is continuously deformed as in figure 4.3(b).

Proposition 4.1. The relation $\alpha \sim \beta$ is an equivalence relation.

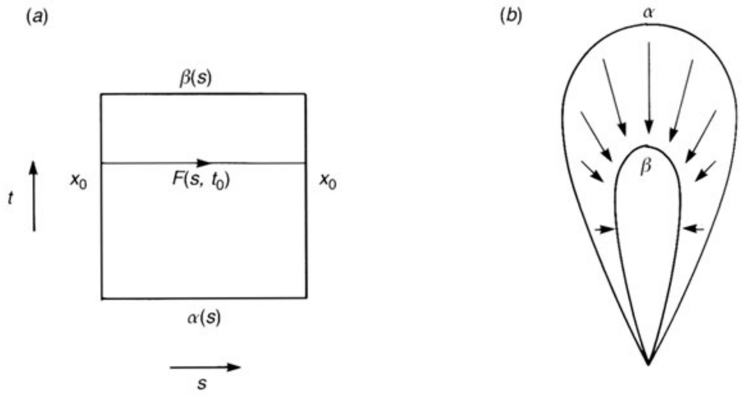


Figure 4.3. (*a*) The square represents a homotopy F interpolating the loops α and β. (*b*) The image of α is continuously deformed to the image of β in real space X.

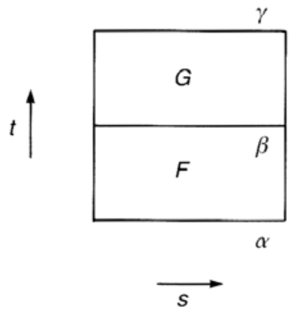


Figure 4.4. A homotopy H between α and γ via β.

Proof. Reflectivity: $\alpha \sim \alpha$. The homotopy may be given by $F(s,t) = \alpha(s)$ for any $t \in I$.

Symmetry: Let $\alpha \sim \beta$ with the homotopy $F(s,t)$ such that $F(s,0) = \alpha(s)$, $F(s,1) = \beta(s)$. Then $\beta \sim \alpha$, where the homotopy is given by $F(s, 1-t)$.

Transitivity: Let $\alpha \sim \beta$ and $\beta \sim \gamma$. Then $\alpha \sim \gamma$. If $F(s,t)$ is a homotopy between α and β and $G(s,t)$ is a homotopy between β and γ, a homotopy between α and γ may be (figure 4.4)

$$H(s,t) = \begin{cases} F(s,2t) & 0 \le t \le \frac{1}{2} \\ G(s, 2t-1) & \frac{1}{2} \le t \le 1. \end{cases} \qquad \square$$

4.1.4 Fundamental groups

The equivalence class of loops is denoted by $[\alpha]$ and is called the **homotopy class** of α. The product between loops naturally defines the product in the set of homotopy classes of loops.

Definition 4.5. Let X be a topological space. The set of homotopy classes of loops at $x_0 \in X$ is denoted by $\pi_1(X, x_0)$ and is called the **fundamental group** (or the **first homotopy group**) of X at x_0. The product of homotopy classes $[\alpha]$ and $[\beta]$ is defined by

$$[\alpha] * [\beta] = [\alpha * \beta]. \tag{4.4}$$

Lemma 4.1. The product of homotopy classes is independent of the representative, that is, if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha * \beta \sim \alpha' * \beta'$.

Proof. Let $F(s, t)$ be a homotopy between α and α' and $G(s, t)$ be a homotopy between β and β'. Then

$$H(s, t) = \begin{cases} F(2s, t) & 0 \le s \le \frac{1}{2} \\ G(2s - 1, t) & \frac{1}{2} \le s \le 1 \end{cases}$$

is a homotopy between $\alpha * \beta$ and $\alpha' * \beta'$, hence $\alpha * \beta \sim \alpha' * \beta'$ and $[\alpha] * [\beta]$ is well defined. □

Theorem 4.1. The fundamental group is a group. Namely, if $\alpha, \beta, \ldots$ are loops at $x \in X$, the following group properties are satisfied:

(1) $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$
(2) $[\alpha] * [c_x] = [\alpha]$ and $[c_x] * [\alpha] = [\alpha]$ (unit element)
(3) $[\alpha] * [\alpha^{-1}] = [c_x]$, hence $[\alpha]^{-1} = [\alpha^{-1}]$ (inverse).

Proof. (1) Let $F(s, t)$ be a homotopy between $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$. It may be given by (figure 4.5(a))

$$F(s, t) = \begin{cases} \alpha\left(\dfrac{4s}{1+t}\right) & 0 \le s \le \dfrac{1+t}{4} \\ \beta(4s - t - 1) & \dfrac{1+t}{4} \le s \le \dfrac{2+t}{4} \\ \gamma\left(\dfrac{4s - t - 2}{2 - t}\right) & \dfrac{2+t}{4} \le s \le 1. \end{cases}$$

Thus, we may simply write $[\alpha * \beta * \gamma]$ to denote $[(\alpha * \beta) * \gamma]$ or $[\alpha * (\beta * \gamma)]$.

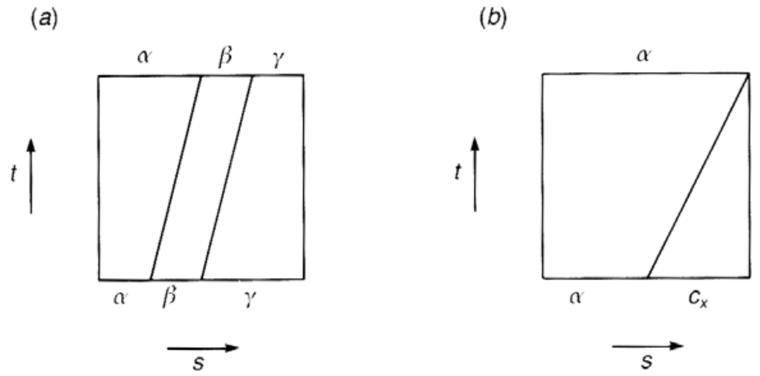


Figure 4.5. (*a*) A homotopy between $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$. (*b*) A homotopy between $\alpha * c_x$ and α.

(2) Define a homotopy $F(s, t)$ by (figure 4.5(*b*))

$$F(s,t) = \begin{cases} \alpha\left(\dfrac{2s}{1+t}\right) & 0 \le s \le \dfrac{t+1}{2} \\ x & \dfrac{t+1}{2} \le s \le 1. \end{cases}$$

Clearly this is a homotopy between $\alpha * c_x$ and α. Similarly, a homotopy between $c_x * \alpha$ and α is given by

$$F(s,t) = \begin{cases} x & 0 \le s \le \dfrac{1-t}{2} \\ \alpha\left(\dfrac{2s-1+t}{1+t}\right) & \dfrac{1-t}{2} \le s \le 1. \end{cases}$$

This shows that $[\alpha] * [c_x] = [\alpha] = [c_x] * [\alpha]$.

(3) Define a map $F : I \times I \to X$ by

$$F(s,t) = \begin{cases} \alpha(2s(1-t)) & 0 \le s \le \frac{1}{2} \\ \alpha(2(1-s)(1-t)) & \frac{1}{2} \le s \le 1. \end{cases}$$

Clearly $F(s, 0) = \alpha * \alpha^{-1}$ and $F(s, 1) = c_x$, hence

$$[\alpha * \alpha^{-1}] = [\alpha] * [\alpha^{-1}] = [c_x].$$

This shows that $[\alpha^{-1}] = [\alpha]^{-1}$. □

In summary, $\pi_1(X, x)$ is a group whose unit element is the homotopy class of the constant loop c_x. The product $[\alpha] * [\beta]$ is well defined and satisfies the

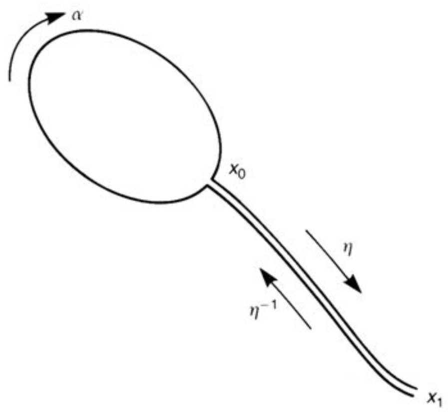


Figure 4.6. From a loop α at x_0, a loop $\eta^{-1} * \alpha * \eta$ at x_1 is constructed.

group axioms. The inverse of $[\alpha]$ is $[\alpha]^{-1} = [\alpha^{-1}]$. In the next section we study the general properties of fundamental groups, which simplify the actual computations.

4.2 General properties of fundamental groups

4.2.1 Arcwise connectedness and fundamental groups

In section 2.3 we defined a topological space X to be arcwise connected if, for any $x_0, x_1 \in X$, there exists a path α such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$.

Theorem 4.2. Let X be an arcwise connected topological space and let $x_0, x_1 \in X$. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Proof. Let $\eta : I \to X$ be a path such that $\eta(0) = x_0$ and $\eta(1) = x_1$. If α is a loop at x_0, then $\eta^{-1} * \alpha * \eta$ is a loop at x_1 (figure 4.6). Given an element $[\alpha] \in \pi_1(X, x_0)$, this correspondence induces a unique element $[\alpha'] = [\eta^{-1} * \alpha * \eta] \in \pi_1(X, x_1)$. We denote this map by $P_\eta : \pi_1(X, x_0) \to \pi_1(X, x_1)$ so that $[\alpha'] = P_\eta([\alpha])$.

We show that P_η is an isomorphism. First, P_η is a *homomorphism*, since for $[\alpha], [\beta] \in \pi_1(X, x_0)$, we have

$$\begin{aligned} P_\eta([\alpha] * [\beta]) &= [\eta^{-1}] * [\alpha] * [\beta] * [\eta] \\ &= [\eta^{-1}] * [\alpha] * [\eta] * [\eta^{-1}] * [\beta] * [\eta] \\ &= P_\eta([\alpha]) * P_\eta([\beta]). \end{aligned}$$

To show that P_η is *bijective*, we introduce the inverse of P_η. Define a map $P_\eta^{-1} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ whose action on $[\alpha']$ is $P_\eta^{-1}([\alpha']) = [\eta * \alpha * \eta^{-1}]$.

Clearly P^{-1} is the inverse of P_η since

$$P_\eta^{-1} \circ P_\eta([\alpha]) = P_\eta^{-1}([\eta^{-1} * \alpha * \eta]) = [\eta * \eta^{-1} * \alpha * \eta * \eta^{-1}] = [\alpha].$$

Thus, $P_\eta^{-1} \circ P_\eta = \mathrm{id}_{\pi_1(X,x_0)}$. From the symmetry, we have $P_\eta \circ P_\eta^{-1} = \mathrm{id}_{\pi_1(X,x_1)}$. We find from exercise 2.3 that P_η is one to one and onto. □

Accordingly, if X is arcwise connected, we do not need to specify the base point since $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any $x_0, x_1 \in X$, and we may simply write $\pi_1(X)$.

Exercise 4.1. (1) Let η and ζ be paths from x_0 to x_1, such that $\eta \sim \zeta$. Show that $P_\eta = P_\zeta$.

(2) Let η and ζ be paths such that $\eta(1) = \zeta(0)$. Show that $P_{\eta*\zeta} = P_\zeta \circ P_\eta$.

4.2.2 Homotopic invariance of fundamental groups

The homotopic equivalence of paths and loops is easily generalized to arbitrary maps. Let $f, g : X \to Y$ be continuous maps. If there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, f is said to be **homotopic** to g, denoted by $f \sim g$. The map F is called a **homotopy** between f and g.

Definition 4.6. Let X and Y be topological spaces. X and Y are of the same **homotopy type**, written as $X \simeq Y$, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. The map f is called the **homotopy equivalence** and g, its **homotopy inverse**. [*Remark*: If X is homeomorphic to Y, X and Y are of the same homotopy type but the converse is not necessarily true. For example, a point $\{p\}$ and the real line $\mathbb{R}$ are of the same homotopy type but $\{p\}$ is not homeomorphic to $\mathbb{R}$.]

Proposition 4.2. 'Of the same homotopy type' is an equivalence relation in the set of topological spaces.

Proof. Reflectivity: $X \simeq X$ where id_X is a homotopy equivalence. *Symmetry:* Let $X \simeq Y$ with the homotopy equivalence $f : X \to Y$. Then $Y \simeq X$, the homotopy equivalence being the homotopy inverse of f. *Transitivity:* Let $X \simeq Y$ and $Y \simeq Z$. Suppose $f : X \to Y$, $g : Y \to Z$ are homotopy equivalences and $f' : Y \to X$, $g' : Z \to Y$, their homotopy inverses. Then

$$(g \circ f)(f' \circ g') = g(f \circ f')g' \sim g \circ \mathrm{id}_Y \circ g' = g \circ g' \sim \mathrm{id}_Z$$
$$(f' \circ g')(g \circ f) = f'(g' \circ g)f \sim f' \circ \mathrm{id}_Y \circ f = f' \circ f \sim \mathrm{id}_X$$

from which it follows $X \simeq Z$. □

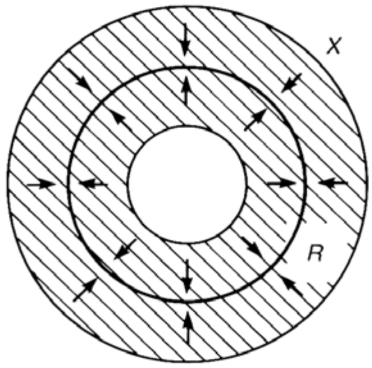


Figure 4.7. The circle R is a retract of the annulus X. The arrows depict the action of the retraction.

One of the most remarkable properties of the fundamental groups is that two topological spaces of the same homotopy type have the same fundamental group.

Theorem 4.3. Let X and Y be topological spaces of the same homotopy type. If $f : X \to Y$ is a homotopy equivalence, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, f(x_0))$.

The following corollary follows directly from theorem 4.3.

Corollary 4.1. A fundamental group is invariant under homeomorphisms, and hence is a topological invariant.

In this sense, we must admit that fundamental groups classify topological spaces in a less strict manner than homeomorphisms. What we claim at most is that if topological spaces X and Y have different fundamental groups, X cannot be homeomorphic to Y. Note, however, that the homotopy groups including the fundamental groups have many applications to physics as we shall see in due course. We should stress that the main usage of the homotopy groups in physics is not to classify spaces but to classify maps or field configurations.

It is rather difficult to appreciate what is meant by 'of the same homotopy type' for an arbitrary pair of X and Y. In practice, however, it often happens that Y is a subspace of X. We then claim that $X \simeq Y$ if Y is obtained by a continuous deformation of X.

Definition 4.7. Let R ($\neq\emptyset$) be a subspace of X. If there exists a continuous map $f : X \to R$ such that $f|_R = \mathrm{id}_R$, R is called a **retract** of X and f a **retraction**.

Note that the whole of X is mapped onto R keeping points in R fixed. Figure 4.7 is an example of a retract and retraction.

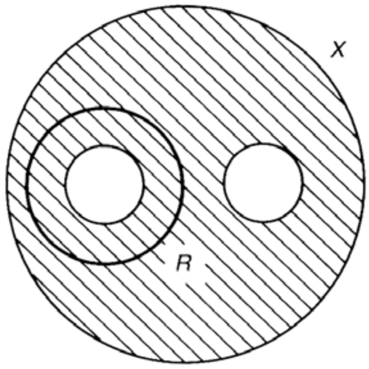


Figure 4.8. The circle R is not a deformation retract of X.

