2021. 4. 8.

1. one-parameter compactification.

$\mathbb{R}^2 \cup \{\infty\} \simeq S^2$

2. $\mathbb{R}/\mathbb{Z} \simeq S^1 \Rightarrow$ Bloch 球.

3. Geometry $H(\phi) \Rightarrow \phi \in [0, 2\pi]$

材料：张庆华.(Heavyside) 和他的科学成就.

$\Downarrow$ Heavyside function

① 德国人将 maxwell $\rightarrow$ 简化到4个.

$\uparrow$

用微分形式.

② 坡印亭矢量.

③ 电子质量源自于电磁场

作业：① Non-Abelian Geometry phase & Yang-Mills eq. 推导.

② 推导 Byers. Yang, PRL. 7. 46. (1961).
Quantized magnetic flux in SC cylinders

(应用) Geometry $\rightarrow$ AB effect $\rightarrow$ SC.

Geometry phase TKNN #. $\sigma = ne^2/h$

推导（三个版本）
① Berry (1984).
② Linear response. (Shen Shun Qin)
③ QFT

$H(t) \to H(R(t))$

$$\begin{cases} H(t) = \vec{B}(t)\cdot\vec{\sigma} \\ H(t) = \frac{P^2}{2m} + \frac{1}{2} m \omega^2(t) x^2 \end{cases}$$

$|\psi_m(R(t))\rangle = |m\rangle$

$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$
$|\psi\rangle = |m\rangle \underbrace{e^{i\theta(t)}}_{U(1)}$ 绝热演化

$$i\frac{\partial}{\partial t}|\psi\rangle = i(\partial_t|m\rangle)e^{i\theta} + i|m\rangle i\dot{\theta} e^{i\theta}$$
$$= H|m\rangle e^{i\theta}$$
$$= E_m|m\rangle e^{i\theta}$$

$$\cancel{i\frac{\partial}{\partial t}|m\rangle} - |m\rangle\dot{\theta} = E_m|m\rangle$$

$$\Rightarrow \dot{\theta} = E_m \Rightarrow \theta = \int_0^t E_m(t)\,dt.$$

令 $\theta = \int_0^T E_m(t)\,dt + \varphi$

$$\dot\theta = E_m + \dot\varphi$$

$$\Rightarrow i\frac{\partial}{\partial t}|m\rangle = \dot\varphi\,|m\rangle$$

$$\therefore \quad \dot\varphi = i\langle m|\frac{\partial}{\partial t}|m\rangle .$$

$$\varphi(T)-\varphi(0) = i\int_0^T \langle m|\frac{\partial}{\partial t}|m\rangle\,dt$$

$$\Rightarrow i\int_{R(0)}^{R(T)} \langle m|\partial/\partial\vec R\,|m\rangle\cdot d\vec R$$

$$= \oint \vec A\cdot d\vec R .$$

$\Downarrow$ 规范不变

$$\oint(\vec A+\nabla\phi)\cdot d\vec x$$

$$= \oint \vec A\,d\vec x + \underbrace{\oint \nabla\phi\cdot d\vec x}_{\text{全微分}}$$

$$= 0 .$$

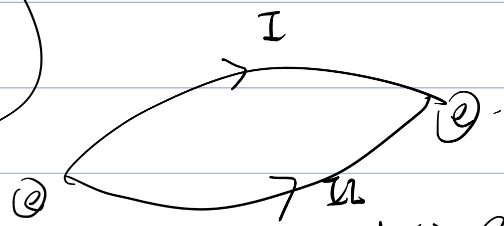
等价类 $\{\vec A\} = \{\vec A + \nabla\phi \mid \phi \sim \ldots\}$

$\langle\tilde m|\nabla|\tilde m\rangle = \langle m|e^{-i\phi_m}\nabla e^{i\phi_m}|m\rangle = \langle m|\nabla|m\rangle + i\nabla\phi_m .$

