Geometry Phase.

如何计算.

经典： $I = \oint \vec{p} \cdot d\vec{q}$. 绝热不变量.

有几何性.

$$\boxed{\begin{array}{l}\text{相空间体积元不变.} \\ \Omega = V/h^{d} \iff S = k_B \ln \Omega\end{array}}$$

量子：$I = \oint \langle \psi(q) | \hat{p} | \psi(q) \rangle \, dq$

q可以是一个参数。(注意)

e.g.

$$H = \vec{B} \cdot \vec{\sigma}$$

$$= B_{//} \cos(\phi) \sigma_x + B_{//} \sin(\phi) \sigma_y + B_{\perp} \sigma_z.$$

$$\phi \to \phi(t)$$

$$= H(\phi)$$

$$\gamma = I = \oint \langle \phi(\vec{R}) | -i\hbar \vec{\nabla}_R | \phi(\vec{R}) \rangle \cdot d\vec{R}$$

闭合曲面，(物理上)

$$\left.\begin{array}{l}
1、\text{one-point compactification.} \\
\qquad \mathbb{R}^2 \cup \{\infty\} = S^2 \\
2、\text{布里渊区}, \quad T^2 \\
3、\quad S^1 \quad \phi \in [0, 2\pi)
\end{array}\right\} \text{manifold}$$

$P(g) \in C^N$

Fibre Bundle

mapping

$$\oint \langle\psi|P|\psi\rangle \cdot dg = \oint P(g)\cdot dg.$$

Topo 不变.

1、从 mapping 角度理解 topology

2、形变和 mapping 之间的关系。

材料：

①孙小庆华、Heavyside 和它的科学成就.

小Ref.

Geometry phase & TKNN number

$$G = n\frac{e^2}{h}$$

Ref: Berry, "Quantum Phase factors accomponing adabatic changes". (1984)

HW.

① Non-Abelian Geometry phase & Yang-Mills. 推导

② 推导 Byers, Yang. PRL, 7, 46 (1961)

推导 Geometry Phase.

$H(\vec{R}(t))$ ① 含时 $\vec{R}(0)=\vec{R}(T)$

② 含时通过参数进入 H

e.g. ① $H=\vec{B}(t)\cdot\vec{\sigma}$

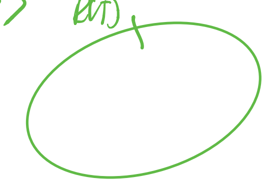
② $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2(t)x^2$

瞬时本征波函数. $|\phi(x,\omega)\rangle$

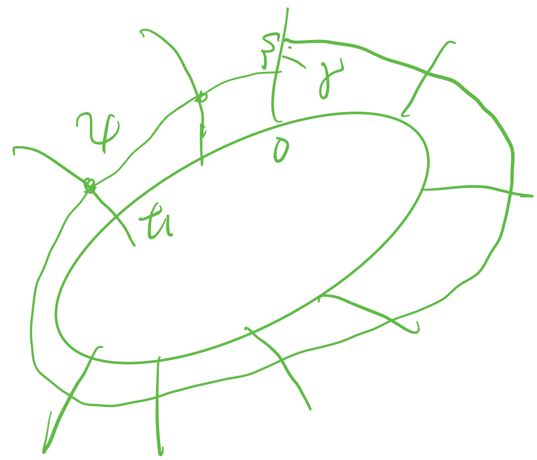
Geometry → AB effect

SC

$\vec{R}(T)$ $\vec{R}(0)$



$$H(\vec{R}(t))|\psi_m(\vec{R}(t))\rangle = \varepsilon_m(\vec{R}(t))|\psi_m(\vec{R}(t))\rangle$$



Non-Abelian phase. 时间反演不变系统.

$$H|\psi_{n\alpha}(\vec{R})\rangle = E_{n\alpha}|\psi_{n\alpha}(\vec{R})\rangle$$

<u>Gauge invariant</u>

简化: $|\psi_m(\vec{R}(t))\rangle \equiv |m\rangle$

$$i\frac{\partial}{\partial t}|\phi\rangle = H|\phi\rangle$$

$$|\phi\rangle = |m\rangle e^{i\theta(t)} \quad \underline{U(1)}$$

$\Downarrow$

$$i\frac{\partial}{\partial t}|\phi\rangle = i\left(\frac{\partial}{\partial t}|m\rangle\right)e^{i\theta} + i\,i\dot{\theta}|m\rangle e^{i\theta}$$

$\Downarrow$

$$i\frac{\partial}{\partial t}|m\rangle - |m\rangle\dot{\theta} = E_m|m\rangle$$

$\Downarrow$

令 $\theta = -\int_0^t E_m(\tau)d\tau$, Dynamic phase.