Definition 4.8. Let R be a subspace of X. If there exists a continuous map $H : X \times I \to X$ such that

$$H(x, 0) = x \qquad H(x, 1) \in R \qquad \text{for any } x \in X \tag{4.5}$$

$$H(x, t) = x \qquad \text{for any } x \in R \text{ and any } t \in I. \tag{4.6}$$

The space R is said to be a **deformation retract** of X. Note that H is a homotopy between id_X and a retraction $f : X \to R$, which leaves all the points in R fixed during deformation.

A retract is not necessarily a deformation retract. In figure 4.8, the circle R is a retract of X but not a deformation retract, since the hole in X is an obstruction to continuous deformation of id_X to the retraction. Since X and R are of the same homotopy type, we have

$$\pi_1(X, a) \cong \pi_1(R, a) \qquad a \in R. \tag{4.7}$$

Example 4.1. Let X be the unit circle and Y be the annulus,

$$X = \{\mathrm{e}^{\mathrm{i}\theta} | 0 \le \theta < 2\pi\} \tag{4.8}$$

$$Y = \{r\mathrm{e}^{\mathrm{i}\theta} | 0 \le \theta < 2\pi, \tfrac{1}{2} \le r \le \tfrac{2}{3}\} \tag{4.9}$$

see figure 4.7. Define $f : X \hookrightarrow Y$ by $f(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta}$ and $g : Y \to X$ by $g(r\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta}$. Then $f \circ g : r\mathrm{e}^{\mathrm{i}\theta} \mapsto \mathrm{e}^{\mathrm{i}\theta}$ and $g \circ f : \mathrm{e}^{\mathrm{i}\theta} \mapsto \mathrm{e}^{\mathrm{i}\theta}$. Observe that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. There exists a homotopy

$$H(r\mathrm{e}^{\mathrm{i}\theta}, t) = \{1 + (r - 1)(1 - t)\}\mathrm{e}^{\mathrm{i}\theta}$$

which interpolates between id_X and $f \circ g$, keeping the points on X fixed. Hence, X is a deformation retract of Y. As for the fundamental groups we have $\pi_1(X, a) \cong \pi_1(Y, a)$ where $a \in X$.

Definition 4.9. If a point $a \in X$ is a deformation retract of X, X is said to be **contractible**.

Let $c_a : X \to \{a\}$ be a constant map. If X is contractible, there exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = c_a(x) = a$ and $H(x, 1) = \mathrm{id}_X(x) = x$ for any $x \in X$ and, moreover, $H(a, t) = a$ for any $t \in I$. The homotopy H is called the **contraction**.

Example 4.2. $X = \mathbb{R}^n$ is contractible to the origin 0. In fact, if we define $H : \mathbb{R}^n \times I \to \mathbb{R}$ by $H(x, t) = tx$, we have (i) $H(x, 0) = 0$ and $H(x, 1) = x$ for any $x \in X$ and (ii) $H(0, 1) = 0$ for any $t \in I$. Now it is clear that any convex subset of $\mathbb{R}^n$ is contractible.

Exercise 4.2. Let $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. Show that the unit circle S^1 is a deformation retract of $D^2 - \{0\}$. Show also that the unit sphere S^n is a deformation retract of $D^{n+1} - \{0\}$, where $D^{n+1} = \{x \in \mathbb{R}^{n+1} | |x| \leq 1\}$.

Theorem 4.4. The fundamental group of a contractible space X is trivial, $\pi_1(X, x_0) \cong \{e\}$. In particular, the fundamental group of $\mathbb{R}^n$ is trivial, $\pi_1(\mathbb{R}^n, x_0) \cong \{e\}$.

Proof. A contractible space has the same fundamental group as a point $\{p\}$ and a point has a trivial fundamental group. □

If an arcwise connected space X has a trivial fundamental group, X is said to be **simply connected**, see section 2.3.

4.3 Examples of fundamental groups

There does not exist a routine procedure to compute the fundamental groups, in general. However, in certain cases, they are obtained by relatively simple considerations. Here we look at the fundamental groups of the circle S^1 and related spaces.

Let us express S^1 as $\{z \in \mathbb{C} | |z| = 1\}$. Define a map $p : \mathbb{R} \to S^1$ by $p : x \mapsto \exp(ix)$. Under p, the point $0 \in \mathbb{R}$ is mapped to $1 \in S^1$, which is taken to be the base point. We imagine that $\mathbb{R}$ wraps around S^1 under p, see figure 4.9. If $x, y \in \mathbb{R}$ satisfies $x - y = 2\pi m (m \in \mathbb{Z})$, they are mapped to the same point in S^1. Then we write $x \sim y$. This is an equivalence relation and the equivalence class $[x] = \{y | x - y = 2\pi m$ for some $m \in \mathbb{Z}\}$ is identified with a point $\exp(ix) \in S^1$. It then follows that $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a continuous map such that $\tilde{f}(0) = 0$ and $\tilde{f}(x + 2\pi) \sim \tilde{f}(x)$. It is obvious that $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2n\pi$ for any $x \in \mathbb{R}$, where n is a fixed integer. If $x \sim y$ $(x - y = 2\pi m)$, we have

$$\begin{aligned}\widetilde{f}(x) - \widetilde{f}(y) &= \widetilde{f}(y + 2\pi m) - \widetilde{f}(y) \\ &= \widetilde{f}(y) + 2\pi mn - \widetilde{f}(y) = 2\pi mn\end{aligned}$$

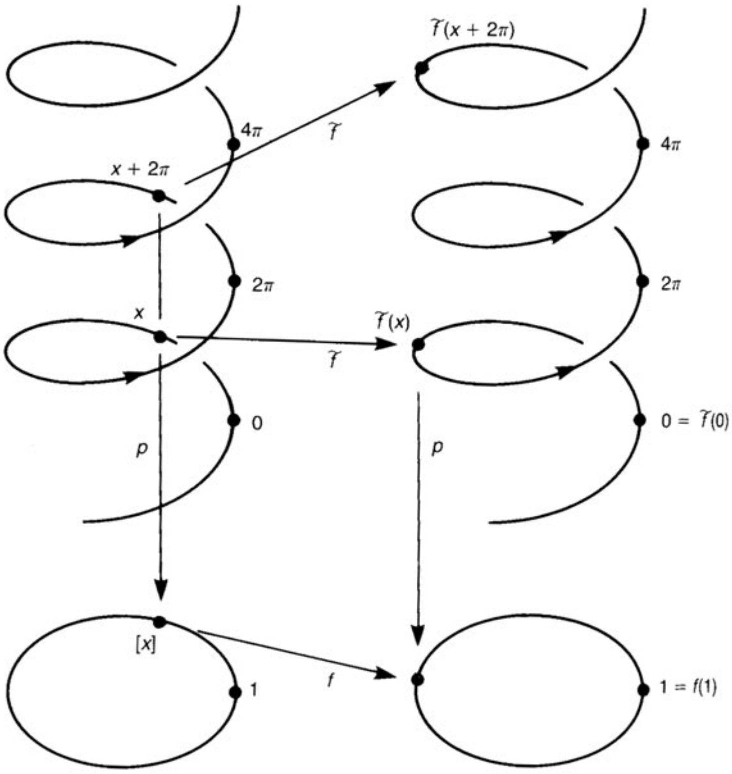


Figure 4.9. The map $p : \mathbb{R} \to S^1$ defined by $x \mapsto \exp(\mathrm{i}x)$ projects $x + 2m\pi$ to the same point on S^1, while $\tilde{f} : \mathbb{R} \to \mathbb{R}$, such that $\tilde{f}(0) = 0$ and $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2n\pi$ for fixed n, defines a map $f : S^1 \to S^1$. The integer n specifies the homotopy class to which f belongs.

hence $\tilde{f}(x) \sim \tilde{f}(y)$. Accordingly, $\tilde{f} : \mathbb{R} \to \mathbb{R}$ uniquely defines a continuous map $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ by $f([x]) = p \circ \tilde{f}(x)$, see figure 4.9. Note that f keeps the base point $1 \in S^1$ fixed. Conversely, given a map $f : S^1 \to S^1$, which leaves $1 \in S^1$ fixed, we may define a map $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\tilde{f}(0) = 0$ and $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi n$.

In summary, there is a one-to-one correspondence between the set of maps from S^1 to S^1 with $f(1) = 1$ and the set of maps from $\mathbb{R}$ to $\mathbb{R}$ such that $\tilde{f}(0) = 0$ and $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi n$. The integer n is called the **degree** of f and is denoted by $\deg(f)$. While x encircles S^1 once, $f(x)$ encircles S^1 n times.

Lemma 4.2. (1) Let $f, g : S^1 \to S^1$ such that $f(1) = g(1) = 1$. Then $\deg(f) = \deg(g)$ if and only if f is homotopic to g.

(2) For any $n \in \mathbb{Z}$, there exists a map $f : S^1 \to S^1$ such that $\deg(f) = n$.

Proof. (1) Let $\deg(f) = \deg(g)$ and $\tilde{f}, \tilde{g} : \mathbb{R} \to \mathbb{R}$ be the corresponding maps. Then $\tilde{F}(x, t) \equiv t\tilde{f}(x) + (1 - t)\tilde{g}(x)$ is a homotopy between $\tilde{f}(x)$ and $\tilde{g}(x)$. It is easy to verify that $F \equiv p \circ \tilde{F}$ is a homotopy between f and g. Conversely, if $f \sim g : S^1 \to S^1$, there exists a homotopy $F : S^1 \times I \to S^1$ such that $F(1, t) = 1$ for any $t \in I$. The corresponding homotopy $\tilde{F} : \mathbb{R} \times I \to \mathbb{R}$ between $\tilde{f}$ and $\tilde{g}$ satisfies $\tilde{F}(x + 2\pi, t) = \tilde{F}(x, t) + 2n\pi$ for some $n \in \mathbb{Z}$. Thus, $\deg(f) = \deg(g)$.

(2) $\tilde{f} : x \mapsto nx$ induces a map $f : S^1 \to S^1$ with $\deg(f) = n$. □

Lemma 4.2 tells us that by assigning an integer $\deg(f)$ to a map $f : S^1 \to S^1$ such that $f(1) = 1$, there is a bijection between $\pi_1(S^1, 1)$ and $\mathbb{Z}$. Moreover, this is an isomorphism. In fact, for $f, g : S^1 \to S^1$, $f * g$, defined as a product of loops, satisfies $\deg(f * g) = \deg(f) + \deg(g)$. [Let $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi n$ and $\tilde{g}(x + 2\pi) = \tilde{g}(x) + 2\pi m$. Then $f * g(x + 2\pi) = f * g(x) + 2\pi(m + n)$. Note that $*$ is not a composite of maps but a product of paths.] We have finally proved the following theorem.

Theorem 4.5. The fundamental group of S^1 is isomorphic to $\mathbb{Z}$,

$$\pi_1(S^1) \cong \mathbb{Z}. \tag{4.10}$$

[Since S^1 is arcwise connected, we may drop the base point.]

Although the proof of the theorem is not too obvious, the statement itself is easily understood even by children. Suppose we encircle a cylinder with an elastic band. If it encircles the cylinder n times, the configuration cannot be continuously deformed into that with m ($\neq n$) encirclements. If an elastic band encircles a cylinder first n times and then m times, it encircles the cylinder $n + m$ times in total.

4.3.1 Fundamental group of torus

Theorem 4.6. Let X and Y be arcwise connected topological spaces. Then $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \oplus \pi_1(Y, y_0)$.

Proof. Define projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. If α is a loop in $X \times Y$ at (x_0, y_0), $\alpha_1 \equiv p_1(\alpha)$ is a loop in X at x_0, and $\alpha_2 \equiv p_2(\alpha)$ is a loop in Y at y_0. Conversely, any pair of loops α_1 of X at x_0 and α_2 of Y at y_0 determines a unique loop $\alpha = (\alpha_1, \alpha_2)$ of $X \times Y$ at (x_0, y_0). Define a homomorphism $\varphi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ by

$$\varphi([\alpha]) = ([\alpha_1], [\alpha_2]).$$

By construction φ has an inverse, hence it is the required isomorphism and $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$. □

Example 4.3. (1) Let $T^2 = S^1 \times S^1$ be a torus. Then

$$\pi_1(T^2) \cong \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{4.11}$$

Similarly, for the n-dimensional torus

$$T^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$$

we have

$$\pi_1(T^n) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n}. \tag{4.12}$$

(2) Let $X = S^1 \times \mathbb{R}$ be a cylinder. Since $\pi_1(\mathbb{R}) \cong \{e\}$, we have

$$\pi_1(X) \cong \mathbb{Z} \oplus \{e\} \cong \mathbb{Z}. \tag{4.13}$$

4.4 Fundamental groups of polyhedra

The computation of fundamental groups in the previous section was, in a sense, *ad hoc* and we certainly need a more systematic way of computing the fundamental groups. Fortunately if a space X is triangulable, we can compute the fundamental group of the polyhedron K, and hence that of X by a routine procedure. Let us start with some aspects of group theories.

4.4.1 Free groups and relations

The free groups that we define here are not necessarily Abelian and we employ multiplicative notation for the group operation. A subset $X = \{x_j\}$ of a group G is called a **free set of generators** of G if any element $g \in G - \{e\}$ is *uniquely* written as

$$g = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \tag{4.14}$$

where n is finite and $i_k \in \mathbb{Z}$. We assume no adjacent x_j are equal; $x_j \neq x_{j+1}$. If $i_j = 1$, ${x_j}^1$ is simply written as x_j. If $i_j = 0$, the term ${x_j}^0$ should be dropped from g. For example, $g = a^3b^{-2}cb^3$ is acceptable but $h = a^3a^{-2}cb^0$ is not. If each element is to be written uniquely, h must be reduced to $h = ac$. If G has a free set of generators, it is called a **free group**.

Conversely, given a set X, we can construct a free group G whose free set of generators is X. Let us call each element of X a **letter**. The product

$$w = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \tag{4.15}$$

is called a **word**, where $x_j \in X$ and $i_j \in \mathbb{Z}$. If $i_j \neq 0$ and $x_j \neq x_{j+1}$ the word is called a **reduced word**. It is always possible to reduce a word by finite steps. For example,

$$a^{-2}b^{-3}b^3a^4b^3c^{-2}c^4 = a^{-2}b^0a^4b^3c^2 = a^2b^3c^2.$$

A word with no letters is called an **empty word** and denoted by 1. For example, it is obtained by reducing $w = a^0$.

A product of words is defined by simply juxtaposing two words. Note that a juxtaposition of reduced words is not necessarily reduced but it is always possible to reduce it. For example, if $v = a^2c^{-3}b^2$ and $w = b^{-2}c^2b^3$, the product vw is reduced as

$$vw = a^2c^{-3}b^2b^{-2}c^2b^3 = a^2c^{-3}c^2b^3 = a^2c^{-1}b^3.$$

Thus, the set of all reduced words form a well-defined free group called the free group generated by X, denoted by $F[X]$. The multiplication is the juxtaposition of two words followed by reduction, the unit element is the empty word and the inverse of

$$w = x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$$

is

$$w^{-1} = x_n^{-i_n}\cdots x_2^{-i_2}x_1^{-i_1}.$$

Exercise 4.3. Let $X = \{a\}$. Show that the free group generated by X is isomorphic to $\mathbb{Z}$.

In general, an arbitrary group G is specified by the generators and certain constraints that these must satisfy. If $\{x_k\}$ is the set of generators, the constraints are most commonly written as

$$r = x_{k_1}^{i_1}x_{k_2}^{i_2}\cdots x_{k_n}^{i_n} = 1 \tag{4.16}$$

and are called **relations**. For example, the cyclic group of order n generated by x (in multiplicative notation) satisfies a relation $x^n = 1$.