$\oint\langle\tilde m|\nabla|\tilde m\rangle = \oint\langle m|\nabla|\phi\rangle$, 波函数相位不影响

(规范不变)

AB效应



$$|\psi\rangle e^{i\int_{I}\vec{A}\cdot d\vec{r}} + |\psi\rangle e^{i\int_{II}\vec{A}\cdot d\vec{r}}$$

$$= |\psi\rangle e^{i\int_{I}\vec{A}\cdot d\vec{r}}\left(1 + e^{i\oint_{I+II}\vec{A}\cdot d\vec{r}}\right)$$

★

$$\gamma = i\oint \langle m|\frac{\partial}{\partial t}|m\rangle dt$$

$$= i\oint \langle m|\partial_{\vec{R}}|m\rangle\cdot d\vec{R}$$

$$= i\oint \langle m|d|m\rangle.$$

$$\begin{cases} w = i\langle m|d|m\rangle \\ dw = i(d\langle m|)\wedge d|m\rangle \end{cases}$$

★

$$\langle m|\nabla|n\rangle = \frac{\langle m|(\nabla H)|n\rangle}{E_n - E_m}, \quad n \neq m.$$

$$\partial_x H\psi - H\partial_x\psi$$

$$[\nabla, H] = (\nabla H) \leftarrow \boxed{(\partial_x H)\psi} + \underbrace{H(\partial_x\psi) - H\partial_x\psi}_{=0}$$

证.

$$\langle m|(\nabla H - H\nabla)|n\rangle = (E_n - E_m)\langle m|\nabla|n\rangle$$

$$\gamma = i\oint \langle m|\partial_{\vec{R}}|m\rangle \cdot d\vec{R}$$

$$= i\oint \nabla\times\langle m|\partial_{\vec{R}}|m\rangle \cdot d\vec{S}$$

$$= -\mathrm{Im}\oint \nabla\times\langle m|\partial_{R}|m\rangle \cdot dS$$

???

$$= -\mathrm{Im}\oint \langle\nabla m|\times|\nabla m\rangle \cdot d\vec{S}$$

$$\sum_n |n\rangle\langle n| = 1$$

$$= -\mathrm{Im}\oint \sum_{n\neq m}\langle\nabla m|n\rangle\times\langle n|\nabla m\rangle \cdot d\vec{S}$$

$$= -\mathrm{Im}\sum_{m\neq n}\oint d\vec{S}\cdot\boxed{\frac{\langle m|(\nabla H)|n\rangle\times\langle n|(\nabla H)|m\rangle}{(E_m-E_n)^2}}$$

$$\boxed{\gamma_m = -\oint d\vec{S}\cdot\vec{V}_m(\vec{R})}$$

例子:

$$H = \begin{pmatrix} B_z & Be^{i\phi(t)} \\ Be^{-i\phi(t)} & -B_z \end{pmatrix}, \quad E = \sqrt{B_z^2+B^2}$$

$$\gamma = i\oint\langle\psi(\phi)|\partial_\phi|\psi(\phi)\rangle\, d\phi$$

$$\begin{pmatrix} B_z & Be^{i\phi} \\ Be^{-i\phi} & -B_z \end{pmatrix}\begin{pmatrix} u\,e^{i\phi} \\ v \end{pmatrix} = -E_g\begin{pmatrix} u\,e^{i\phi} \\ v \end{pmatrix}$$

$$\begin{cases} B_z u \cancel{e^{i\phi}} + B \cancel{e^{i\phi}} v = -E_g u \cancel{e^{i\phi}} \\ B u - B_z v = -E_g v \end{cases}$$

$$i \langle \psi(\phi) | \partial_\phi | \psi(\phi) \rangle = i \left(u e^{-i\phi}, v \right) \begin{pmatrix} u i e^{i\phi} \\ 0 \end{pmatrix} = -|u|^2$$