$+\varphi$

degenerate space (提示).

$$|\phi\rangle \equiv U_{ab}(t)|\psi_{mb}\rangle$$

基态空间

$$\langle\phi|\phi\rangle \Leftrightarrow \underline{U^\dagger U = 1}.$$

$$|\psi_m(t)\rangle \underset{\text{Rotation}}{=} R(t)|\psi_m\rangle$$

$\Downarrow$. $\dot{\theta} = -E_m + \varphi$ $\Rightarrow$ $i\frac{\partial}{\partial t}|m\rangle = \dot{\varphi}|m\rangle$

$$\dot{\varphi} = i\langle m|\frac{\partial}{\partial t}|m\rangle$$

$\Downarrow$.

$$\varphi(T) - \varphi(0) = i\int_0^T \langle m|\frac{\partial}{\partial t}|m\rangle dt$$

注意： $|m\rangle = |m(R(t))\rangle \Rightarrow \frac{\partial}{\partial t}|m\rangle = \frac{\partial \vec{R}}{\partial t}\cdot\vec{\nabla}_R|m\rangle$

$$\Rightarrow \varphi(T) - \varphi(0) = i\oint \underbrace{\langle m|\vec{\nabla}_R|m\rangle d\vec{R}}_{\text{一个 1-form}}$$

$$\equiv i\oint \vec{A}\cdot d\vec{R}$$

证明： AB effect $\oint \vec{A}\cdot d\vec{l} = \oint(\vec{A}+\vec{\nabla}\phi)\cdot d\vec{l}$

$$= \oint \vec{A}\cdot d\vec{l} + \underbrace{\oint(\vec{\nabla}\phi)d\vec{l}}_{=0}$$

$$= \oint \vec{A}\cdot d\vec{l}$$

$$\boxed{\oint_S d\phi = \int_{\partial S}\phi = 0}$$

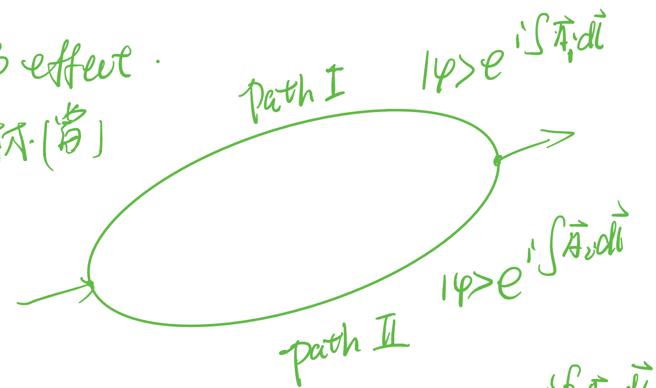
Gauge Transformation.

$$\langle m|\nabla_R|m\rangle \Rightarrow \langle m|e^{-i\Phi_m}\nabla_R e^{i\Phi_m}|m\rangle$$

$$= \langle m|\nabla_R|m\rangle + i\nabla_R \Xi_m$$

自然地我们可以注意到 Geometry Phase 是 Gauge Invariant

AB effect.
教材P1(首)



$$\Rightarrow |\psi\rangle e^{i\int_{I} \vec{A}_1\cdot d\vec{l}} + |\psi\rangle e^{i\int_{II} \vec{A}_2\cdot d\vec{l}}$$

$$= |\psi\rangle e^{i\int \vec{A}_1\cdot d\vec{l}}\left[1 + e^{\underline{i\oint \vec{A}\cdot d\vec{l}}}\right]$$

AB相位.

表示 $\gamma = i\oint\langle m|\frac{\partial}{\partial t}|m\rangle\cdot dt$

$$= i\oint\langle m|\nabla_R|m\rangle\cdot d\vec{R}$$

$$= \underline{i\oint\langle m|d|m\rangle}$$, d 微分算子.