More formally, let G be a group which is generated by $X = \{x_k\}$. Any element $g \in G$ is written as $g = x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$, where we do not require that the expression be unique (G is not necessarily free). For example, we have $x^i = x^{n+1}$ in $\mathbb{Z}$. Let $F[X]$ be the free group generated by X. Then there is a natural homomorphism φ from $F[X]$ onto G defined by

$$x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \xrightarrow{\varphi} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \in G. \tag{4.17}$$

Note that this is not an isomorphism since the LHS is not unique. φ is onto since X generates both $F[X]$ and G. Although $F[X]$ is not isomorphic to G, $F[X]/\ker\varphi$ is (see theorem 3.1),

$$F[X]/\ker\varphi \cong G. \tag{4.18}$$

In this sense, the set of generators X and $\ker\varphi$ completely determine the group G. [$\ker\varphi$ is a normal subgroup. Lemma 3.1 claims that $\ker\varphi$ is a subgroup of $F[X]$. Let $r \in \ker\varphi$, that is, $r \in F[X]$ and $\varphi(r) = 1$. For any element $x \in F[X]$, we have $\varphi(x^{-1}rx) = \varphi(x^{-1})\varphi(r)\varphi(x) = \varphi(x)^{-1}\varphi(r)\varphi(x) = 1$, hence $x^{-1}rx \in \ker\varphi$.]

In this way, a group G generated by X is specified by the relations. The juxtaposition of generators and relations

$$(x_1, \ldots, x_p; r_1, \ldots, r_q) \tag{4.19}$$

is called the **presentation** of G. For example, $\mathbb{Z}_n = (x; x^n)$ and $\mathbb{Z} = (x; \emptyset)$.

Example 4.4. Let $\mathbb{Z} \oplus \mathbb{Z} = \{x^n y^m | n, m \in \mathbb{Z}\}$ be a free Abelian group generated by $X = \{x, y\}$. Then we have $xy = yx$. Since $xyx^{-1}y^{-1} = 1$, we have a relation $r = xyx^{-1}y^{-1}$. The presentation of $\mathbb{Z} \oplus \mathbb{Z}$ is $(x, y : xyx^{-1}y^{-1})$.

4.4.2 Calculating fundamental groups of polyhedra

We shall be sketchy here to avoid getting into the technical details. We shall follow Armstrong (1983); the interested reader should consult this book or any textbook on algebraic topology. As noted in the previous chapter, a polyhedron $|K|$ is a nice approximation of a given topological space X within a homeomorphism. Since fundamental groups are topological invariants, we have $\pi_1(X) = \pi_1(|K|)$. We assume X is an arcwise connected space and drop the base point. Accordingly, if we have a systematic way of computing $\pi_1(|K|)$, we can also find $\pi_1(X)$.

We first define the edge group of a simplicial complex, which corresponds to the fundamental group of a topological space, then introduce a convenient way of computing it. Let $f : |K| \to X$ be a triangulation of a topological space X. If we note that an element of the fundamental group of X can be represented by loops in X, we expect that similar loops must exist in $|K|$ as well. Since any loop in $|K|$ is made up of 1-simplexes, we look at the set of all 1-simplexes in $|K|$, which can be endowed with a group structure called the edge group of K.

An **edge path** in a simplicial complex K is a sequence $v_0 v_1 \ldots v_k$ of vertices of $|K|$, in which the consecutive pair $v_i v_{i+1}$ is a 0- or 1-simplex of $|K|$. [For technical reasons, we allow the possibility $v_i = v_{i+1}$, in which case the relevant simplex is a 0-simplex $v_i = v_{i+1}$.] If $v_0 = v_k$ $(=v)$, the edge path is called an **edge loop** at v. We classify these loops into equivalence classes according to some equivalence relation. We define two edge loops α and β to be equivalent if one is obtained from the other by repeating the following operations a finite number of times.

(1) If the vertices u, v and w span a 2-simplex in K, the edge path uvw may be replaced by uw and *vice versa*; see figure 4.10(a).

(2) As a special case, if $u = w$ in (1), the edge path uvw corresponds to traversing along uv first then reversing backwards from v to $w = u$. This edge path uvu may be replaced by a 0-simplex u and *vice versa*, see figure 4.10(b).

Let us denote the equivalence class of edge loops at v, to which $vv_1 \ldots v_{k-1}v$ belongs, by $\{vv_1 \ldots v_{k-1}v\}$. The set of equivalence classes forms a group under the product operation defined by

$$\{vu_1 \ldots u_{k-1}v\} * \{vv_1 \ldots v_{i-1}v\} = \{vu_1 \ldots u_{k-1}vv_1 \ldots v_{i-1}v\}. \tag{4.20}$$

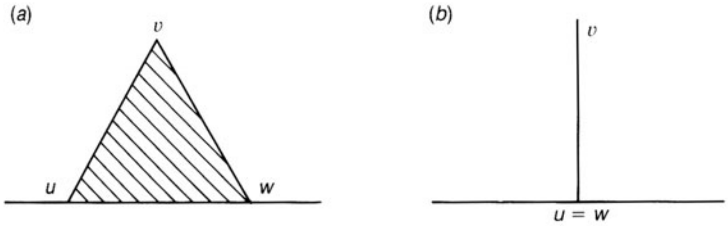


Figure 4.10. Possible deformations of the edge loops. In (*a*), uvw is replaced by uw. In (*b*), uvu is replaced by u.

The unit element is an equivalence class $\{v\}$ while the inverse of $\{vv_1 \dots v_{k-1}v\}$ is $\{vv_{k-1} \dots v_1 v\}$. This group is called the **edge group** of K at v and denoted by $E(K; v)$.

Theorem 4.7. $E(K; v)$ is isomorphic to $\pi_1(|K|; v)$.

The proof is found in Armstrong (1983), for example. This isomorphism $\varphi : E(K; v) \to \pi_1(|K|; v)$ is given by identifying an edge loop in K with a loop in $|K|$. To find $E(K; v)$, we need to read off the generators and relations. Let L be a simplicial subcomplex of K, such that

(a) L contains *all the vertices* (0-simplexes) of K;
(b) the polyhedron $|L|$ is *arcwise connected* and *simply connected*.

Given an arcwise-connected simplicial complex K, there always exists a subcomplex L that satisfies these conditions. A one-dimensional simplicial complex that is arcwise connected and simply connected is called a **tree**. A tree T_{M} is called the **maximal tree** of K if it is not a proper subset of other trees.

Lemma 4.3. A maximal tree T_{M} contains all the vertices of K and hence satisfies conditions (a) and (b) above.

Proof. Suppose T_{M} does not contain some vertex w. Since K is arcwise connected, there is a 1-simplex vw in K such that $v \in T_{\text{M}}$ and $w \notin T_{\text{M}}$. $T_{\text{M}} \cup \{vw\} \cup \{w\}$ is a one-dimensional subcomplex of K which is arcwise connected, simply connected and contains T_{M}, which contradicts the assumption. □

Suppose we have somehow obtained the subcomplex L. Since $|L|$ is simply connected, the edge loops in $|L|$ do not contribute to $E(K; v)$. Thus, we can effectively ignore the simplexes in L in our calculations. Let v_0 $(=v)$, $v_1, \dots, v_n$ be the vertices of K. Assign an 'object' g_{ij} for each ordered pair of vertices v_i, v_j if $\langle v_i v_j \rangle$ is a 1-simplex of K. Let $G(K; L)$ be a group that is generated by all g_{ij}. What about the relations? We have the following.

(1) Since we ignore those simplexes in L, we assign $g_{ij} = 1$ if $\langle v_i v_j \rangle \in L$.
(2) If $\langle v_i v_j v_k \rangle$ is a 2-simplex of K, there are no non-trivial loops around $v_i v_j v_k$ and we have the relation $g_{ij} g_{jk} g_{ki} = 1$.

The generators $\{g_{ij}\}$ and the set of relations completely determine the group $G(K; L)$.

Theorem 4.8. $G(K; L)$ is isomorphic to $E(K; v) \simeq \pi_1(|K|; v)$.

In fact, we can be more efficient than is apparent. For example, g_{ii} should be set equal to 1 since g_{ii} corresponds to the vertex v_i which is an element of L. Moreover, from $g_{ij} g_{ji} = g_{ii} = 1$, we have $g_{ij} = g_{ji}^{-1}$. Therefore, we only need to introduce those generators g_{ij} for each pair of vertices v_i, v_j such that $\langle v_i v_j \rangle \in K - L$ and $i < j$. Since there are no generators g_{ij} such that $\langle v_i v_j \rangle \in L$, we can ignore the first type of relation. If $\langle v_i v_j v_k \rangle$ is a 2-simplex of $K - L$ such that $i < j < k$, the corresponding relation is *uniquely* given by $g_{ij} g_{jk} = g_{ik}$ since we are only concerned with simplexes $\langle v_i v_j \rangle$ such that $i < j$.

To summarize, the rules of the game are as follows.

(1) First, find a triangulation $f : |K| \to X$.
(2) Find the subcomplex L that is arcwise connected, simply connected and contains all the vertices of K.
(3) Assign a generator g_{ij} to each 1-simplex $\langle v_i v_j \rangle$ of $K - L$, for which $i < j$.
(4) Impose a relation $g_{ij} g_{jk} = g_{ik}$ if there is a 2-simplex $\langle v_i v_j v_k \rangle$ such that $i < j < k$. If two of the vertices v_i, v_j and v_k form a 1-simplex of L, the corresponding generator should be set equal to 1.
(5) Now $\pi_1(X)$ is isomorphic to $G(K; L)$ which is a group generated by $\{g_{ij}\}$ with the relations obtained in (4).

Let us work out several examples.

Example 4.5. From our construction, it should be clear that $E(K; v)$ and $G(K; L)$ involve only the 0-, 1- and 2-simplexes of K. Accordingly, if $K^{(2)}$ denotes a **2-skeleton** of K, which is defined to be the set of all 0-, 1- and 2-simplexes in K, we should have

$$\pi_1(|K|) \cong \pi_1(|K^{(2)}|). \tag{4.21}$$

This is quite useful in actual computations. For example, a 3-simplex and its boundary have the same 2-skeleton. A 3-simplex is a polyhedron $|K|$ of the solid ball D^3, while its boundary $|L|$ is a polyhedron of the sphere S^2. Since D^3 is contractible, $\pi_1(|K|) \cong \{e\}$. From (4.21) we find $\pi_1(S^2) \cong \pi_1(|K|) \cong \{e\}$. In general, for $n \geq 2$, the $(n+1)$-simplex σ_{n+1} and the boundary of σ_{n+1} have the same 2-skeleton. If we note that σ_{n+1} is contractible and the boundary of σ_{n+1} is a polyhedron of S^n, we find the formula

$$\pi_1(S^n) \cong \{e\} \qquad n \geq 2. \tag{4.22}$$

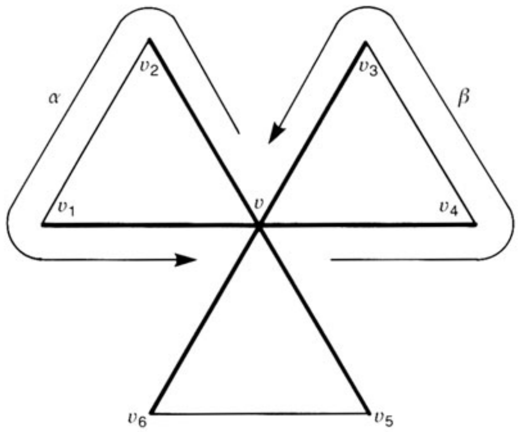


Figure 4.11. A triangulation of a 3-bouquet. The bold lines denote the maximal tree L.

Example 4.6. Let $K \equiv \{v_1, v_2, v_3, \langle v_1 v_2\rangle, \langle v_1 v_3\rangle, \langle v_2 v_3\rangle\}$ be a simplicial complex of a circle S^1. We take v_1 as the base point. A maximal tree may be $L = \{v_1, v_2, v_3, \langle v_1 v_2\rangle, \langle v_1 v_3\rangle\}$. There is only one generator g_{23}. Since there are no 2-simplexes in K, the relation is empty. Hence,

$$\pi_1(S^1) \cong G(K; L) = (g_{23}; \emptyset) \cong \mathbb{Z} \tag{4.23}$$

in agreement with theorem 4.5.

Example 4.7. An ***n*-bouquet** is defined by the one-point union of n circles. For example, figure 4.11 is a triangulation of a 3-bouquet. Take the common point v as the base point. The bold lines in figure 4.11 form a maximal tree L. The generators of $G(K; L)$ are g_{12}, g_{34} and g_{56}. There are no relations and we find

$$\pi_1(\text{3-bouquet}) = G(K; L) = (x, y, z; \emptyset). \tag{4.24}$$

Note that this is a free group but not free *Abelian*. The non-commutativity can be shown as follows. Consider loops α and β at v encircling different holes. Obviously the product $\alpha * \beta * \alpha^{-1}$ cannot be continuously deformed into β, hence $[\alpha] * [\beta] * [\alpha]^{-1} \neq [\beta]$, or

$$[\alpha] * [\beta] \neq [\beta] * [\alpha]. \tag{4.25}$$

In general, an n-bouquet has n generators $g_{12}, \ldots, g_{2n-1\,2n}$ and the fundamental group is isomorphic to the free group with n generators with no relations.

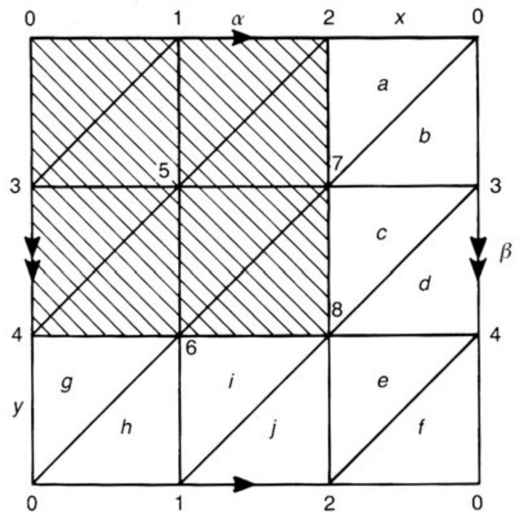


Figure 4.12. A triangulation of the torus.

Example 4.8. Let D^2 be a two-dimensional disc. A triangulation K of D^2 is given by a triangle with its interior included. Clearly K itself may be L and $K - L$ is empty. Thus, we find $\pi_1(K) \cong \{e\}$.