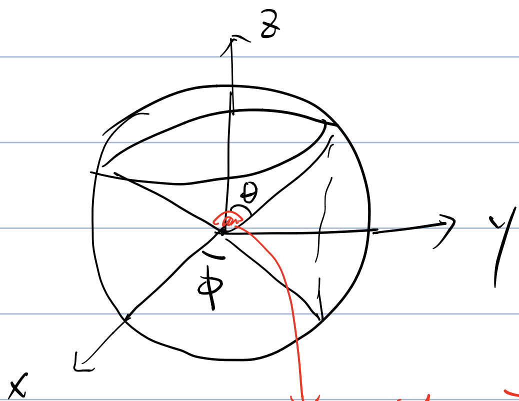
$$\gamma = i \oint \vec{A} \cdot d\phi = -|u|^2 \oint d\phi = -2\pi |u|^2$$

$-2\pi (\sin\theta/2)^2$

$= -2\pi(1-\cos\theta)$

立体角

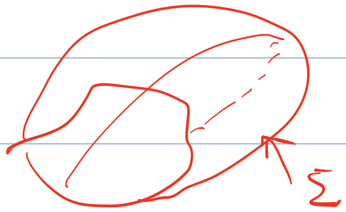
$$\begin{cases} B_z = R\cos\theta \\ B = R\sin\theta \end{cases} \Rightarrow u = \sin\theta/2$$



立体角 → γ

$$\gamma = -\int_\Sigma d\vec{S} \cdot \vec{V}_m(\vec{R})$$

① 有限R时



② $S \to \infty$ (loop, ∞大)

$\mathbb{R}^2 \cup \{\infty\} \to S^2$

球面上的积分

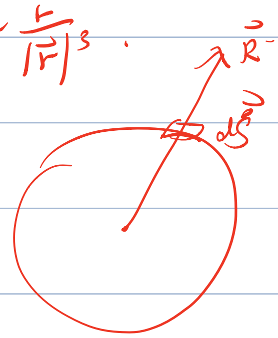
$$\gamma = -\int_{S^2} d\vec{S} \cdot \vec{V}_m(\vec{R})$$

$$= -\int_{S^2} d\vec{S} \cdot (\nabla \times \vec{A}_m) \neq 0$$

$$V_m = \frac{g\vec{R}}{|\vec{R}|^3} \to \vec{E} \sim \frac{\vec{r}}{|\vec{r}|^3}$$

$$\Rightarrow -g \frac{R}{R^3} 4\pi R^2$$

$$= -4\pi g$$



$$\oint_{S^2} d\vec{S}\cdot\vec{B} = \langle S^2, dw\rangle = \langle \partial S^2, w\rangle = 0.$$

$\to \quad \nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0$ 与上面等价

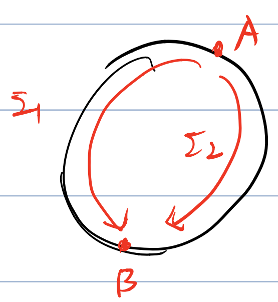
用的都是 Stokes 定理

$$\frac{1}{2\pi i}\oint \frac{dz}{z} = \frac{1}{2\pi i}\left(\int_{\Sigma_1}\frac{dz}{z} + \int_{\Sigma_2}\frac{dz}{z}\right)$$

$$= \frac{1}{2\pi i}\left(\ln z\Big|_{B}^{A}{}_{\Sigma_1} + \ln z\Big|_{B}^{A}{}_{\Sigma_2}\right)$$

$$= \frac{1}{2\pi i}\left(\ln z\Big|_{A\,\Sigma_1} - \ln z\Big|_{A\,\Sigma_2}\right)$$

$$\frac{(n-m)2\pi i}{2\pi i}$$



$$= \frac{2\pi\tilde{n} i}{2\pi i} = (n-m)$$

$$\frac{dz}{z}\Big|_A, \quad z \to z e^{i2\pi n}.$$