TKNN

预备: $\langle m|\nabla|n\rangle = \dfrac{\langle m|(\nabla H)|n\rangle}{E_n - E_m}$ $(n\neq m)$

证明: $[\nabla, H] = \nabla H$

$$\Rightarrow \langle m|\nabla H|n\rangle = \langle m|(\nabla H - H\nabla)|n\rangle$$

$$= (E_n - E_m)\langle m|\nabla|n\rangle$$

$$\gamma_m = -\mathrm{Im}\oint \langle m|\vec{\nabla}_R|m\rangle \cdot d\vec{R}$$

$$= -\mathrm{Im}\int d\vec{S}\cdot \nabla\times\vec{A}$$

$$= -\mathrm{Im}\oint d\vec{S}\cdot\vec{\nabla}\times\langle m|\vec{\nabla}_R|m\rangle$$

$\vec{A}$ 形式上与磁场一致。
$\nabla\cdot\vec{B}=0$
参数空间 $\Rightarrow$ Topo Insulator

$$= -\mathrm{Im}\oint d\vec{S}\cdot\langle\nabla m|\times|\nabla m\rangle$$

$A = i\langle m|d|m\rangle$

$dA = i\langle dm|dm\rangle$

$I = \sum_n |n\rangle\langle n|$

$$= -\mathrm{Im}\sum_{n\neq m}\oint d\vec{S}\cdot\langle\vec{\nabla} m|n\rangle\times\langle n|\vec{\nabla} m\rangle$$

$$= -\mathrm{Im}\sum_{n\neq m}\oint d\vec{S}\cdot\frac{\langle m|(\vec{\nabla}H)|n\rangle\times\langle n|(\nabla H)|m\rangle}{(E_m-E_n)^2}$$

$$\gamma_m = -\mathrm{Im}\oint d\vec{S}\cdot\vec{V}_m(\vec{R})$$

例子：$H=\begin{pmatrix}B_z & Be^{i\phi(t)}\\ Be^{-i\phi(t)} & -B_z\end{pmatrix} \Rightarrow \lambda_\pm = \sqrt{B_z^2+B^2}$

$$\gamma = i\oint\langle\psi(\phi)|\frac{\partial}{\partial R}|\psi(\phi)\rangle\cdot d\phi$$

一个本征波函数. $\begin{pmatrix}ue^{i\phi}\\ v\end{pmatrix}$

$$\Rightarrow \gamma = i\oint i|u|^2 d\varphi = -|u|^2\oint d\varphi = -2\pi|u|^2$$

$$\begin{cases} B_z = R\cos\theta \\ B = R\sin\theta \end{cases}$$

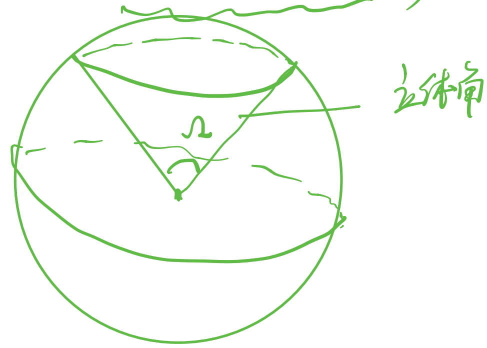
$$\Rightarrow H = R\begin{pmatrix}\cos\theta & \sin\theta e^{i\phi}\\ \sin\theta e^{-i\phi} & -\cos\theta\end{pmatrix}$$

$$\Rightarrow \lambda^2-\cos^2\theta-\sin^2\theta=0$$

$$\Rightarrow \lambda=\pm 1$$

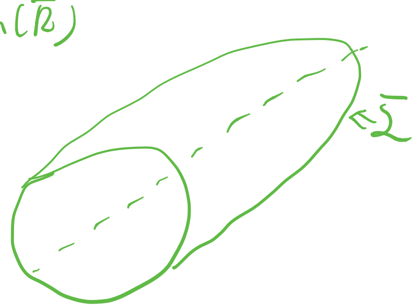
$$\Rightarrow |\psi_+\rangle=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} \qquad |\psi_-\rangle=\begin{pmatrix}-\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

$$\Rightarrow \gamma=-2\pi\sin^2\frac{\theta}{2}=\underline{-\pi[1-\cos\theta]}$$，轨迹围成的立体角.