Example 4.9. Figure 4.12 is a triangulation of the torus T^2. The shaded area is chosen to be the subcomplex L. [Verify that it contains all the vertices and is both arcwise and simply connected.] There are 11 generators with ten relations. Let us take $x = g_{02}$ and $y = g_{04}$ and write down the relations

$$\text{(a)} \quad \underset{x}{g_{02}} \; \underset{1}{g_{27}} = g_{07} \quad \rightarrow \quad g_{07} = x$$

$$\text{(b)} \quad \underset{1}{g_{03}} \; g_{37} = \underset{x}{g_{07}} \quad \rightarrow \quad g_{37} = x$$

$$\text{(c)} \quad \underset{x}{g_{37}} \; \underset{1}{g_{78}} = g_{38} \quad \rightarrow \quad g_{38} = x$$

$$\text{(d)} \quad \underset{1}{g_{34}} \; g_{48} = \underset{x}{g_{38}} \quad \rightarrow \quad g_{48} = x$$

$$\text{(e)} \quad g_{24} \; \underset{x}{g_{48}} = g_{28} \quad \rightarrow \quad g_{24}x = g_{28}$$

$$\text{(f)} \quad \underset{x}{g_{02}} \; g_{24} = \underset{y}{g_{04}} \quad \rightarrow \quad xg_{24} = y$$

(g) $$\underset{y}{g_{04}}\ \underset{1}{g_{46}} = g_{06} \to g_{06} = y$$

(h) $$\underset{1}{g_{01}}\ g_{16} = \underset{y}{g_{06}} \to g_{16} = y$$

(i) $$\underset{y}{g_{16}}\ \underset{1}{g_{68}} = g_{18} \to g_{18} = y$$

(j) $$\underset{1}{g_{12}}\ g_{28} = \underset{y}{g_{18}} \to g_{28} = y\ .$$

It follows from (e) and (f) that $x^{-1}yx = g_{28}$. We finally have

$$\begin{aligned} &g_{02} = g_{07} = g_{37} = g_{38} = g_{48} = x \\ &g_{04} = g_{06} = g_{16} = g_{18} = g_{28} = y \\ &g_{24} = x^{-1}y \end{aligned}$$

with a relation $x^{-1}yx = y$ or

$$xyx^{-1}y^{-1} = 1. \tag{4.26}$$

This shows that $G(K; L)$ is generated by two commutative generators (note $xy = yx$), hence (cf example 4.4)

$$G(K; L) = (x, y; xyx^{-1}y^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z} \tag{4.27}$$

in agreement with (4.11).

We have the following intuitive picture. Consider loops $\alpha = 0 \to 1 \to 2 \to 0$ and $\beta = 0 \to 3 \to 4 \to 0$. The loop α is identified with $x = g_{02}$ since $g_{12} = g_{01} = 1$ and β with $y = g_{04}$. They generate $\pi_1(T^2)$ since α and β are independent non-trivial loops. In terms of these, the relation is written as

$$\alpha * \beta * \alpha^{-1} * \beta^{-1} \sim c_v \tag{4.28}$$

where c_v is a constant loop at v, see figure 4.13.

More generally, let Σ_g be the torus with genus g. As we have shown in problem 2.1, Σ_g is expressed as a subset of $\mathbb{R}^2$ with proper identifications at the boundary. The fundamental group of Σ_g is generated by $2g$ loops α_i, β_i $(1 \le i \le g)$. Similarly, to (4.28), we verify that

$$\prod_{i=1}^{g} (\alpha_i * \beta_i * \alpha_i^{-1} * \beta_i^{-1}) \sim c_v \tag{4.29}$$

If we denote the generators corresponding to α_i by x_i and β_i by y_i, there is only one relation among them,

$$\prod_{i=1}^{g} (x_i y_i x_i^{-1} y_i^{-1}) = 1. \tag{4.30}$$

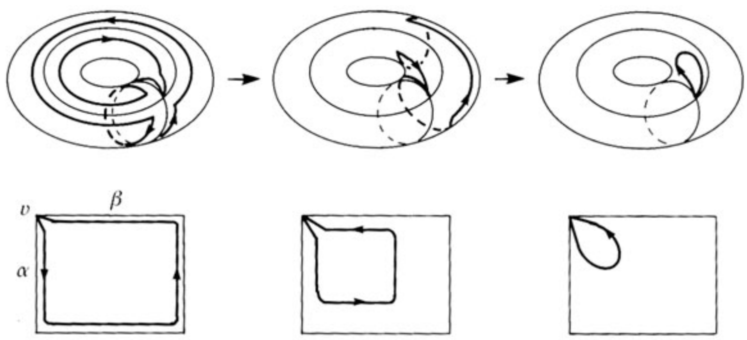


Figure 4.13. The loops α and β satisfy the relation $\alpha * \beta * \alpha^{-1} * \beta^{-1} \sim c_v$.

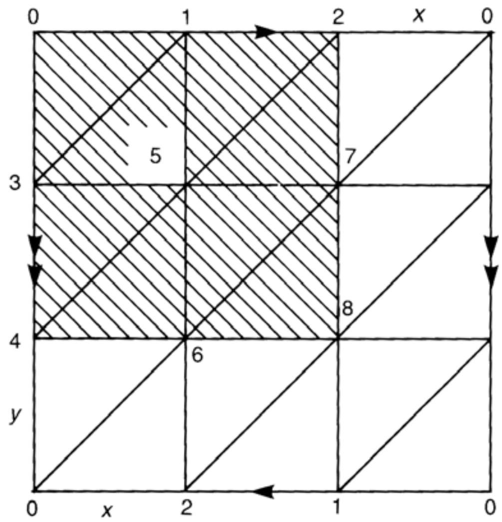


Figure 4.14. A triangulation of the Klein bottle.

Exercise 4.4. Figure 4.14 is a triangulation of the Klein bottle. The shaded area is the subcomplex L. There are 11 generators and ten relations. Take $x = g_{02}$ and $y = g_{04}$ and write down the relations for 2-simplexes to show that

$$\pi_1(\text{Klein bottle}) \cong (x, y; xyxy^{-1}). \tag{4.31}$$

Example 4.10. Figure 4.15 is a triangulation of the projective plane $\mathbb{R}P^2$. The shaded area is the subcomplex L. There are seven generators and six relations.

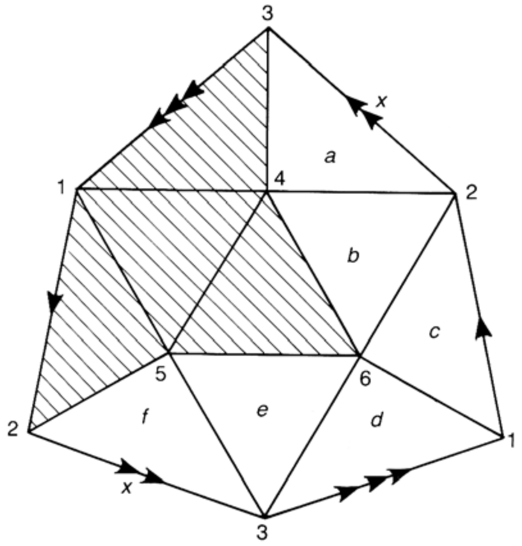


Figure 4.15. A triangulation of the projective plane.

Let us take $x = g_{23}$ and write down the relations

$$\begin{array}{llllll}
\text{(a)} & \underset{x}{g_{23}} & \underset{1}{g_{34}} & = g_{24} & \to & g_{24} = x \\
\text{(b)} & \underset{x}{g_{24}} & \underset{1}{g_{46}} & = g_{26} & \to & g_{26} = x \\
\text{(c)} & \underset{1}{g_{12}} & \underset{x}{g_{26}} & = g_{16} & \to & g_{16} = x \\
\text{(d)} & \underset{1}{g_{13}} & g_{36} & = \underset{x}{g_{16}} & \to & g_{36} = x \\
\text{(e)} & g_{35} & \underset{1}{g_{56}} & = \underset{x}{g_{36}} & \to & g_{35} = x \\
\text{(f)} & \underset{x}{g_{23}} & \underset{x}{g_{35}} & = \underset{1}{g_{25}} & \to & x^2 = 1.
\end{array}$$

Hence, we find that

$$\pi_1(\mathbb{R}P^2) \cong (x; x^2) \cong \mathbb{Z}_2. \tag{4.32}$$

Intuitively, the appearance of a cyclic group is understood as follows. Figure 4.16(a) is a schematic picture of $\mathbb{R}P^2$. Take loops α and β. It is easy to see that α is continuously deformed to a point, and hence is a trivial element of $\pi_1(\mathbb{R}P^2)$. Since diametrically opposite points are identified in $\mathbb{R}P^2$, β is actually

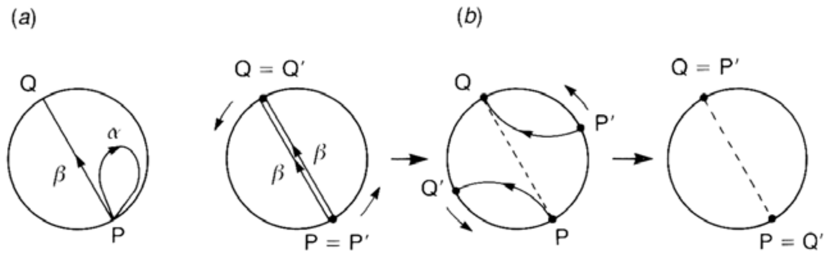


Figure 4.16. (a) α is a trivial loop while the loop β cannot be shrunk to a point. (b) $\beta * \beta$ is continuously shrunk to a point.

a closed loop. Since it cannot be shrunk to a point, it is a non-trivial element of $\pi_1(\mathbb{R}P^2)$. What about the product? $\beta * \beta$ is a loop which traverses from P to Q $\sim$ P twice. It can be read off from figure 4.16(b) that $\beta * \beta$ is continuously shrunk to a point, and thus belongs to the trivial class. This shows that the generator x, corresponding to the homotopy class of the loop β, satisfies the relation $x^2 = 1$, which verifies our result.

The same pictures can be used to show that

$$\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2 \tag{4.33}$$

where $\mathbb{R}P^3$ is identified as S^3 with diametrically opposite points identified, $\mathbb{R}P^3 = S^3/(x \sim -x)$. If we take the hemisphere of S^3 as the representative, $\mathbb{R}P^3$ can be expressed as a solid ball D^3 with diametrically opposite points on the surface identified. If the discs D^2 in figure 4.16 are interpreted as solid balls D^3, the same pictures verify (4.33).

Exercise 4.5. A triangulation of the Möbius strip is given by figure 3.8. Find the maximal tree and show that

$$\pi_1(\text{Möbius strip}) \cong \mathbb{Z}. \tag{4.34}$$

[*Note*: Of course the Möbius strip is of the same homotopy type as S^1, hence (4.34) is trivial. The reader is asked to obtain this result through routine procedures.]

4.4.3 Relations between $H_1(K)$ and $\pi_1(|K|)$

The reader might have noticed that there is a certain similarity between the first homology group $H_1(K)$ and the fundamental group $\pi_1(|K|)$. For example, the fundamental groups of many spaces (circle, disc, n-spheres, torus and many more) are identical to the corresponding first homology group. In some cases, however, they are different: $H_1(\text{2-bouquet}) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_1(\text{2-bouquet}) = (x, y : \emptyset)$, for example. Note that H_1(2-bouquet) is a free *Abelian* group while π_1(2-bouquet) is a free group. The following theorem relates $\pi_1(|K|)$ to $H_1(K)$.

Theorem 4.9. Let K be a connected simplicial complex. Then $H_1(K)$ is isomorphic to $\pi_1(|K|)/F$, where F is the commutator subgroup (see later) of $\pi_1(|K|)$.

Let G be a group whose presentation is $(x_i; r_m)$. The **commutator subgroup** F of G is a group generated by the elements of the form $x_i x_j x_i^{-1} x_j^{-1}$. Thus, G/F is a group generated by $\{x_i\}$ with the set of relations $\{r_m\}$ and $\{x_i x_j x_i^{-1} x_j^{-1}\}$. The theorem states that if $\pi_1(|K|) = (x_i : r_m)$, then $H_1(K) \cong (x_i : r_m, x_i x_j x_i^{-1} x_j^{-1})$. For example, from π_1(2-bouquet) $= (x, y : \emptyset)$, we find

$$\pi_1(\text{2-bouquet})/F \cong (x, y; xyx^{-1}y^{-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

which is isomorphic to H_1(2-bouquet).

The proof of theorem 4.9 is found in Greenberg and Harper (1981) and also outlined in Croom (1978).

Example 4.11. From π_1(Klein bottle) $\cong (x, y; xyxy^{-1})$, we have

$$\pi_1(\text{Klein bottle})/F \cong (x, y; xyxy^{-1}, xyx^{-1}y^{-1}).$$

Two relations are replaced by $x^2 = 1$ and $xyx^{-1}y^{-1} = 1$ to yield

$$\begin{aligned}\pi_1(\text{Klein bottle})/F &\cong (x, y; xyx^{-1}y^{-1}, x^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \\ &\cong H_1(\text{Klein bottle})\end{aligned}$$

where the factor $\mathbb{Z}$ is generated by y and $\mathbb{Z}_2$ by x.

Corollary 4.2. Let X be a connected topological space. Then $\pi_1(X)$ is isomorphic to $H_1(X)$ if and only if $\pi_1(X)$ is commutative. In particular, if $\pi_1(X)$ is generated by one generator, $\pi_1(X)$ is always isomorphic to $H_1(X)$. [Use theorem 4.9.]

Corollary 4.3. If X and Y are of the same homotopy type, their first homology groups are identical: $H_1(X) = H_1(Y)$. [Use theorems 4.9 and 4.3.]

4.5 Higher homotopy groups

The fundamental group classifies the homotopy classes of loops in a topological space X. There are many ways to assign other groups to X. For example, we may classify homotopy classes of the spheres in X or homotopy classes of the tori in X. It turns out that the homotopy classes of the sphere S^n $(n \geq 2)$ form a group similar to the fundamental group.

4.5.1 Definitions

Let I^n ($n \geq 1$) denote the unit n-cube $I \times \cdots \times I$,

$$I^n = \{(s_1, \ldots, s_n) | 0 \leq s_i \leq 1 \ (1 \leq i \leq n)\}. \tag{4.35}$$

The boundary ∂I^n is the geometrical boundary of I^n,

$$\partial I^n = \{(s_1, \ldots, s_n) \in I^n | \text{ some } s_i = 0 \text{ or } 1\}. \tag{4.36}$$

We recall that in the fundamental group, the boundary ∂I of $I = [0, 1]$ is mapped to the base point x_0. Similarly, we assume here that we shall be concerned with continuous maps $\alpha : I^n \to X$, which map the boundary ∂I^n to a point $x_0 \in X$. Since the boundary is mapped to a single point x_0, we have effectively obtained S^n from I^n; cf figure 2.8. If $I^n/\partial I^n$ denotes the cube I^n whose boundary ∂I^n is shrunk to a point, we have $I^n/\partial I^n \cong S^n$. The map α is called an ***n*-loop** at x_0. A straightforward generalization of definition 4.4 is as follows.

Definition 4.10. Let X be a topological space and $\alpha, \beta : I^n \to X$ be n-loops at $x_0 \in X$. The map α is **homotopic** to β, denoted by $\alpha \sim \beta$, if there exists a continuous map $F : I^n \times I \to X$ such that

$$F(s_1, \ldots, s_n, 0) = \alpha(s_1, \ldots, s_n) \tag{4.37a}$$
$$F(s_1, \ldots, s_n, 1) = \beta(s_1, \ldots, s_n) \tag{4.37b}$$
$$F(s_1, \ldots, s_n, t) = x_0 \qquad \text{for } (s_1, \ldots, s_n) \in \partial I^n, t \in I. \tag{4.37c}$$

F is called a **homotopy** between α and β.

Exercise 4.6. Show that $\alpha \sim \beta$ is an equivalence relation. The equivalence class to which α belongs is called the **homotopy class** of α and is denoted by $[\alpha]$.

Let us define the group operations. The product $\alpha * \beta$ of n-loops α and β is defined by

$$\alpha * \beta(s_1, \ldots, s_n) = \begin{cases} \alpha(2s_1, \ldots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ \beta(2s_1 - 1, \ldots, s_n) & \frac{1}{2} \leq s_1 \leq 1. \end{cases} \tag{4.38}$$

The product $\alpha * \beta$ looks like figure 4.17(a) in X. It is helpful to express it as figure 4.17(b). If we define α^{-1} by

$$\alpha^{-1}(s_1, \ldots, s_n) \equiv \alpha(1 - s_1, \ldots, s_n) \tag{4.39}$$

it satisfies

$$\alpha^{-1} * \alpha(s_1, \ldots, s_n) \sim \alpha * \alpha^{-1}(s_1, \ldots, s_n) \sim c_{x_0}(s_1, \ldots, s_n) \tag{4.40}$$

where c_{x_0} is a constant n-loop at $x_0 \in X$, $c_{x_0} : (s_1, \ldots, s_n) \mapsto x_0$. Verify that both $\alpha * \beta$ and α^{-1} are n-loops at x_0.

