$$|\psi_n(t)\rangle = e^{i\gamma_n(t)} \exp\left[-\frac{i}{\hbar}\int_0^t dt' \varepsilon_n(\boldsymbol{R}(t'))\right]|n(\boldsymbol{R}(t))\rangle, \tag{1.3}$$

where the second exponential is known as the dynamical phase factor. Inserting Eq. (1.3) into the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi_n(t)\rangle = H(\boldsymbol{R}(t))|\psi_n(t)\rangle \tag{1.4}$$

and multiplying it from the left by $\langle n(\boldsymbol{R}(t))|$, one finds that γ_n can be expressed as a path integral in the parameter space

$$\gamma_n = \int_{\mathcal{C}} d\boldsymbol{R}\cdot\mathcal{A}_n(\boldsymbol{R}), \tag{1.5}$$

where $\mathcal{A}_n(\boldsymbol{R})$ is a vector-valued function

$$\mathcal{A}_n(\boldsymbol{R}) = i\langle n(\boldsymbol{R})|\frac{\partial}{\partial\boldsymbol{R}}|n(\boldsymbol{R})\rangle. \tag{1.6}$$

This vector $\mathcal{A}_n(\boldsymbol{R})$ is called the Berry connection or the Berry vector potential. Equation (1.5) shows that, in addition to the dynamical phase, the quantum state will acquire an additional phase γ_n during the adiabatic evolution.

Obviously, $\mathcal{A}_n(\boldsymbol{R})$ is gauge dependent. If we make a gauge transformation

$$|n(\boldsymbol{R})\rangle \to e^{i\zeta(\boldsymbol{R})}|n(\boldsymbol{R})\rangle, \tag{1.7}$$

with $\zeta(\boldsymbol{R})$ an arbitrary smooth function and $\mathcal{A}_n(\boldsymbol{R})$ transforms according to

$$\mathcal{A}_n(\boldsymbol{R}) \to \mathcal{A}_n(\boldsymbol{R}) - \frac{\partial}{\partial\boldsymbol{R}}\zeta(\boldsymbol{R}). \tag{1.8}$$

Consequently, the phase γ_n given by Eq. (1.5) will be changed by $\zeta(\boldsymbol{R}(0))-\zeta(\boldsymbol{R}(T))$ after the transformation, where $\boldsymbol{R}(0)$ and $\boldsymbol{R}(T)$ are the initial and final points of the path $\mathcal{C}$. This observation has led Fock (1928) to conclude that one can always choose a suitable $\zeta(\boldsymbol{R})$ such that γ_n accumulated along the path $\mathcal{C}$ is canceled out, leaving Eq. (1.3) with only the dynamical phase. Because of this, the phase γ_n has long been deemed unimportant and it was usually neglected in the theoretical treatment of time-dependent problems.

This conclusion remained unchallenged until Berry (1984) reconsidered the cyclic evolution of the system along a *closed* path $\mathcal{C}$ with $\boldsymbol{R}(T)=\boldsymbol{R}(0)$. The phase choice we made earlier on the basis function $|n(\boldsymbol{R})\rangle$ requires $e^{i\zeta(\boldsymbol{R})}$ in the gauge transformation [Eq. (1.7)] to be single valued, which implies

$$\zeta(\boldsymbol{R}(0)) - \zeta(\boldsymbol{R}(T)) = 2\pi\times\text{integer}. \tag{1.9}$$

This shows that γ_n can be only changed by an integer multiple of 2π under the gauge transformation [Eq. (1.7)] and it cannot be removed. Therefore for a closed path, γ_n becomes a gauge-invariant physical quantity, now known as the Berry phase or geometric phase in general; it is given by

$$\gamma_n = \oint_{\mathcal{C}} d\boldsymbol{R}\cdot\mathcal{A}_n(\boldsymbol{R}). \tag{1.10}$$

From the above definition, we can see that the Berry phase only depends on the geometric aspect of the closed path and is independent of how $\boldsymbol{R}(t)$ varies in time. The explicit time dependence is thus not essential in the description of the Berry phase and will be dropped in the following discussion.

2. Berry curvature

It is useful to define, in analogy to electrodynamics, a gauge-field tensor derived from the Berry vector potential:

$$\begin{aligned}\Omega^n_{\mu\nu}(\boldsymbol{R}) &= \frac{\partial}{\partial R^\mu}\mathcal{A}^n_\nu(\boldsymbol{R}) - \frac{\partial}{\partial R_\nu}\mathcal{A}^n_\mu(\boldsymbol{R})\\ &= i\left[\left\langle\frac{\partial n(\boldsymbol{R})}{\partial R^\mu}\middle|\frac{\partial n(\boldsymbol{R})}{\partial R^\nu}\right\rangle - (\nu\leftrightarrow\mu)\right].\end{aligned} \tag{1.11}$$

This field is called the Berry curvature. Then according to Stokes's theorem the Berry phase can be written as a surface integral

$$\gamma_n = \int_{\mathcal{S}} dR^\mu \wedge dR^\nu \tfrac{1}{2}\Omega^n_{\mu\nu}(\boldsymbol{R}), \tag{1.12}$$

where $\mathcal{S}$ is an arbitrary surface enclosed by the path $\mathcal{C}$. It can be verified from Eq. (1.11) that, unlike the Berry vector potential, the Berry curvature is gauge invariant and thus observable.

If the parameter space is three dimensional, Eqs. (1.11) and (1.12) can be recast into a vector form

$$\boldsymbol{\Omega}_n(\boldsymbol{R}) = \boldsymbol{\nabla}_{\boldsymbol{R}}\times\mathcal{A}_n(\boldsymbol{R}), \tag{1.11$'$}$$

$$\gamma_n = \int_{\mathcal{S}} d\boldsymbol{S}\cdot\boldsymbol{\Omega}_n(\boldsymbol{R}). \tag{1.12$'$}$$

The Berry curvature tensor $\Omega^n_{\mu\nu}$ and vector $\boldsymbol{\Omega}_n$ are related by $\Omega^n_{\mu\nu}=\epsilon_{\mu\nu\xi}(\boldsymbol{\Omega}_n)_\xi$ with $\epsilon_{\mu\nu\xi}$ the Levi-Cività antisymmetry tensor. The vector form gives us an intuitive picture of the Berry curvature: it can be viewed as the magnetic field in the parameter space.

Besides the differential formula given in Eq. (1.11), the Berry curvature can be also written as a summation over the eigenstates:

$$\Omega^n_{\mu\nu}(\boldsymbol{R}) = i\sum_{n'\neq n}\frac{\langle n|\partial H/\partial R^\mu|n'\rangle\langle n'|\partial H/\partial R^\nu|n\rangle - (\nu\leftrightarrow\mu)}{(\varepsilon_n-\varepsilon_{n'})^2}. \tag{1.13}$$

The curvature can be obtained from Eq. (1.11) using $\langle n|\partial H/\partial\boldsymbol{R}|n'\rangle=\langle\partial n/\partial\boldsymbol{R}|n'\rangle(\varepsilon_n-\varepsilon_{n'})$ for $n'\neq n$. The summation formula has the advantage that no differentiation on the wave function is involved, therefore it can be evaluated under any gauge choice. This property is par-