$$\gamma=-\int d\vec{S}\cdot\vec{V}_n(\vec{R})$$

1、有限尺寸



2、loop是无穷大

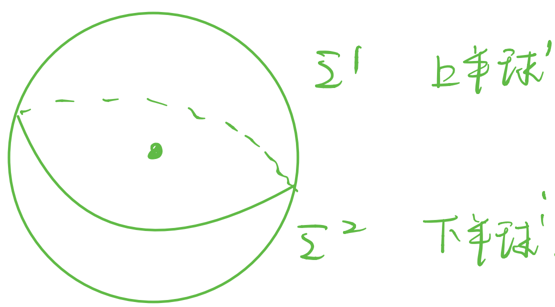
$R^2\cup\{\infty\}\to S^2$. 球面上的积分.

$$\gamma=-\int_{S^2}d\vec{S}\cdot\vec{V}_n(\vec{R})$$
$$=-\int_{S^2}d\vec{S}(\nabla\times\vec{A}_n)$$

$$V_m = \frac{g\vec{R}}{|\vec{R}|^3} \to \vec{E} \cup \frac{\vec{r}}{|\vec{r}|^3}$$

$$\Rightarrow \gamma = -4\pi g$$

Wu-Yang. PRD. 12. 3845. (1975)



$$\int_{S^2} = \underset{A_1}{\int_{\Sigma^1}} + \underset{A_2}{\int_{\Sigma^2}}$$

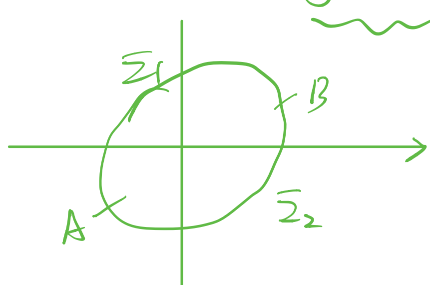
$$\begin{cases} \oint_{\Sigma^1} \vec{B}\cdot d\vec{S} = \oint_{\partial\Sigma^1} \vec{A}_1\cdot d\vec{l} \\ \oint_{\Sigma^2} \vec{B}\cdot d\vec{S} = \oint_{\partial\Sigma^2} \vec{A}_2\cdot d\vec{l} \end{cases} \qquad \partial\Sigma^1 + \partial\Sigma^2 = \partial\Sigma$$

$$\Rightarrow \oint d\vec{S}\cdot\vec{V}_m = \oint_{\partial\Sigma^1} (\vec{A}_1 - \vec{A}_2)\cdot d\vec{l}$$

, 需满足 $\nabla\times\vec{A}_1 = \nabla\times\vec{A}_2$

$\Rightarrow \vec{A}_1 - \vec{A}_2 = \nabla\phi$

$$= \oint \nabla\phi\cdot d\vec{l} = 2\pi n$$



$$\frac{1}{2\pi i}\oint \frac{1}{z}dz = \frac{1}{2\pi i}\left(\int_{\Sigma_1}\frac{dz}{z} + \int_{\Sigma_2}\frac{dz}{z}\right)$$

$$= \frac{1}{2\pi i}\left(\ln z\Big|_B^A + \ln z\Big|_A^B\right)$$

$\underbrace{\qquad}_{z_1}$ $\qquad$ $\underbrace{\qquad}_{z_2}$

$\alpha \to \alpha e^{i2\pi n}$

$$= \frac{1}{2\pi i}(\ln \alpha_A - \ln \alpha_B)$$

$$= \underline{\underline{n}}, \quad \in \underline{\underline{\mathbb{Z}}}$$

$$\frac{1}{2\pi i}\oint d\alpha = \frac{1}{2\pi i}\oint d\theta = \oint (\vec{\nabla}\theta)\cdot d\vec{l}$$

Summary: ① Geometry phase. $\Rightarrow$ $\boxed{\text{提供了一种构造参数空间物理量的方法}}$

$\updownarrow$ Stokes定理

TKNN number.

$\oint d\vec{S}\cdot\vec{B}$

$\downarrow$

monopole $\Longleftrightarrow$ 参数空间的磁单极子。