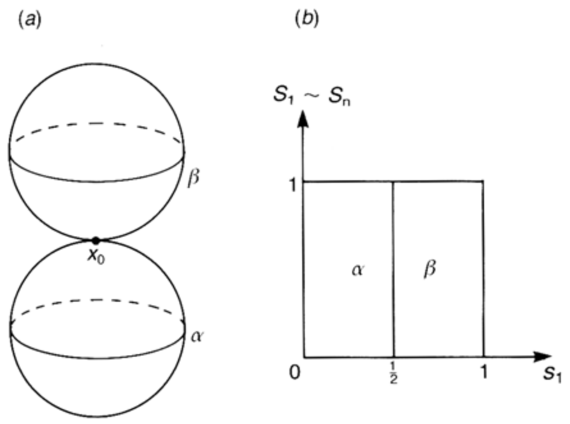


Figure 4.17. A product $\alpha * \beta$ of n-loops α and β.

Definition 4.11. Let X be a topological space. The set of homotopy classes of n-loops ($n \geq 1$) at $x_0 \in X$ is denoted by $\pi_n(X, x_0)$ and called the ***n*th homotopy group** at x_0. $\pi_n(x, x_0)$ is called the *higher* homotopy group if $n \geq 2$.

The product $\alpha * \beta$ just defined naturally induces a product of homotopy classes defined by

$$[\alpha] * [\beta] \equiv [\alpha * \beta] \tag{4.41}$$

where α and β are n-loops at x_0. The following exercises verify that this product is well defined and satisfies the group axioms.

Exercise 4.7. Show that the product of n-loops defined by (4.41) is independent of the representatives: cf lemma 4.1.

Exercise 4.8. Show that the nth homotopy group is a group. To prove this, the following facts may be verified; cf theorem 4.1.

(1) $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$.
(2) $[\alpha] * [c_x] = [c_x] * [\alpha] = [\alpha]$.
(3) $[\alpha] * [\alpha^{-1}] = [c_x]$, which defines the inverse $[\alpha]^{-1} = [\alpha^{-1}]$.

We have excluded $\pi_0(X, x_0)$ so far. Let us classify maps from I^0 to X. We note $I^0 = \{0\}$ and $\partial I^0 = \emptyset$. Let $\alpha, \beta : \{0\} \to X$ be such that $\alpha(0) = x$ and $\beta(0) = y$. We define $\alpha \sim \beta$ if there exists a continuous map $F : \{0\} \times I \to X$ such that $F(0, 0) = x$ and $F(0, 1) = y$. This shows that $\alpha \sim \beta$ if and only if x and y are connected by a curve in X, namely they are in the same (arcwise) connected component. Clearly this equivalence relation is independent of x_0 and we simply denote the zeroth homology group by $\pi_0(X)$. Note, however, that $\pi_0(X)$ is not a group and denotes the number of (arcwise) connected components of X.

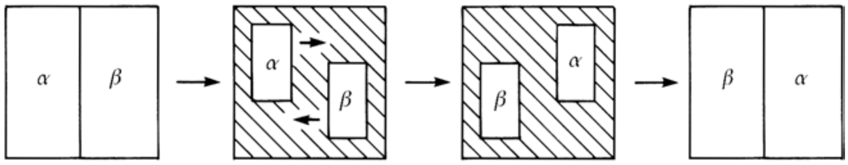


Figure 4.18. Higher homotopy groups are always commutative, $\alpha * \beta \sim \beta * \alpha$.

4.6 General properties of higher homotopy groups

4.6.1 Abelian nature of higher homotopy groups

Higher homotopy groups are always Abelian; for any n-loops α and β at $x_0 \in X$, $[\alpha]$ and $[\beta]$ satisfy

$$[\alpha] * [\beta] = [\beta] * [\alpha]. \tag{4.42}$$

To verify this assertion let us observe figure 4.18. Clearly the deformation is homotopic at each step of the sequence. This shows that $\alpha * \beta \sim \beta * \alpha$, namely $[\alpha] * [\beta] = [\beta] * [\alpha]$.

4.6.2 Arcwise connectedness and higher homotopy groups

If a topological space X is arcwise connected, $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_1)$ for any pair $x_0, x_1 \in X$. The proof is parallel to that of theorem 4.2. Accordingly, if X is arcwise connected, the base point need not be specified.

4.6.3 Homotopy invariance of higher homotopy groups

Let X and Y be topological spaces of the same homotopy type; see definition 4.6. If $f : X \to Y$ is a homotopy equivalence, the homotopy group $\pi_n(X, x_0)$ is isomorphic to $\pi_n(Y, f(x_0))$; cf theorem 4.3. Topological invariance of higher homotopy groups is the direct consequence of this fact. In particular, if X is contractible, the homotopy groups are trivial: $\pi_n(X, x_0) = \{e\}, n > 1$.

4.6.4 Higher homotopy groups of a product space

Let X and Y be arcwise connected topological spaces. Then

$$\pi_n(X \times Y) \cong \pi_n(X) \oplus \pi_n(Y) \tag{4.43}$$

cf theorem 4.6.

4.6.5 Universal covering spaces and higher homotopy groups

There are several cases in which the homotopy groups of one space are given by the known homotopy groups of the other space. There is a remarkable property between the higher homotopy groups of a topological space and its *universal covering space*.

Definition 4.12. Let X and $\widetilde{X}$ be connected topological spaces. The pair $(\widetilde{X}, p)$, or simply $\widetilde{X}$, is called the **covering space** of X if there exists a continuous map $p : \widetilde{X} \to X$ such that

(1) p is surjective (onto)
(2) for each $x \in X$, there exists a connected open set $U \subset X$ containing x, such that $p^{-1}(U)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by p.

In particular, if $\widetilde{X}$ is *simply* connected, $(\widetilde{X}, p)$ is called the **universal covering space** of X. [*Remarks*: Certain groups are known to be topological spaces. They are called topological groups. For example SO(n) and SU(n) are topological groups. If X and $\widetilde{X}$ in definition 4.12 happen to be topological groups and $p : \widetilde{X} \to X$ to be a group homomorphism, the (universal) covering space is called the (**universal**) **covering group**.]

For example, $\mathbb{R}$ is the universal covering space of S^1, see section 4.3. Since S^1 is identified with U(1), $\mathbb{R}$ is a universal covering group of U(1) if $\mathbb{R}$ is regarded as an additive group. The map $p : \mathbb{R} \to \mathrm{U}(1)$ may be $p : x \to \mathrm{e}^{\mathrm{i}2\pi x}$. Clearly p is surjective and if $U = \{\mathrm{e}^{\mathrm{i}2\pi x} | x \in (x_0 - 0.1, x_0 + 0.1)\}$, then

$$p^{-1}(U) = \bigcup_{n\in\mathbb{Z}} (x_0 - 0.1 + n, x_0 + 0.1 + n)$$

which is a disjoint union of open sets of $\mathbb{R}$. It is easy to show that p is also a homomorphism with respect to addition in $\mathbb{R}$ and multiplication in U(1). Hence, $(\mathbb{R}, p)$ is the universal covering group of $\mathrm{U}(1) = S^1$.

Theorem 4.10. Let $(\widetilde{X}, p)$ be the universal covering space of a connected topological space X. If $x_0 \in X$ and $\tilde{x}_0 \in \widetilde{X}$ are base points such that $p(\tilde{x}_0) = x_0$, the induced homomorphism

$$p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, x_0) \tag{4.44}$$

is an isomorphism for $n \geq 2$. [*Warning*: This theorem cannot be applied if $n = 1$; $\pi_1(\mathbb{R}) = \{e\}$ while $\pi_1(S^1) = \mathbb{Z}$.]

The proof is given in Croom (1978). For example, we have $\pi_n(\mathbb{R}) = \{e\}$ since $\mathbb{R}$ is contractible. Then we find

$$\pi_n(S^1) \cong \pi_n(\mathrm{U}(1)) = \{e\} \qquad n \geq 2. \tag{4.45}$$

Example 4.12. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid |x|^2 = 1\}$. The real projective space $\mathbb{R}P^n$ is obtained from S^n by identifying the pair of antipodal points $(x, -x)$. It is easy to see that S^n is a covering space of $\mathbb{R}P^n$ for $n \geq 2$. Since $\pi_1(S^n) = \{e\}$ for $n \geq 2$, S^n is the universal covering space of $\mathbb{R}P^n$ and we have

$$\pi_n(\mathbb{R}P^m) \cong \pi_n(S^m). \tag{4.46}$$

It is interesting to note that $\mathbb{R}P^3$ is identified with SO(3). To see this let us specify an element of SO(3) by a rotation about an axis $\boldsymbol{n}$ by an angle θ $(0 < \theta < \pi)$ and assign a 'vector' $\boldsymbol{\Omega} \equiv \theta \boldsymbol{n}$ to this element. Ω takes its value in the disc D^3 of radius π. Moreover, $\pi \boldsymbol{n}$ and $-\pi \boldsymbol{n}$ represent the same rotation and should be identified. Thus, the space to which Ω belongs is a disc D^3 whose anti-podal points on the surface S^2 are identified. Note also that we may express $\mathbb{R}P^3$ as the northern hemisphere D^3 of S^3, whose anti-podal points on the boundary S^2 are identified. This shows that $\mathbb{R}P^3$ is identified with SO(3).

It is also interesting to see that S^3 is identified with SU(2). First note that any element $g \in \mathrm{SU}(2)$ is written as

$$g = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1. \tag{4.47}$$

If we write $a = u + \mathrm{i}v$ and $b = x + \mathrm{i}y$, this becomes S^3,

$$u^2 + v^2 + x^2 + y^2 = 1.$$

Collecting these results, we find

$$\pi_n(\mathrm{SO}(3)) = \pi_n(\mathbb{R}P^3) = \pi_n(S^3) = \pi_n(\mathrm{SU}(2)) \qquad n \geq 2. \tag{4.48}$$

More generally, the universal covering group Spin(n) of SO(n) is called the **spin group**. For small n, they are

$$\mathrm{Spin}(3) = \mathrm{SU}(2) \tag{4.49}$$

$$\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{4.50}$$

$$\mathrm{Spin}(5) = \mathrm{USp}(4) \tag{4.51}$$

$$\mathrm{Spin}(6) = \mathrm{SU}(4). \tag{4.52}$$

Here USp($2N$) stands for the compact group of $2N \times 2N$ matrices A satisfying $A^{\mathrm{t}} J A = J$, where

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}.$$

4.7 Examples of higher homotopy groups

In general, there are no algorithms to compute higher homotopy groups $\pi_n(X)$. An *ad hoc* method is required for each topological space for $n \geq 2$. Here, we study several examples in which higher homotopy groups may be obtained by intuitive arguments. We also collect useful results in table 4.1.

Table 4.1. Useful homotopy groups.

		π_1	π_2	π_3	π_4	π_5	π_6
SO(3)		$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SO(4)		$\mathbb{Z}_2$	0	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_2+\mathbb{Z}_2$	$\mathbb{Z}_{12}+\mathbb{Z}_{12}$
SO(5)		$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
SO(6)		$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
SO(n)	$n > 6$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
U(1)		$\mathbb{Z}$	0	0	0	0	0
SU(2)		0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
SU(3)		0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$
SU(n)	$n > 3$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
S^2		0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^3		0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
S^4		0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
G_2		0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_3$
F_4		0	0	$\mathbb{Z}$	0	0	0
E_6		0	0	$\mathbb{Z}$	0	0	0
E_7		0	0	$\mathbb{Z}$	0	0	0
E_8		0	0	$\mathbb{Z}$	0	0	0

Example 4.13. If we note that $\pi_n(X, x_0)$ is the set of the homotopy classes of n-loops S^n in X, we immediately find that

$$\pi_n(S^n, x_0) \cong \mathbb{Z} \qquad n \geq 1. \tag{4.53}$$

If α maps S^n onto a point $x_0 \in S^n$, $[\alpha]$ is the unit element $0 \in \mathbb{Z}$. Since both $I^n/\partial I^n$ and S^n are orientable, we may assign orientations to them. If α maps $I^n/\partial I^n$ homeomorphically to S^n in the same sense of orientation, then $[\alpha]$ is assigned an element $1 \in \mathbb{Z}$. If a homeomorphism α maps $I^n/\partial I^n$ onto S^n in an orientation of opposite sense, $[\alpha]$ corresponds to an element -1. For example, let $n = 2$. Since $I^2/\partial I^2 \cong S^2$, the point in I^2 can be expressed by the polar coordinate (θ, ϕ), see figure 4.19. Similarly, $X = S^2$ can be expressed by the polar coordinate (θ', ϕ'). Let $\alpha : (\theta, \phi) \to (\theta', \phi')$ be a 2-loop in X. If $\theta' = \theta$ and $\phi' = \phi$, the point (θ', ϕ') sweeps S^2 once while the point (θ, ϕ) scans I^2 once in the same orientation. This 2-loop belongs to the class $+1 \in \pi_2(S^2, x_0)$. If $\alpha : (\theta, \phi) \to (\theta', \phi')$ is given by $\theta' = \theta$ and $\phi' = 2\phi$, the point (θ', ϕ') sweeps S^2 twice while (θ, ϕ) scans I^2 once. This 2-loop belongs to the class $2 \in \pi_2(S^2, x_0)$. In general, the map $(\theta, \phi) \mapsto (\theta, k\phi)$, $k \in \mathbb{Z}$, corresponds to the class k of $\pi_2(S^2, x_0)$. A similar argument verifies (4.53) for general $n > 2$.

Example 4.14. Noting that S^n is a universal covering space of $\mathbb{R}P^n$ for $n > 2$, we find

$$\pi_n(\mathbb{R}P^n) \cong \pi_n(S^n) \cong \mathbb{Z} \qquad n \geq 2. \tag{4.54}$$

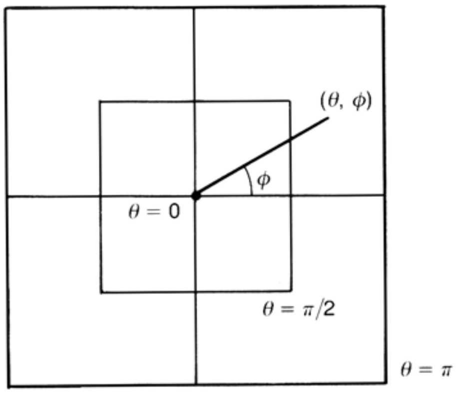


Figure 4.19. A point in I^2 may be expressed by polar coordinates (θ, ϕ).

[Of course this happens to be true for $n = 1$, since $\mathbb{R}P^1 = S^1$.] For example, we have $\pi_2(\mathbb{R}P^2) \cong \pi_2(S^2) \cong \mathbb{Z}$. Since $SU(2) = S^3$ is the universal covering group of $SO(3) = \mathbb{R}P^3$, it follows from theorem 4.10 that (see also (4.48))

$$\pi_3(SO(3)) \cong \pi_3(SU(2)) \cong \pi_3(S^3) \cong \mathbb{Z}. \tag{4.55}$$

Shankar's monopoles in superfluid ^{3}He-A correspond to non-trivial elements of these homotopy classes, see section 4.10. $\pi_3(SU(2))$ is also employed in the classification of instantons in example 9.8.

