由哈密顿形式导出带电粒子运动方程

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带电粒子的哈密顿量为

$$\hat{H} = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 + e\phi. \tag{0.1}$$

我们利用哈密顿正则方程

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{\partial H}{\partial \boldsymbol{p}} \\ \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{q}}, \end{aligned} \tag{0.2}$$

则有

$$\dot{\boldsymbol{q}} = \frac{1}{m}(\boldsymbol{p} - e\boldsymbol{A}) \tag{0.3}$$

$$\dot{\boldsymbol{p}} = -\left(e\boldsymbol{\nabla}\phi + \frac{1}{2m}\boldsymbol{\nabla}[(\boldsymbol{p} - e\boldsymbol{A})^2]\right) \tag{0.4}$$

又注意到上式有

$$\begin{aligned} \dot{\boldsymbol{p}} &= -\left(e\boldsymbol{\nabla}\phi + \frac{1}{2m}\boldsymbol{\nabla}[(\boldsymbol{p} - e\boldsymbol{A})^2]\right) \\ &= -e\boldsymbol{\nabla}\phi - \frac{1}{m}\left\{[(\boldsymbol{p} - e\boldsymbol{A}) \cdot \boldsymbol{\nabla}](\boldsymbol{p} - e\boldsymbol{A}) + (\boldsymbol{p} - e\boldsymbol{A}) \times [\boldsymbol{\nabla} \times (\boldsymbol{p} - e\boldsymbol{A})]\right\} \\ &= -e\boldsymbol{\nabla}\phi + e(\boldsymbol{v} \cdot \nabla)\boldsymbol{A} + e\boldsymbol{v} \times (\nabla \times \boldsymbol{A}). \end{aligned} \tag{0.5}$$

上式中，我们利用了 $m\boldsymbol{v} = \boldsymbol{p} - e\boldsymbol{A}$ 以及矢量恒等式 $\boldsymbol{\nabla}(\boldsymbol{a} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a} + \boldsymbol{b} \times$

$(\boldsymbol{\nabla} \times \boldsymbol{a}) + \boldsymbol{a} \times (\boldsymbol{\nabla} \times \boldsymbol{b})$。自然地，我们就有

$$
\begin{aligned}
m\ddot{\boldsymbol{q}} &= -e\boldsymbol{\nabla}\phi + e(\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + e\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) - e\frac{d\boldsymbol{A}}{dt} \\
&= -e\boldsymbol{\nabla}\phi + e(\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + e\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) - e\frac{\partial\boldsymbol{A}}{\partial t} - e(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A} \\
&= -e\boldsymbol{\nabla}\phi - e\frac{\partial\boldsymbol{A}}{\partial t} + e\boldsymbol{v}\times(\nabla\times\boldsymbol{A}) \\
&= e\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right)
\end{aligned}
\tag{0.6}
$$