In summary, we have table 4.1. In this table, other useful homotopy groups are also listed. We comment on several interesting facts.

(a) Since $\text{Spin}(4) = SU(2) \times SU(2)$ is the universal covering group of SO(4), we have $\pi_n(SO(4)) = \pi_n(SU(2)) \oplus \pi_n(SU(2))$ for $n > 2$.
(b) There exists a map J called the ***J*-homomorphism** $J : \pi_k(SO(n)) \to \pi_{k+n}(S^n)$, see Whitehead (1978). In particular, if $k = 1$, the homomorphism is known to be an isomorphism and we have $\pi_1(SO(n)) = \pi_{n+1}(S^n)$. For example, we find

$$\pi_1(SO(2)) \cong \pi_3(S^2) \cong \mathbb{Z}$$
$$\pi_1(SO(3)) \cong \pi_4(S^3) \cong \pi_4(SU(2)) \cong \pi_4(SO(3)) \cong \mathbb{Z}_2.$$

(c) The **Bott periodicity theorem** states that

$$\pi_k(U(n)) \cong \pi_k(SU(n)) \cong \begin{cases} \{e\} & \text{if } k \text{ is even} \\ \mathbb{Z} & \text{if } k \text{ is odd} \end{cases} \tag{4.56}$$

for $n \geq (k+1)/2$. Similarly,

$$\pi_k(\mathrm{O}(n)) \cong \pi_k(\mathrm{SO}(n)) \cong \begin{cases} \{e\} & \text{if } k \equiv 2, 4, 5, 6 \pmod 8 \\ \mathbb{Z}_2 & \text{if } k \equiv 0, 1 \pmod 8 \\ \mathbb{Z} & \text{if } k \equiv 3, 7 \pmod 8 \end{cases} \tag{4.50}$$

for $n \geq k+2$. Similar periodicity holds for symplectic groups which we shall not give here.

Many more will be found in appendix A, table 6 of Ito (1987).

4.8 Orders in condensed matter systems

Recently topological methods have played increasingly important roles in condensed matter physics. For example, homotopy theory has been employed to classify possible forms of extended objects, such as solitons, vortices, monopoles and so on, in condensed systems. These classifications will be studied in sections 4.8–4.10. Here, we briefly look at the order parameters of condensed systems that undergo phase transitions.

4.8.1 Order parameter

Let H be a Hamiltonian describing a condensed matter system. We assume H is invariant under a certain symmetry operation. The ground state of the system need not preserve the symmetry of H. If this is the case, we say the system undergoes **spontaneous symmetry breakdown**.

To illustrate this phenomenon, we consider the **Heisenberg Hamiltonian**

$$H = -J \sum_{(i,j)} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + \boldsymbol{h} \cdot \sum_i \boldsymbol{S}_i \tag{4.57}$$

which describes N ferromagnetic Heisenberg spins $\{\boldsymbol{S}_i\}$. The parameter J is a positive constant, the summation is over the pair of the nearest-neighbour sites (i, j) and $\boldsymbol{h}$ is the uniform external magnetic field. The partition function is $Z = \mathrm{tr}\,\mathrm{e}^{-\beta H}$, where $\beta = 1/T$ is the inverse temperature. The free energy F is defined by $\exp(-\beta F) = Z$. The average magnetization per spin is

$$\boldsymbol{m} \equiv \frac{1}{N} \sum_i \langle \boldsymbol{S}_i \rangle = \frac{1}{N\beta} \frac{\partial F}{\partial \boldsymbol{h}} \tag{4.58}$$

where $\langle \ldots \rangle \equiv \mathrm{tr}(\ldots \mathrm{e}^{-\beta H})/Z$. Let us consider the limit $\boldsymbol{h} \to 0$. Although H is invariant under the SO(3) rotations of all $\boldsymbol{S}_i$ in this limit, it is well known that $\boldsymbol{m}$ does not vanish for large enough β and the system does not observe the SO(3) symmetry. It is said that the system exhibits **spontaneous magnetization** and the maximum temperature, such that $\boldsymbol{m} \neq 0$ is called the **critical temperature**.

The vector $\boldsymbol{m}$ is the **order parameter** describing the phase transition between the ordered state ($\boldsymbol{m} \neq 0$) and the disordered state ($\boldsymbol{m} = 0$). The system is still symmetric under SO(2) rotations around the magnetization axis $\boldsymbol{m}$.

What is the mechanism underlying the phase transition? The free energy is $F = \langle H \rangle - TS$, S being the entropy. At low temperature, the term TS in F may be negligible and the minimum of F is attained by minimizing $\langle H \rangle$, which is realized if all $\boldsymbol{S}_i$ align in the same direction. At high temperature, however, the entropy term dominates F and the minimum of F is attained by maximizing S, which is realized if the directions of $\boldsymbol{S}_i$ are totally random.

If the system is at a uniform temperature, the magnitude $|\boldsymbol{m}|$ is independent of the position and $\boldsymbol{m}$ is specified by its direction only. In the ground state, $\boldsymbol{m}$ itself is expected to be independent of position. It is convenient to introduce the polar coordinate (θ, ϕ) to specify the direction of $\boldsymbol{m}$. There is a one-to-one correspondence between $\boldsymbol{m}$ and a point on the sphere S^2. Suppose $\boldsymbol{m}$ varies as a function of position: $\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{x})$. At each point $\boldsymbol{x}$ of the space, a point (θ, ϕ) of S^2 is assigned and we have a map $(\theta(\boldsymbol{x}), \phi(\boldsymbol{x}))$ from the space to S^2. Besides the ground state (and excited states that are described by small oscillations (spin waves) around the ground state) the system may carry various excited states that cannot be obtained from the ground state by small perturbations. What kinds of excitation are possible depends on the dimension of the space and the order parameter. For example, if the space is two dimensional, the Heisenberg ferromagnet may admit an excitation called the **Belavin–Polyakov monopole** shown in figure 4.20 (Belavin and Polyakov 1975). Observe that $\boldsymbol{m}$ approaches a constant vector ($\hat{z}$ in this case) so the energy does not diverge. This condition guarantees the stability of this excitation; it is impossible to deform this configuration into the uniform one with $\boldsymbol{m}$ far from the origin kept fixed. These kinds of excitation whose stability depends on topological arguments are called **topological excitations**. Note that the field $\boldsymbol{m}(\boldsymbol{x})$ defines a map $\boldsymbol{m} : S^2 \to S^2$ and, hence, are classified by the homotopy group $\pi_2(S^2) = \mathbb{Z}$.

4.8.2 Superfluid ^{4}He and superconductors

In Bogoliubov's theory, the order parameter of superfluid ^{4}He is the expectation value

$$\langle \phi(\boldsymbol{x}) \rangle = \Psi(\boldsymbol{r}) = \Delta_0(\boldsymbol{x}) \mathrm{e}^{\mathrm{i}\alpha(\boldsymbol{x})} \tag{4.59}$$

where $\phi(\boldsymbol{x})$ is the field operator. In the operator formalism,

$$\phi(\boldsymbol{x}) \sim (\text{creation operator}) + (\text{annihilation operator})$$

from which we find the number of particles is not conserved if $\Psi(\boldsymbol{x}) \neq 0$. This is related to the spontaneous breakdown of the global gauge symmetry. The

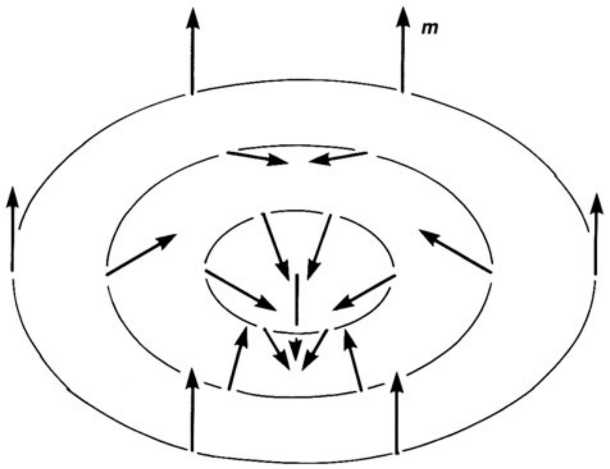


Figure 4.20. A sketch of the Belavin–Polyakov monopole. The vector $\boldsymbol{m}$ approaches $\hat{z}$ as $|\boldsymbol{x}| \to \infty$.

Hamiltonian of ^{4}He is

$$H = \int \mathrm{d}\boldsymbol{x}\, \phi^\dagger(\boldsymbol{x}) \left(-\frac{\nabla^2}{2m} - \mu \right) \phi(\boldsymbol{x}) + \frac{1}{2} \int \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{y}\, \phi^\dagger(\boldsymbol{y}) \phi(\boldsymbol{y}) V(|\boldsymbol{x} - \boldsymbol{y}|) \phi^\dagger(\boldsymbol{x}) \phi(\boldsymbol{x}). \tag{4.60}$$

Clearly H is invariant under the global gauge transformation

$$\phi(\boldsymbol{x}) \to \mathrm{e}^{\mathrm{i}\chi} \phi(\boldsymbol{x}). \tag{4.61}$$

The order parameter, however, transforms as

$$\Psi(\boldsymbol{x}) \to \mathrm{e}^{\mathrm{i}\chi} \Phi(\boldsymbol{x}) \tag{4.62}$$

and hence does not observe the symmetry of the Hamiltonian. The phenomenological free energy describing ^{4}He is made up of two contributions. The main contribution is the **condensation energy**

$$\mathcal{F}_0 \equiv \frac{\alpha}{2!} |\Psi(\boldsymbol{x})|^2 + \frac{\beta}{4!} |\Psi(\boldsymbol{x})|^4 \tag{4.63a}$$

where $\alpha \sim \alpha_0(T - T_\mathrm{c})$ changes sign at the critical temperature $T \sim 4$ K. Figure 4.21 sketches $\mathcal{F}_0$ for $T > T_\mathrm{c}$ and $T < T_\mathrm{c}$. If $T > T_\mathrm{c}$, the minimum of $\mathcal{F}_0$ is attained at $\Psi(\boldsymbol{x}) = 0$ while if $T < T_\mathrm{c}$ at $|\Psi| = \Delta_0 \equiv [-(6\alpha/\beta)]^{1/2}$. If $\Psi(\boldsymbol{x})$ depends on $\boldsymbol{x}$, we have an additional contribution called the **gradient energy**

$$\mathcal{F}_{\mathrm{grad}} \equiv \tfrac{1}{2} K \overline{\nabla \Psi(\boldsymbol{x})} \cdot \nabla \Psi(\boldsymbol{x}) \tag{4.63b}$$

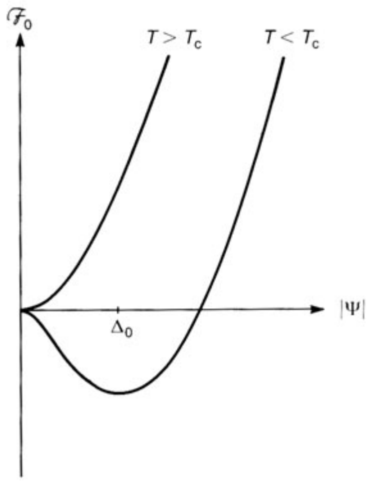


Figure 4.21. The free energy has a minimum at $|\Psi| = 0$ for $T > T_c$ and at $|\Psi| = \Delta_0$ for $T < T_c$.

K being a positive constant. If the spatial variation of $\Psi(\boldsymbol{x})$ is mild enough, we may assume Δ_0 is constant (the London limit).

In the BCS theory of superconductors, the order parameter is given by (Tsuneto 1982)

$$\Psi_{\alpha\beta} \equiv \langle \psi_\alpha(\boldsymbol{x})\psi_\beta(\boldsymbol{x})\rangle \tag{4.64}$$

$\psi_\alpha(\boldsymbol{x})$ being the (non-relativistic) electron field operator of spin $\alpha = (\uparrow, \downarrow)$. It should be noted, however, that (4.64) is not an irreducible representation of the spin algebra. To see this, we examine the behaviour of $\Psi_{\alpha\beta}$ under a spin rotation. Consider an infinitesimal spin rotation around an axis $\boldsymbol{n}$ by an angle θ, whose matrix representation is

$$R = I_2 + \mathrm{i}\frac{\theta}{2}n^\mu\sigma_\mu,$$

σ_μ being the Pauli matrices. Since ψ_α transforms as $\psi_\alpha \to R_\alpha{}^\beta\psi_\beta$ we have

$$\begin{aligned}\Psi_{\alpha\beta} \to R_\alpha{}^{\alpha'}\Psi_{\alpha'\beta'}R_\beta{}^{\beta'} &= (R\cdot\Psi\cdot R^{\mathrm{t}})_{\alpha\beta}\\ &= \left[\Psi + \mathrm{i}\frac{\delta}{2}\boldsymbol{n}(\boldsymbol{\sigma}\Psi\sigma_2 - \Psi\sigma_2\boldsymbol{\sigma})\right]_{\alpha\beta}\end{aligned}$$

where we note that $\sigma_\mu^{\mathrm{t}} = -\sigma_2\sigma_\mu\sigma_2$. Suppose $\Psi_{\alpha\beta} \propto \mathrm{i}(\sigma_2)_{\alpha\beta}$. Then Ψ does not change under this rotation, hence it represents the spin-singlet pairing. We write

$$\Psi_{\alpha\beta}(\boldsymbol{x}) = \Delta(\boldsymbol{x})(\mathrm{i}\sigma_2)_{\alpha\beta} = \Delta_0(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{x})}(\mathrm{i}\sigma_2)_{\alpha\beta}. \tag{4.65a}$$

If, however, we take

$$\Psi_{\alpha\beta}(\boldsymbol{x}) = \Delta^\mu(\boldsymbol{x})\mathrm{i}(\sigma_\mu\cdot\sigma_2)_{\alpha\beta} \tag{4.65b}$$

we have

$$\Psi_{\alpha\beta} \to [\Delta^\mu + \delta\varepsilon^{\mu\nu\lambda} n_\nu \Delta_\lambda](\mathrm{i}\sigma_\mu \cdot \sigma_2)_{\alpha\beta}.$$

This shows that Δ^μ is a vector in spin space, hence (4.65b) represents the spin-triplet pairing.

The order parameter of a conventional superconductor is of the form (4.65a) and we restrict the analysis to this case for the moment. In (4.65a), $\Delta(\boldsymbol{x})$ assumes the same form as $\Psi(\boldsymbol{x})$ of superfluid ^{4}He and the free energy is again given by (4.63). This similarity is attributed to the Cooper pair. In the superfluid state, a macroscopic number of ^{4}He atoms occupy the ground state (Bose–Einstein condensation) which then behaves like a huge molecule due to the quantum coherence. In this state creating elementary excitations requires a finite amount of energy and the flow cannot decay unless this critical energy is supplied. Since an electron is a fermion there is, at first sight, no Bose–Einstein condensation. The key observation is the Cooper pair. By the exchange of phonons, a pair of electrons feels an attractive force that barely overcomes the Coulomb repulsion. This tiny attractive force makes it possible for electrons to form a pair (in momentum space) that obeys Bose statistics. The pairs then condense to form the superfluid state of the Cooper pairs of electric charge $2e$.

An electromagnetic field couples to the system through the minimal coupling

$$\mathcal{F}_{\text{grad}} = \tfrac{1}{2}K \left|(\partial_\mu - \mathrm{i}2eA_\mu)\Delta(\boldsymbol{x})\right|^2. \qquad (4.66)$$

(The term $2e$ is used since the Cooper pair carries charge $2e$.) Superconductors are roughly divided into two types according to their behaviour in applied magnetic fields. The type-I superconductor forms an intermediate state in which normal and superconducting regions coexist in strong magnetic fields. The type-II superconductor forms a vortex lattice (**Abrikosov lattice**) to confine the magnetic fields within the cores of the vortices with other regions remaining in the superconducting state. A similar vortex lattice has been observed in rotating superfluid ^{4}He in a cylinder.

4.8.3 General consideration

In the next two sections, we study applications of homotopy groups to the classification of defects in ordered media. The analysis of this section is based on Toulouse and Kléman (1976), Mermin (1979) and Mineev (1980).

As we saw in the previous subsections, when a condensed matter system undergoes a phase transition, the symmetry of the system is reduced and this reduction is described by the order parameter. For definiteness, let us consider the three-dimensional medium of a superconductor. The order parameter takes the form $\psi(\boldsymbol{x}) = \Delta_0(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\varphi(\boldsymbol{x})}$. Let us consider a homogeneous system under uniform external conditions (temperature, pressure etc). The amplitude Δ_0 is uniquely fixed by minimizing the condensation free energy. Note that there are still a large number of degrees of freedom left. ψ may take any value in the circle $S^1 \cong \mathrm{U}(1)$

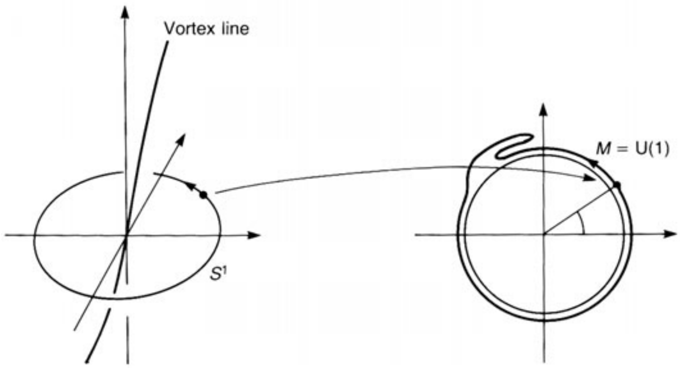


Figure 4.22. A circle S^1 surrounding a line defect (vortex) is mapped to $\mathrm{U}(1) = S^1$. This map is classified by the fundamental group $\pi_1(\mathrm{U}(1))$.

determined by the phase $e^{i\varphi}$. In this way, a uniform system takes its value in a certain region M called the **order parameter space**. For a superconductor, $M = U(1)$. For the Heisenberg spin system, $M = S^2$. The nematic liquid crystal has $M = \mathbb{R}P^2$ while $M = S^2 \times \mathrm{SO}(3)$ for the superfluid ^{3}He-A, see sections 4.9–4.10.

If the system is in an inhomogeneous state, the gradient free energy cannot be negligible and ψ may not be in M. If the characteristic size of the variation of the order parameter is much larger than the coherence length, however, we may still assume that the order parameter takes its value in M, where the value is a function of position this time. If this is the case, there may be points, lines or surfaces in the medium on which the order parameter is not uniquely defined. They are called the **defects**. We have **point defects** (**monopoles**), **line defects** (**vortices**) and **surface defects** (**domain walls**) according to their dimensionalities. These defects are classified by the homotopy groups.

To be more mathematical, let X be a space which is filled with the medium under consideration. The order parameter is a classical field $\psi(x)$, which is also regarded as a *map* $\psi : X \to M$. Suppose there is a defect in the medium. For concreteness, we consider a line defect in the three-dimensional medium of a superconductor. Imagine a circle S^1 which encircles the line defect. If each part of S^1 is far from the line defect, much further than the coherence length ξ, we may assume the order parameter along S^1 takes its value in the order parameter space $M = \mathrm{U}(1)$, see figure 4.22. This is how the fundamental group comes into the problem; we talk of loops in a topological space U(1). The map $S^1 \to \mathrm{U}(1)$ is classified by the homotopy classes. Take a point $r_0 \in S^1$ and require that r_0 be mapped to $x_0 \in M$. By noting that $\pi_1(\mathrm{U}(1), x_0) = \mathbb{Z}$, we may assign an integer to the line defect. This integer is called the **winding number** since it counts how many times the image of S^1 winds the space U(1). If two line defects have the same winding number, one can be continuously deformed to the other. If two line defects A and B merge together, the new line defect belongs to the homotopy class of the product of the homotopy classes to which A and B belonged before coalescence. Since the group operation in $\mathbb{Z}$ is an addition, the new winding number is a sum of the old winding numbers. A uniform distribution of the order parameter corresponds to the constant map $\psi(x) = x_0 \in M$, which belongs to the unit element $0 \in \mathbb{Z}$. If two line defects of opposite winding numbers merge together, the new line defect can be continuously deformed into the defect-free configuration.

What about the other homotopy groups? We first consider the dimensionality of the defect and the sphere S^n which *surrounds* it. For example, consider a point defect in a three-dimensional medium. It can be surrounded by S^2 and the defect is classified by $\pi_2(M, x_0)$. If M has many components, $\pi_0(M)$ is non-trivia1. Let us consider a three-dimensional Ising model for which $M = \{\downarrow\} \cup \{\uparrow\}$. Then there is a domain wall on which the order parameter is not defined. For example, if $\boldsymbol{S} = \uparrow$ for $x < 0$ and $\boldsymbol{S} = \downarrow$ for $x > 0$, there is a domain wall in the yz-plane at $x = 0$. In general, an m-dimensional defect in a d-dimensional medium is classified by the homotopy group $\pi_n(M, x_0)$ where

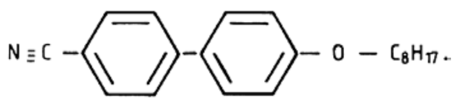
$$n = d - m - 1. \tag{4.67}$$

In the case of the lsing model, $d = 3, m = 2$; hence $n = 0$.

4.9 Defects in nematic liquid crystals

4.9.1 Order parameter of nematic liquid crystals

Certain organic crystals exhibit quite interesting optical properties when they are in their fluid phases. They are called liquid crystals and they are characterized by their optical anisotropy. Here we are interested in so-called nematic liquid crystals. An example of this is *octyloxy-cyanobiphenyl* whose molecular structure is



The molecule of a nematic liquid crystal is very much like a rod and the order parameter, called the **director**, is given by the average direction of the rod. Even though the molecule itself has a head and a tail, the director has an inversion symmetry; it does not make sense to distinguish the directors $\boldsymbol{n} = \rightarrow$ and $-\boldsymbol{n} = \leftarrow$. We are tempted to assign a point on S^2 to specify the director. This works except for one point. Two antipodal points $\boldsymbol{n} = (\theta, \phi)$ and $-\boldsymbol{n} = (\pi - \theta, \pi + \phi)$ represent the same state; see figure 4.23. Accordingly, the order parameter of the nematic liquid crystal is the **projective plane** $\mathbb{R}P^2$. The director field in general

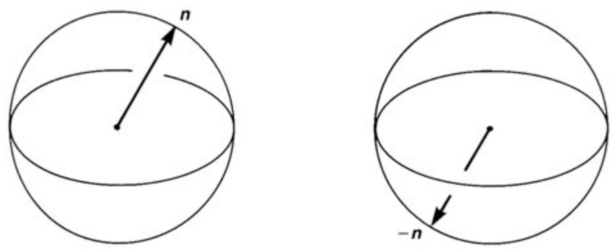


Figure 4.23. Since the director $\boldsymbol{n}$ has no head or tail, one cannot distinguish $\boldsymbol{n}$ from $-\boldsymbol{n}$. Therefore, these two pictures correspond to the same order-parameter configuration.

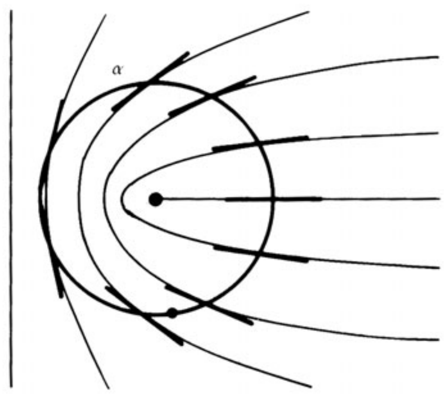


Figure 4.24. A vortex in a nematic liquid crystal, which corresponds to the non-trivial element of $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$.

depends on the position $\boldsymbol{r}$. Then we may define a map $f : \mathbb{R}^3 \to \mathbb{R}P^2$. This map is called the **texture**. The actual order-parameter configuration in $\mathbb{R}^3$ is also called the texture.

4.9.2 Line defects in nematic liquid crystals

From example 4.10 we have $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 = \{0, 1\}$. There exist two kinds of line defect in nematic liquid crystals; one can be continuously deformed into a uniform configuration while the other cannot. The latter represents a stable vortex, whose texture is sketched in figure 4.24. The reader should observe how the loop α is mapped to $\mathbb{R}P^2$ by this texture.

Exercise 4.9. Show that the line 'defect' in figure 4.25 is fictitious, namely the singularity at the centre may be eliminated by a continuous deformation of directors with directors at the boundary fixed. This corresponds to the operation $1 + 1 = 0$.

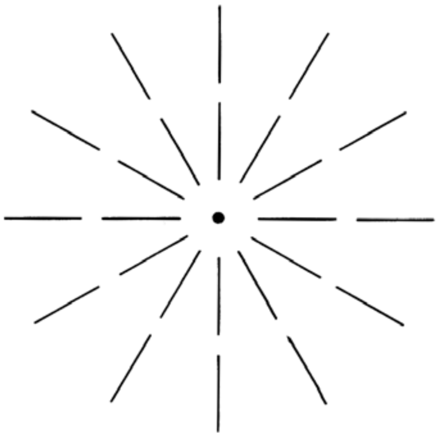


Figure 4.25. A line defect which may be continuously deformed into a uniform configuration.

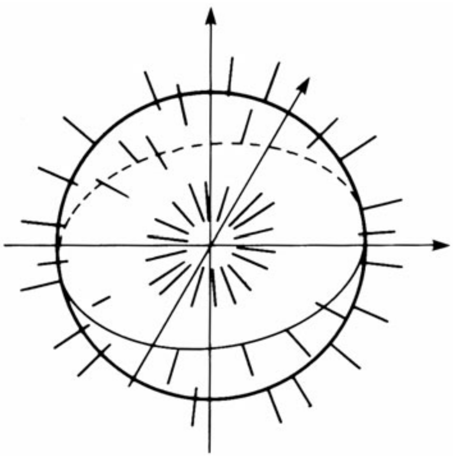


Figure 4.26. The texture of a point defect in a nematic liquid crystal.

4.9.3 Point defects in nematic liquid crystals

From example 4.14, we have $\pi_2(\mathbb{R}P^2) = \mathbb{Z}$. Accordingly, there are stable point defects in the nematic liquid crystal. Figure 4.26 shows the texture of the point defects that belong to the class $1 \in \mathbb{Z}$.

It is interesting to point out that a line defect and a point defect may be combined into a **ring defect**, which is specified by both $\pi_1(\mathbb{R}P^2)$ and $\pi_2(\mathbb{R}P^2)$, see Mineev (1980). If the ring defect is observed from far away, it looks like

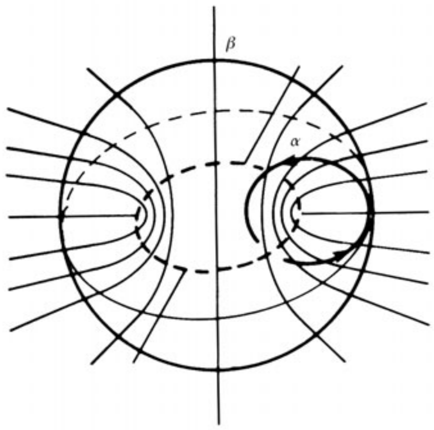


Figure 4.27. The texture of a ring defect in a nematic liquid crystal. The loop α classifies $\pi_1(\mathbb{R}P^2)$ while the sphere (2-loop) β classifies $\pi_2(\mathbb{R}P^2)$.

a point defect, while its local structure along the ring is specified by $\pi_1(\mathbb{R}P^2)$. Figure 4.27 is an example of such a ring defect. The loop α classifies $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ while the sphere (2-loop) β classifies $\pi_2(\mathbb{R}P^2) = \mathbb{Z}$.

4.9.4 Higher dimensional texture

The third homotopy group $\pi(\mathbb{R}P^2) \cong \mathbb{Z}$ leads to an interesting singularity-free texture in a three-dimensional medium of nematic liquid crystal. Suppose the director field approaches an asymptotic configuration, say $\boldsymbol{n} = (1, 0, 0)^{\mathrm{t}}$, as $|\boldsymbol{r}| \to \infty$. Then the medium is effectively compactified into the three-dimensional sphere S^3 and the topological structure of the texture is classified by $\pi_3(\mathbb{R}P^2) \cong \mathbb{Z}$. What is the texture corresponding to a non-trivial element of the homotopy group?

An arbitrary rotation in $\mathbb{R}^3$ is specified by a unit vector $\boldsymbol{e}$, around which the rotation is carried out, and the rotation angle α. It is possible to assign a 'vector' $\Omega = \alpha\boldsymbol{e}$ to this rotation. It is not exactly a vector since $\Omega = \pi\boldsymbol{e}$ and $-\Omega = -\pi\boldsymbol{e}$ are the same rotation and hence should be identified. Therefore, Ω belongs to the real projective space $\mathbb{R}P^3$. Suppose we take $\boldsymbol{n}_0 = (1, 0, 0)^{\mathrm{t}}$ as a standard director. Then an arbitrary director configuration is specified by rotating $\boldsymbol{n}_0$ around some axis $\boldsymbol{e}$ by an angle α: $\boldsymbol{n} = R(\boldsymbol{e}, \alpha)\boldsymbol{n}_0$, where $R(\boldsymbol{e}, \alpha)$ is the corresponding rotation matrix in SO(3). Suppose a texture field is given by applying the rotation

$$\alpha\boldsymbol{e}(\boldsymbol{r}) = f(r)\hat{\boldsymbol{r}} \tag{4.68}$$

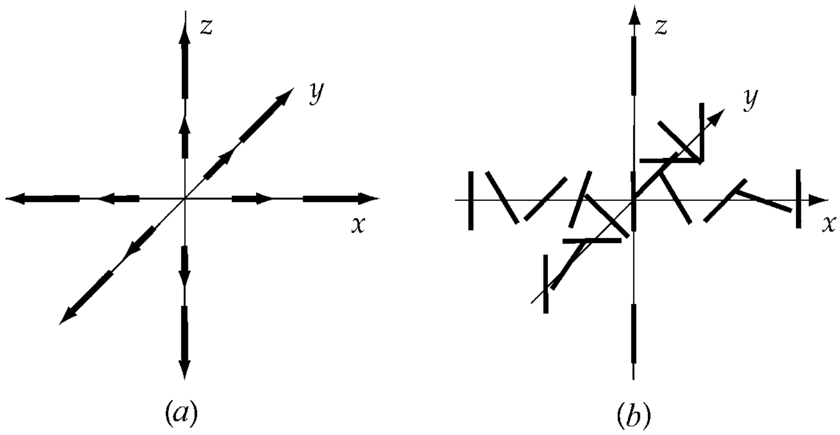


Figure 4.28. The texture of the non-trivial element of $\pi_3(\mathbb{R}P^2) \cong \mathbb{Z}$. (*a*) shows the rotation 'vector' $\alpha \boldsymbol{e}$. The length α approaches π as $|\boldsymbol{r}| \to \infty$. (*b*) shows the corresponding director field.

to $\boldsymbol{n}_0$, where $\hat{\boldsymbol{r}}$ is the unit vector in the direction of the position vector $\boldsymbol{r}$ and

$$f(r) = \begin{cases} 0 & r = 0 \\ \pi & r \to \infty. \end{cases}$$

Figure 4.28 shows the director field of this texture. Note that although there is no singularity in the texture, it is impossible to 'wind off' this to a uniform configuration.

4.10 Textures in superfluid ^{3}He-A

4.10.1 Superfluid ^{3}He-A

Here comes the last and most interesting example. Before 1972 the only example of the BCS superfluid was the conventional superconductor (apart from indirect observations of superfluid neutrons in neutron stars). Figure 4.29 is the phase diagram of superfluid ^{3}He without an external magnetic field. From NMR and other observations, it turns out that the superfluid is in the spin-triplet p-wave state. Instead of the field operators (see (4.65b)), we define the order parameter in terms of the creation and annihilation operators. The most general form of the triplet superfluid order parameter is

$$\langle c_{\alpha,\boldsymbol{k}} c_{\beta,-\boldsymbol{k}} \rangle \propto \sum_{\mu=1}^{3} (\mathrm{i}\sigma_2\sigma_\mu)_{\alpha\beta} d_\mu(\boldsymbol{k}) \tag{4.69a}$$

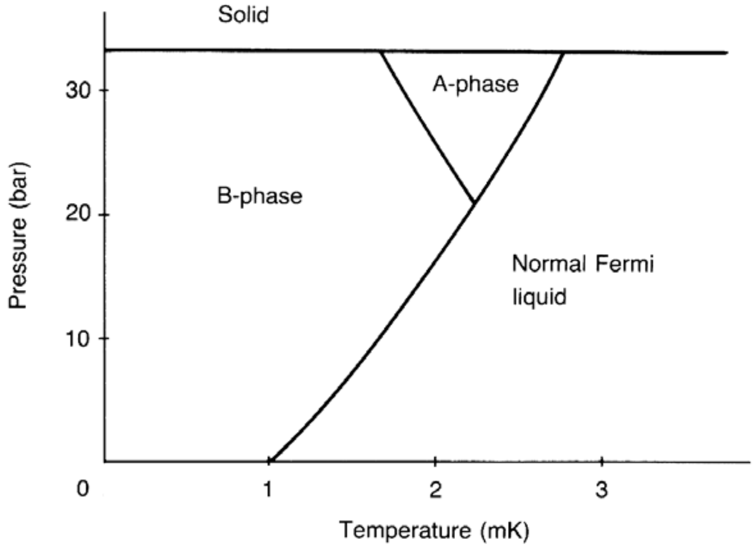


Figure 4.29. The phase diagram of superfluid ^{3}He.

where α and β are spin indices. The Cooper pair forms in the p-wave state hence $d_\mu(\boldsymbol{k})$ is proportional to $Y_{1m} \sim k_i$,

$$d_\mu(\boldsymbol{k}) = \sum_{i=1}^{3} \Delta_0 A_{\mu i} k_i. \tag{4.69b}$$

The bulk energy has several minima. The absolute minimum depends on the pressure and the temperature. We are particularly interested in the A phase in figure 4.29.

The A-phase order parameter takes the form

$$A_{\mu i} = d_\mu(\boldsymbol{\Delta}_1 + \mathrm{i}\boldsymbol{\Delta}_2)_i \tag{4.70}$$

where $\boldsymbol{d}$ is a unit vector along which the spin projection of the Cooper pair vanishes and $(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2)$ is a pair of orthonormal unit vectors. The vector $\boldsymbol{d}$ takes its value in S^2. If we define $\boldsymbol{l} \equiv \boldsymbol{\Delta}_1 \times \boldsymbol{\Delta}_2$, the triad $(\boldsymbol{\Delta}_1, \boldsymbol{\Delta}_2, \boldsymbol{l})$ forms an orthonormal frame at each point of the medium. Since any orthonormal frame can be obtained from a standard orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ by an application of a three-dimensional rotation matrix, we conclude that the order parameter of ^{3}He-A is $S^2 \times \mathrm{SO}(3)$. The vector $\boldsymbol{l}$ introduced here is the axis of the angular momentum of the Cooper pair.

For simplicity, we neglect the variation of the $\hat{\boldsymbol{d}}$-vector. [In fact, $\hat{\boldsymbol{d}}$ is locked along $\hat{\boldsymbol{l}}$ due to the dipole force.] The order parameter assumes the form

$$A_i = \Delta_0(\hat{\boldsymbol{\Delta}}_1 + \hat{\boldsymbol{\Delta}}_2)_i \tag{4.71}$$

where $\hat{\boldsymbol{\Delta}}_1$, $\hat{\boldsymbol{\Delta}}_2$ and $\hat{\boldsymbol{l}} \equiv \hat{\boldsymbol{\Delta}}_1 \times \hat{\boldsymbol{\Delta}}_2$ form an orthonormal frame at each point of the medium. Let us take a standard orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_2)$. The frame $(\hat{\boldsymbol{\Delta}}_1, \hat{\boldsymbol{\Delta}}_2, \hat{\boldsymbol{l}})$ is obtained by applying an element $g \in \mathrm{SO}(3)$ to the standard frame,

$$g : (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_2) \to (\hat{\boldsymbol{\Delta}}_1, \hat{\boldsymbol{\Delta}}_2, \hat{\boldsymbol{l}}). \tag{4.72}$$

Since g depends on the coordinate x, the configuration $(\hat{\boldsymbol{\Delta}}_1(x), \hat{\boldsymbol{\Delta}}_2(x), \hat{\boldsymbol{l}}(x))$ defines a map $\psi : X \to \mathrm{SO}(3)$ as $x \mapsto g(x)$. The map ψ is called the **texture** of a superfluid ^{3}He.[1] The relevant homotopy groups for classifying defects in superfluid ^{3}He-A are $\pi_n(\mathrm{SO}(3))$.

If a container is filled with ^{3}He-A, the boundary poses certain conditions on the texture. The vector $\hat{\boldsymbol{l}}$ is understood as the direction of the angular momentum of the Cooper pair. The pair should rotate in the plane parallel to the boundary wall, thus $\hat{\boldsymbol{l}}$ should be perpendicular to the wall. [*Remark:* If the wall is *diffuse*, the orbital motion of Cooper pairs is disturbed and there is a depression in the amplitude of the order parameter in the vicinity of the wall. We assume, for simplicity, that the wall is *specularly smooth* so that Cooper pairs may execute orbital motion with no disturbance.] There are several kinds of free energy and the texture is determined by solving the Euler–Lagrange equation derived from the total free energy under given boundary conditions.

Reviews on superfluid ^{3}He are found in Anderson and Brinkman (1975), Leggett (1975) and Mermin (1978).

4.10.2 Line defects and non-singular vortices in ^{3}He-A

The fundamental group of $\mathrm{SO}(3) \cong \mathbb{R}P^3$ is $\pi_1(\mathbb{R}P^3) \cong \mathbb{Z}_2 \cong \{0, 1\}$. Textures which belong to class 0 can be continuously deformed into the uniform configuration. Configurations in class 1 are called **disgyrations** and have been analysed by Maki and Tsuneto (1977) and Buchholtz and Fetter (1977). Figure 4.30 describes these disgyrations in their lowest free energy configurations.

A remarkable property of $\mathbb{Z}_2$ is the addition $1 + 1 = 0$; the coalescence of two disgyrations produces a trivial texture. By merging two disgyrations, we may construct a texture that looks like a vortex of double vorticity (homotopy class '2') without a singular core; see figure 4.31(*a*). It is easy to verify that the image of the loop α traverses $\mathbb{R}P^3$ twice while that of the smaller loop β may be shrunk to a point. This texture is called the **Anderson–Toulouse vortex** (Anderson and Toulouse 1977). Mermin and Ho (1976) pointed out that if the medium is in a cylinder, the boundary imposes the condition $\hat{\boldsymbol{l}} \perp$ (boundary) and the vortex is cut at the surface, see figure 4.31(*b*) (the **Mermin–Ho vortex**).

[1] The name 'texture' is, in fact, borrowed from the order-parameter configuration in liquid crystals, see section 4.9.

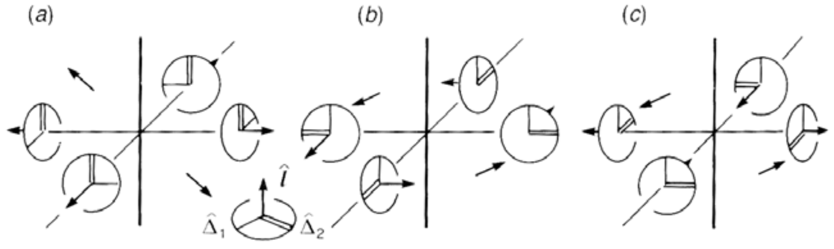


Figure 4.30. Disgyrations in ^{3}He-A.

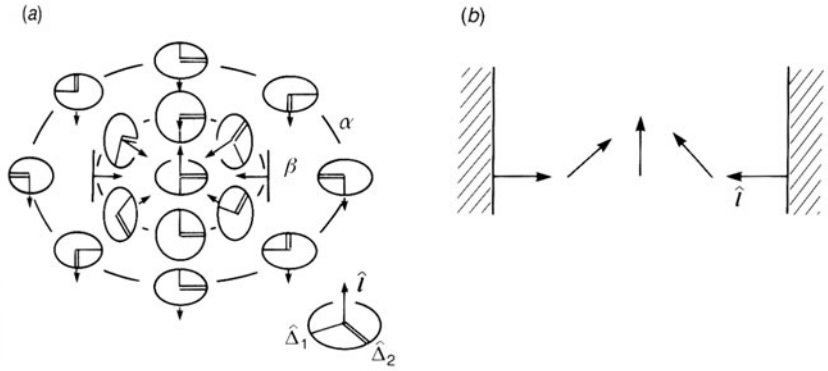


Figure 4.31. The Anderson–Toulouse vortex (*a*) and the Mermin–Ho vortex (*b*). In (*b*) the boundary forces $\hat{\boldsymbol{l}}$ to be perpendicular to the wall.

Since $\pi_2(\mathbb{R}P^3) \cong \{e\}$, there are no point defects in ^{3}He-A. However, $\pi_3(\mathbb{R}P^3) \cong \mathbb{Z}$ introduces a new type of pointlike structure called the Shankar monopole, which we will study next.

4.10.3 Shankar monopole in ^{3}He-A

Shankar (1977) pointed out that there exists a pointlike singularity-free object in ^{3}He-A. Consider an infinite medium of ^{3}He-A. We assume the medium is asymptotically uniform, that is, $(\hat{\boldsymbol{\Delta}}_1, \hat{\boldsymbol{\Delta}}_2, \hat{\boldsymbol{l}})$ approaches a standard orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ as $|x| \to \infty$. Since all the points far from the origin are mapped to a single point, we have compactified $\mathbb{R}^3$ to S^3. Then the texture is classified according to $\pi_3(\mathbb{R}P^3) = \mathbb{Z}$. Let us specify an element of SO(3) by a 'vector' $\boldsymbol{\Omega} = \theta\boldsymbol{n}$ in $\mathbb{R}P^3$ as before (example 4.12). Shankar (1977) proposed a texture,

$$\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{\boldsymbol{r}}{r} \cdot f(r) \tag{4.73}$$

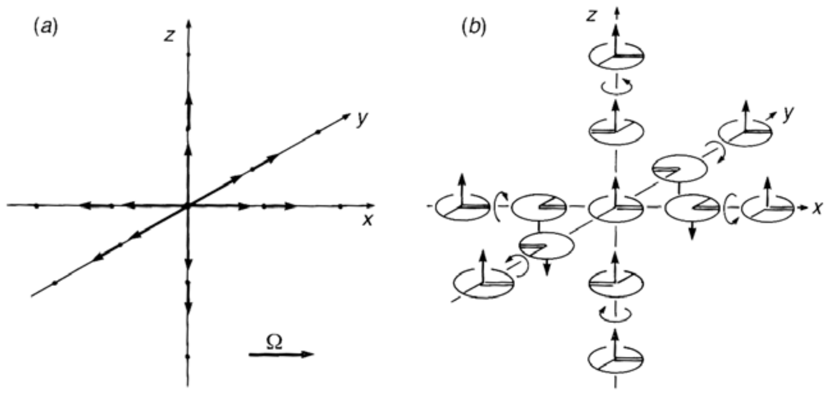


Figure 4.32. The Shankar monopole: (a) shows the 'vectors' $\boldsymbol{\Omega}(\boldsymbol{r})$ and (b) shows the triad $(\hat{\boldsymbol{\Delta}}_1, \hat{\boldsymbol{\Delta}}_2, \hat{\boldsymbol{l}})$. Note that as $|\boldsymbol{r}| \to \infty$ the triad approaches the same configuration.

where $f(r)$ is a monotonically decreasing function such that

$$f(r) = \begin{cases} 2\pi & r = 0 \\ 0 & r = \infty. \end{cases} \tag{4.74}$$

We formally extend the radius of $\mathbb{R}P^3$ to 2π and define the rotation angle modulo 2π. This texture is called the **Shankar monopole**, see figure 4.32(a). At first sight it appears that there is a singularity at the origin. Note, however, that the length of $\boldsymbol{\Omega}$ is 2π there and it is equivalent to the unit element of SO(3). Figure 4.32(b) describes the triad field. Since $\boldsymbol{\Omega}(\boldsymbol{r}) = 0$ as $r \to \infty$, irrespective of the direction, the space $\mathbb{R}^3$ is compactified to S^3. As we scan the whole space, $\boldsymbol{\Omega}(\boldsymbol{r})$ sweeps SO(3) twice and this texture corresponds to class 1 of $\pi_3(\mathrm{SO}(3)) \cong \mathbb{Z}$.

Exercise 4.10. Sketch the Shankar monopole which belongs to the class -1 of $\pi_3(\mathbb{R}P^3)$. [You cannot simply reverse the arrows in figure 4.32.]

Exercise 4.11. Consider classical Heisenberg spins defined in $\mathbb{R}^2$, see section 4.8. Suppose spins take the asymptotic value

$$\boldsymbol{n}(x) \to \boldsymbol{e}_z \qquad |x| \geq L \tag{4.75}$$

for the total energy to be finite, see figure 4.20. Show that the extended objects in this system are classified by $\pi_2(S^2)$. Sketch examples of spin configurations for the classes -1 and $+2$.

Problems

4.1 Show that the n-sphere S^n is a deformation retract of punctured Euclidean space $R^{n+1} - \{0\}$. Find a retraction.

4.2 Let D^2 be the two-dimensional closed disc and $S^1 = \partial D^2$ be its boundary. Let $f : D^2 \to D^2$ be a smooth map. Suppose f has no fixed points, namely $f(p) \neq p$ for any $p \in D^2$. Consider a semi-line starting at p through $f(p)$ (this semi-line is always well defined if $p \neq f(p)$). The line crosses the boundary at some point $q \in S^1$. Then define $\tilde{f} : D^2 \to S^1$ by $\tilde{f}(p) = q$. Use $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = \{0\}$ to show that such an $\tilde{f}$ does not exist and hence, that f must have fixed points. [*Hint*: Show that if such an $\tilde{f}$ existed, D^2 and S^1 would be of the same homotopy type.] This is the two-dimensional version of the **Brouwer fixed-point theorem**.

4.3 Construct a map $f : S^3 \to S^2$ which belongs to the elements 0 and 1 of $\pi_3(S^2) \cong \mathbb{Z}$. See also example 9.9.